STRINGS AND PATTERN MATCHING

- Brute Force, Rabin-Karp, Knuth-Morris-Pratt

String Searching

- The previous slide is not a great example of what is meant by "String Searching." Nor is it meant to ridicule people without eyes....
- The object of string searching is to find the location of a specific text pattern within a larger body of text (e.g., a sentence, a paragraph, a book, etc.).
- As with most algorithms, the main considerations for string searching are speed and efficiency.
- There are a number of string searching algorithms in existence today, but the two we shall review are Brute Force and Rabin-Karp.

Brute Force

- The Brute Force algorithm compares the pattern to the text, one character at a time, until unmatching characters are found:

```
TWO ROADS DIVERGED IN A YELLOW WOOD
ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
 ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
  ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
   ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
    ROADS
```

  - Compared characters are italicized.
  - Correct matches are in boldface type.

- The algorithm can be designed to stop on either the first occurrence of the pattern, or upon reaching the end of the text.

Brute Force Pseudo-Code

- Here's the pseudo-code

```
do
  if (text letter == pattern letter)
    compare next letter of pattern to next
    letter of text
  else
    move pattern down text by one letter
while (entire pattern found or end of text)
```

```
tetththeheehthtehtheththehehtht
the
tetththeheehthtehtheththehehtht
 the
tetththeheehthtehtheththehehtht
  the
tetththeheehthtehtheththehehtht
   the
tetththeheehthtehtheththehehtht
    the
tetththeheehthtehtheththehehtht
     the
```

Brute Force-Complexity

- Given a pattern M characters in length, and a text N characters in length...
- **Worst case**: compares pattern to each substring of text of length M. For example, M=5.

1) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
2) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
3) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
4) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
5) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
....
N) AAAAAAAAAAAAAAAAAAAAAAAAAA*AAAAH*
5 comparisons made *AAAAH*

- Total number of comparisons: M (N-M+1)
- Worst case time complexity: O(MN)

Brute Force-Complexity(cont.)

- Given a pattern M characters in length, and a text N characters in length...
- **Best case if pattern found**: Finds pattern in first M positions of text. For example, M=5.

1) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAH
AAAAA **5 comparisons made**

- Total number of comparisons: M
- Best case time complexity: O(M)

Brute Force-Complexity(cont.)

- Given a pattern M characters in length, and a text N characters in length...
- **Best case if pattern not found**: Always mismatch on first character. For example, M=5.

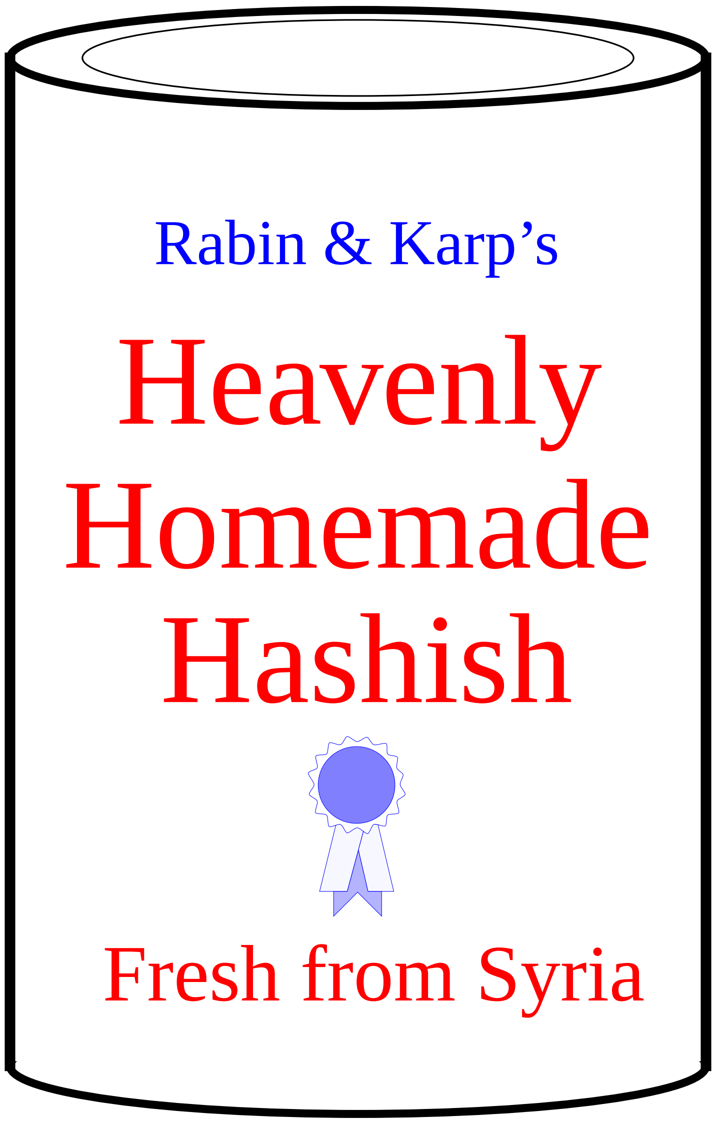
1) *A*AAAAAAAAAAAAAAAAAAAAAAAAAAH
*O*OOOH **1 comparison made**
2) A*A*AAAAAAAAAAAAAAAAAAAAAAAAAH
*O*OOOH **1 comparison made**
3) AA*A*AAAAAAAAAAAAAAAAAAAAAAAAH
*O*OOOH **1 comparison made**
4) AAA*A*AAAAAAAAAAAAAAAAAAAAAAAH
*O*OOOH **1 comparison made**
5) AAAA*A*AAAAAAAAAAAAAAAAAAAAAAH
*O*OOOH **1 comparison made**
...

