ADVANCED ANALYSIS:

AMORTIZATION AND RECURRECNE RELATIONS

- amortized time complexity
- accounting method
- Java vectors
- Recurrence Relations

Amortized Running Time

- Amortized running time considers interactions between operations by studying the total running time of a series of operations.
- **Example:** a **Clearable Stack**: supports the usual stack methods plus operation

  clearStack(): Empty the stack by removing all all its elements
  Input: None; Output: None

  clearStack takes O(n) time in the worst case
- **Proposition:** A series of n operations on an initially empty clearable stack implemented with an array takes overall O(n) time
- **Justification:**
  - Let $M_0 \ldots, M_{n-1}$ be the series of operations and $M_{i_0} \ldots, M_{i_k-1}$ be the k-th clearStack operations in the series
  - We define $i_{-1} = -1$
  - The run time of operation M_{i_j} is $O(i_j - i_{j-1})$ since at most $i_j - i_{j-1}$ elements can be on the stack

Amortized Running Time (cont)

- Thus the running time of all the clearStack operations is

$$O\left(\sum_{j=0}^{k-1}(i_j - i_{j-1})\right)$$

  which is a telescoping sum.
- So the run time is O(n)

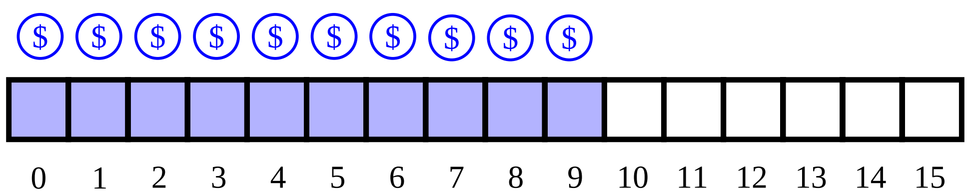
• **Definition:** the **amortized running time** of an operation within a series of operations is the worst-case running time of the entire series of operations divided by the number of operations.

Accounting Method

- The **accounting method** performs an amoritzation analysis with a system of credits and debits
- Let’s view the computer as vending machine that requires one cyber-dollar for a constant amount of computing time.
- An operation consists of a series of constant-time **primitive operations** that cost one cyber-dollar each.
- We will overcharge an operation that executes few primitives and use the profit to pay for operations that execute many primitives.
- We will need to set up a scheme for charging operations. This is known as the amoritization scheme.

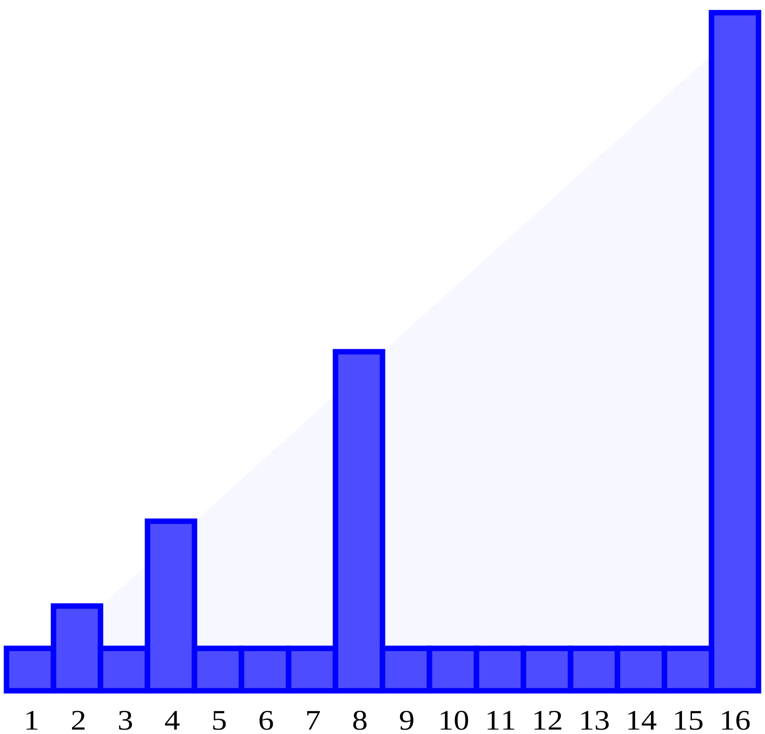
Amortization Scheme Example for a ClearableStack

- Assume one cyber-dollar is enough to pay for the push, pop, top, size, or isEmpty and for the time spent by the clearStack to dereference one element.
- We will charge 2 cyber-dollars though.
- So we undercharge clearStack but overcharge the other operations. When a clearStack operation is executed, the cyber-dollars stored in the stack are used to pay for derefencing the items.