N) AAAAAAAAAAAAAAAAAAAAAAA*A*AAAH
1 comparison made *O*OOOH

- Total number of comparisons: N
- Best case time complexity: O(N)

Rabin-Karp

- The Rabin-Karp string searching algorithm uses a hash function to speed up the search.



Rabin-Karp

- The Rabin-Karp string searching algorithm calculates a **hash value** for the pattern, and for each M-character subsequence of text to be compared.
- If the hash values are unequal, the algorithm will calculate the hash value for next M-character sequence.
- If the hash values are equal, the algorithm will do a Brute Force comparison between the pattern and the M-character sequence.
- In this way, there is only one comparison per text subsequence, and Brute Force is only needed when hash values match.
- Perhaps a figure will clarify some things...

Rabin-Karp Example

Hash value of “AAAAA” is 37

Hash value of “AAAAH” is 100

1) AAAAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

2) AAAAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

3) AAAAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

4) AAAAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

...

N) AAAAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
6 comparisons made 100=100

Rabin-Karp Pseudo-Code

pattern is M characters long

hash_p=hash value of pattern
hash_t=hash value of first M letters in
body of text

```
do
  if (hash_p == hash_t)
    brute force comparison of pattern
    and selected section of text
  hash_t = hash value of next section of
              text, one character over
while (end of text or
        brute force comparison == true)
```

Rabin-Karp

- Common Rabin-Karp questions:

  "What is the hash function used to calculate values for character sequences?"

  "Isn't it time consuming to hash every one of the M-character sequences in the text body?"

  "Is this going to be on the final?"

- To answer some of these questions, we'll have to get mathematical.

Rabin-Karp Math

- Consider an M-character sequence as an M-digit number in base $\boldsymbol{b}$, where $\boldsymbol{b}$ is the number of letters in the alphabet. The text subsequence t[i .. i+M-1] is mapped to the number

  $$\boldsymbol{x}(i) = \boldsymbol{t}[i]\cdot\boldsymbol{b}^{M-1} + \boldsymbol{t}[i+1]\cdot\boldsymbol{b}^{M-2} + \ldots + \boldsymbol{t}[i+M-1]$$

- Furthermore, given x(i) we can compute x(i+1) for the next subsequence t[i+1 .. i+M] in constant time, as follows:

  $$\boldsymbol{x}(i+1) = \boldsymbol{t}[i+1]\cdot\boldsymbol{b}^{M-1} + \boldsymbol{t}[i+2]\cdot\boldsymbol{b}^{M-2} + \ldots + \boldsymbol{t}[i+M]$$

  $\boldsymbol{x}(i+1) = \boldsymbol{x}(i)\cdot\boldsymbol{b}$ Shift left one digit

  $- \boldsymbol{t}[i]\cdot\boldsymbol{b}^{M}$ Subtract leftmost digit

  $+ \boldsymbol{t}[i+M]$ Add new rightmost digit

- In this way, we never explicitly compute a new value. We simply adjust the existing value as we move over one character.