Java Vectors

- The java.util.vector class provides a convenient expandable data type in Java.
- A vector is a wrapper around an array that holds a variable called capacityIncrement. When the user inserts the n+1st element into a vector of size n, the size of the array is increased by capacityIncrement if it is positive, or doubled if capacityIncrement is 0.
- Consider the case of capacityIncrement =0:
  - Copying an array into a larger array takes O(n) time, but this only happens for log(n) insertions.
  - Each insertion has O(1) amortized running time



Java Vectors (contd.)

- Justification:
  The array doubles in size with the insertion of every 2^ith element (1^{st}, 2^{nd}, 4^{th}, etc.)
  Worst case: we insert exactly $n = 2^i$ elements, so the last operation involves copying the entire array over again.
  We have n insertions, and n elements copied in the last insertion. We also have i-1 previous expansions of the array, which perform the following number of element-copy operations:

$$\sum_{k=1}^{i-1} 2^k = 2^i - 1 = n - 1$$

- The overall time complexity is proportional to $3n$-1, which is O(n)
- But what if the capacityIncrement is, say, 3?
  Do we still have the same amortization?
  - No! Copying an array into a larger array is O(n), but this happens once every n/capacityIncrement insertions.
  - Each insertion is amortized to O(n)

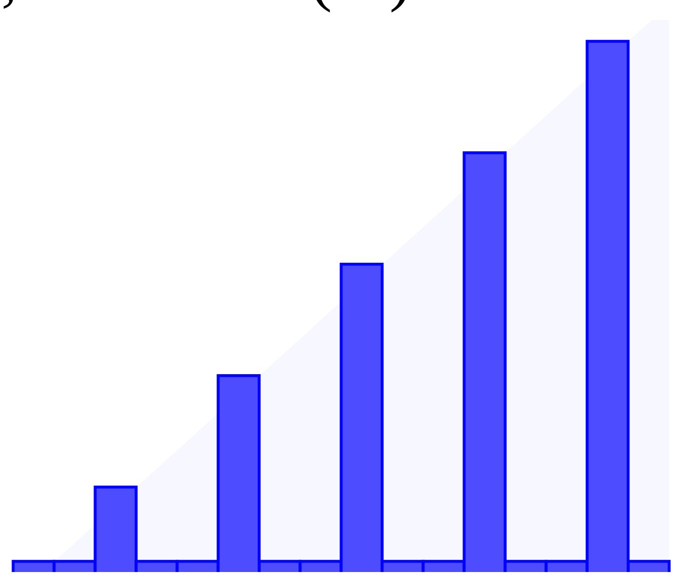
Java Vectors (contd.)

- Justification:(c = capacityIncrement)
  Let us assume that the original vector size is 0.
  The vector increases in size by the insertion of every (ic)th element (1st, cth, 2*cth, etc.)
  Worst case: we insert exactly n = ic elements, so the last operation involves copying the entire array.
  We have n insertions, and n elements copied on the last insertion.
  We also have i-1 other array copies, for a total of:

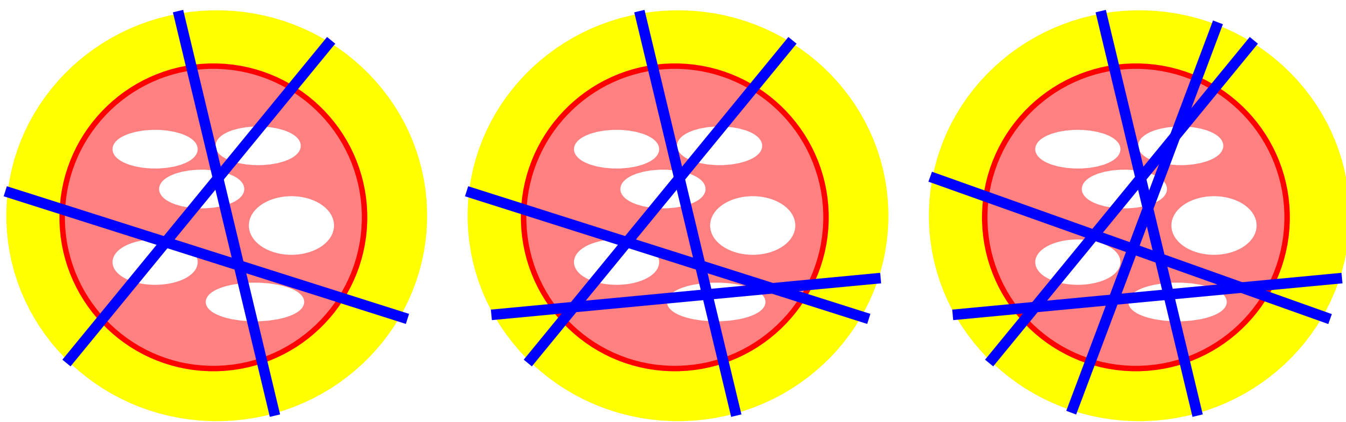
$$\sum_{k=0}^{i-1} ck = c\sum_{k=0}^{i-1} k = c\frac{i(i-1)}{2}$$

  previous element copies.

- The overall time complexity is proportional to $n(n-1/(2c))$, which is $O(n^2)$

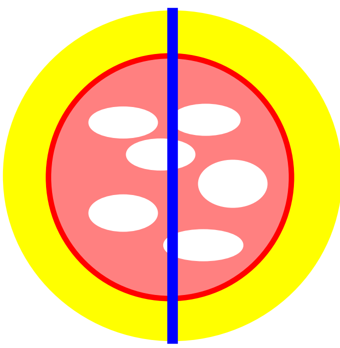


Recurrence Relations

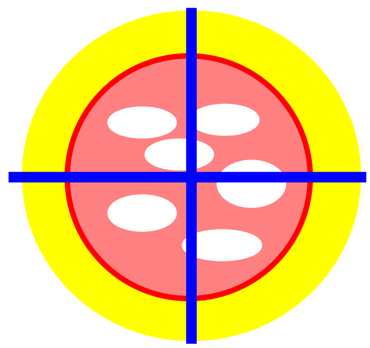


The Pizza Slicing Problem

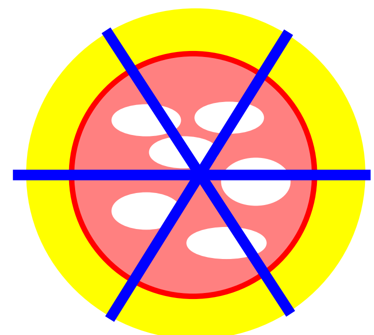
How many pieces of pizza can you get with
N straight cuts ?



1 cut
2 slices



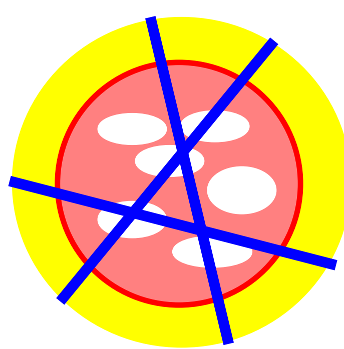
2 cuts
4 slices



3 cuts
6 slices

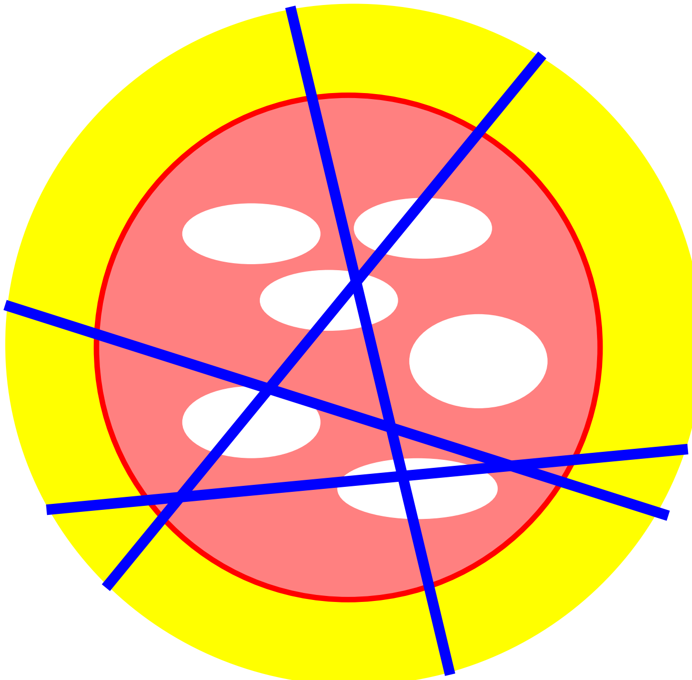
...   N cuts
2N slices

But ... who said you should cut
through the center every time?

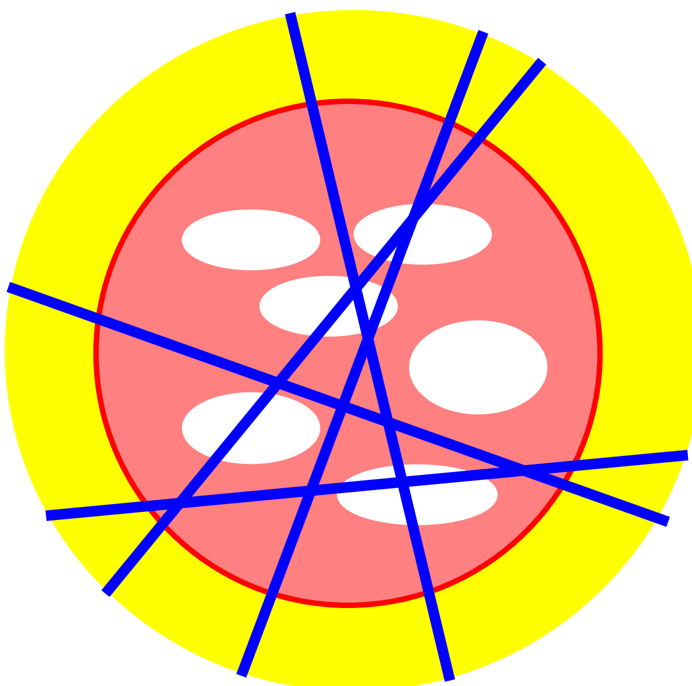


A Better Slicing Method ...

When cutting, intersect all previous cuts and avoid previous intersection points!

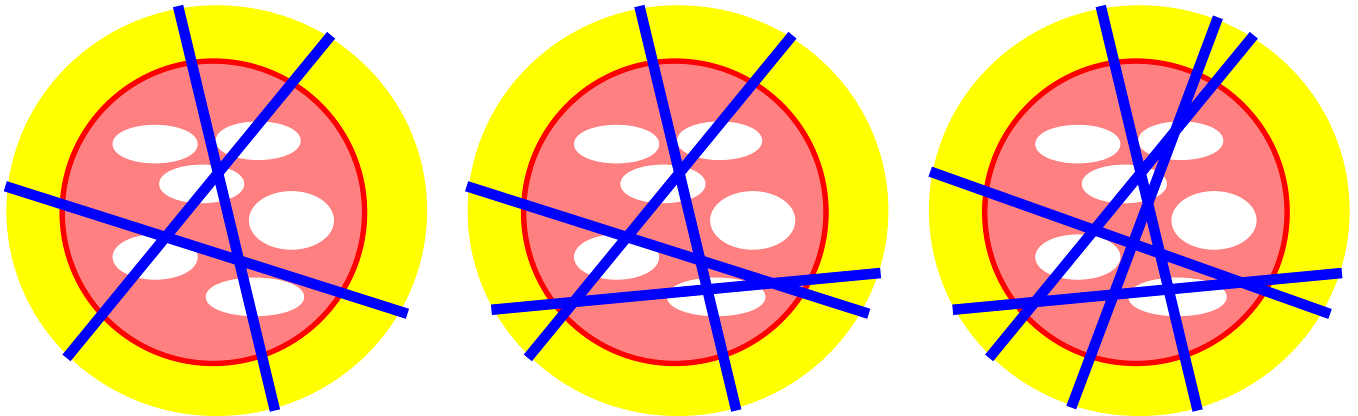


4 cuts
11 slices!!



5 cuts
16 slices!!

So ... How Many Pieces?



3 cuts
7 slices

4 cuts
11 slices

5 cuts
16 slices

The N-th cut creates N new pieces.
Hence, the total number of pieces given by N cuts, denoted P(N), is given by the following two rules:

- P(1) = 2
- P(N) = P(N-1) + N

Recursive definition of P(N)!

Recurrence Relations

- The pizza-cutting problem is an example of **recurrence relation**, where a function $f(N)$ is recursively defined.

(Base Case) $f(1) = 2$

(Recursive Case) $f(N) = f(N-1) + N$ for $N \geq 2$

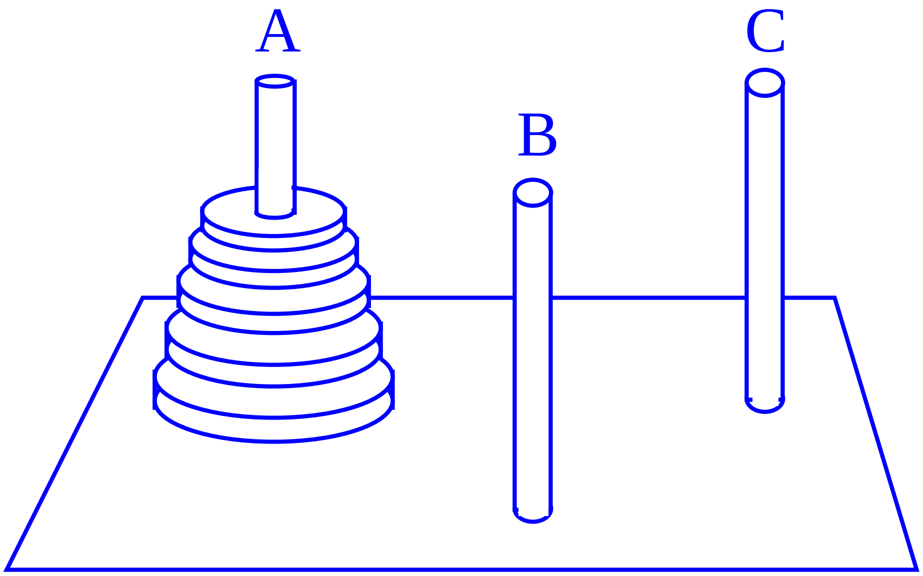
- The standard method for solving recurrence relations, called"unfolding", makes repeated substitutions applying the recursive rule until the base case is reached.

$$f(N) = f(N-1) + N$$
$$f(N) = f(N-2) + (N-1) + N$$
$$f(N) = f(N-3) + (N-2) + (N-1) + N$$
$$\dots$$
$$f(N) = f(N-\boldsymbol{i}) + (N-\boldsymbol{i}+1) + \dots + (N-1) + N$$

The base case is reached when $\boldsymbol{i} = N - 1$

$$f(N) = 2 + 2 + 3 + \dots + (N-2) + (N-1) + N$$
$$f(N) = N\frac{(N+1)}{2} + 1 = O(N^2)$$

Towers of Hanoi



Goal: transfer all N disks from peg A to peg C

Rules:

- move one disk at a time
- never place larger disk above smaller one

Recursive solution:

- transfer $N - 1$ disks from A to B
- move largest disk from A to C
- transfer $N - 1$ disks from B to C

Total number of moves:

- $T(N) = 2\,T(N - 1) + 1$

Solution of the Recurrence for Towers of Hanoi

Recurrence relation:

- $T(N) = 2\,T(N-1) + 1$
- $T(1) = 1$

Solution by unfolding:

$$
\begin{aligned}
T(N) &= 2\,(\,2\,T(N-2) + 1\,) + 1 = \\
&= 4\,T(N-2) + 2 + 1 = \\
&= 4\,(\,2\,T(N-3) + 1\,) + 2 + 1 = \\
&= 8\,T(N-3) + 4 + 2 + 1 = \\
&\dots \\
&= 2^{i}\,T(N-i) + 2^{i-1} + 2^{i-2} + \dots + 2^{1} + 2^{0}
\end{aligned}
$$

the expansion stops when $i = N - 1$

$$T(N) = 2^{N-1} + 2^{N-2} + 2^{N-3} + \dots + 2^{1} + 2^{0}$$

This is a ***geometric sum***, so that we have:

$$T(N) = 2^{N} - 1 = O(2^{N})$$

Another Recurrence

$$T(N) = 2T\left(\frac{N}{2}\right) + N \qquad \text{for } N \geq 2$$

$$T(1) = 1$$

$$\begin{aligned}
T(N) &= 2\left(2T\left(\frac{N}{4}\right) + \frac{N}{2}\right) + N \\
&= 4T\left(\frac{N}{4}\right) + 2N \\
&= 4\left(2T\left(\frac{N}{8}\right) + \frac{N}{4}\right) + 2N \\
&= 8T\left(\frac{N}{8}\right) + 3N \\
&\ldots \\
&= 2^i T\left(\frac{N}{2^i}\right) + iN
\end{aligned}$$

$T(1) = 1$

The expansion stops for $i = \log N$, so that

$$T(N) = N + N \log N$$

Solving Recurrences by "Guess and Prove"

$$T(N) = 2T\left(\frac{N}{2}\right) + N \qquad \text{for} \quad N \geq 2$$

$$T(1) = 1$$

Step 1: Take a wild guess that

$$T(N) = N + N \log N$$

Step 2: Prove it by induction:

Basis

$$T(1) = 1 + \log 1 = 1$$

Inductive Step

$$T(N) = 2T\left(\frac{N}{2}\right) + N = 2\left(\frac{N}{2} + \frac{N}{2}\log\frac{N}{2}\right) + N$$

$$T(N) = N + N(\log N - 1) + N = N + N \log N$$

A More Difficult Example

$T(N) = 2T(\sqrt{N}) + 1 \qquad T(2) = 0$

$$2T(N^{1/2}) + 1$$

$$2(2T(N^{1/4}) + 1) + 1$$

$$4T(N^{1/4}) + 1 + 2$$

$$8T(N^{1/8}) + 1 + 2 + 4$$

$$\ldots$$

$$2^i T\left(N^{\frac{1}{2^i}}\right) + 2^0 + 2^1 + \ldots + 2^{i-1}$$

The expansion stops for $N^{\frac{1}{2^i}} = 2$
i.e., $i = \log\log N$

$T(N) = 2^0 + 2^1 + \ldots + 2^{\log\log N - 1} = \log N$.

Proofs by Induction

We want to show that property P is true for all integers $n \geq n_0$

Basis:

prove that P is true for n_0.

Inductive Step:

prove that

if P is true for all k such that $n_0 \leq k \leq n-1$

then P is also true for n

An Example of Proof by Induction

$$S(n) = \sum_{i=1}^{n} i = n\frac{(n+1)}{2} \quad \text{for } n \geq 1$$

Basis:

$$S(1) = 1\frac{(1+1)}{2} = 1 \quad \text{Easy, Right?}$$

Inductive Step:

Assume $S(k) = k\frac{(k+1)}{2}$ for $1 \leq k \leq \boldsymbol{n-1}$

$$S(\boldsymbol{n}) = \sum_{i=1}^{n} i = \sum_{i=1}^{n-1} i + n = S(n-1) + n$$

$$= (n-1)\frac{(n-1+1)}{2} + n = \frac{(n^2 - n + 2n)}{2}$$

$$= n\frac{(n+1)}{2}$$