Rabin-Karp Mods

- If M is large, then the resulting value (~bM) will be enormous. For this reason, we hash the value by taking it **mod** a prime number ***q***.
- The **mod** function (% in Java) is particularly useful in this case due to several of its inherent properties:
  - $[(x \bmod q) + (y \bmod q)] \bmod q = (x+y) \bmod q$
  - $(x \bmod q) \bmod q = x \bmod q$
- For these reasons:

$$h(i) = ((t[i]\cdot b^{M-1} \bmod q) + (t[i+1]\cdot b^{M-2} \bmod q) + \ldots + (t[i+M-1] \bmod q)) \bmod q$$

$h(i+1) = (h(i)\cdot b \bmod q$

Shift left one digit

$-t[i]\cdot b^{M} \bmod q$

Subtract leftmost digit

$+t[i+M] \bmod q)$

Add new rightmost digit

$\bmod q$

Rabin-Karp Pseudo-Code

pattern is M characters long

hash_p=hash value of pattern

hash_t =hash value of first M letters in body of text

```
do
  if (hash_p == hash_t)
    brute force comparison of pattern
    and selected section of text
  hash_t = hash value of next section of
             text, one character over
while (end of text or
         brute force comparison == true)
```

Rabin-Karp Complexity

- If a sufficiently large prime number is used for the *hash function*, the hashed values of two different patterns will usually be distinct.
- If this is the case, searching takes O(N) time, where N is the number of characters in the larger body of text.
- It is always possible to construct a scenario with a worst case complexity of O(MN). This, however, is likely to happen only if the prime number used for hashing is small.

The Knuth-Morris-Pratt Algorithm

- The Knuth-Morris-Pratt (KMP) string searching algorithm differs from the brute-force algorithm by keeping track of information gained from previous comparisons.
- A failure function (*f*) is computed that indicates how much of the last comparison can be reused if it fais.
- Specifically, *f* is defined to be the longest prefix of the pattern P[0,..,j] that is also a suffix of P[1,..,j]
  - Note: **not** a suffix of P[0,..,j]
- Example:
  - value of the KMP failure function:

j	0	1	2	3	4	5
P[*j*]	a	b	a	b	a	c
f(*j*)	0	0	1	2	3	0

- This shows how much of the beginning of the string matches up to the portion immediately preceding a failed comparison.
  - if the comparison fails at (4), we know the a,b in positions 2,3 is identical to positions 0,1

The KMP Algorithm (contd.)

- Time Complexity Analysis
- define $k = i - j$
- In every iteration through the while loop, one of three things happens.
  - 1) if $T[i] = P[j]$, then i increases by 1, as does j k remains the same.
  - 2) if $T[i] \mathrel{!=} P[j]$ and $j > 0$, then i does not change and k increases by at least 1, since k changes from $i - j$ to $i - f(j-1)$
  - 3) if $T[i] \mathrel{!=} P[j]$ and $j = 0$, then i increases by 1 and k increases by 1 since j remains the same.
- Thus, each time through the loop, either i or k increases by at least 1, so the greatest possible number of loops is $2n$
- This of course assumes that f has already been computed.
- However, f is computed in much the same manner as KMPMatch so the time complexity argument is analogous. KMPFailureFunction is $\boldsymbol{O}(m)$
- Total Time Complexity: $\boldsymbol{O}(n + m)$

The KMP Algorithm (contd.)

- the KMP string matching algorithm: Pseudo-Code

```
Algorithm KMPMatch(T,P)
  Input: Strings T (text) with n characters and P
    (pattern) with m characters.
  Output: Starting index of the first substring of T
    matching P, or an indication that P is not a
    substring of T.

  f ← KMPFailureFunction(P) {build failure function}
  i ← 0
  j ← 0
  while i < n do
    if P[j] = T[i] then
      if j = m - 1 then
        return i - m - 1 {a match}
      i ← i + 1
      j ← j + 1
    else if j > 0 then {no match, but we have advanced}
      j ← f(j-1) {j indexes just after matching prefix in P}
    else
      i ← i + 1
  return "There is no substring of T matching P"
```

The KMP Algorithm (contd.)

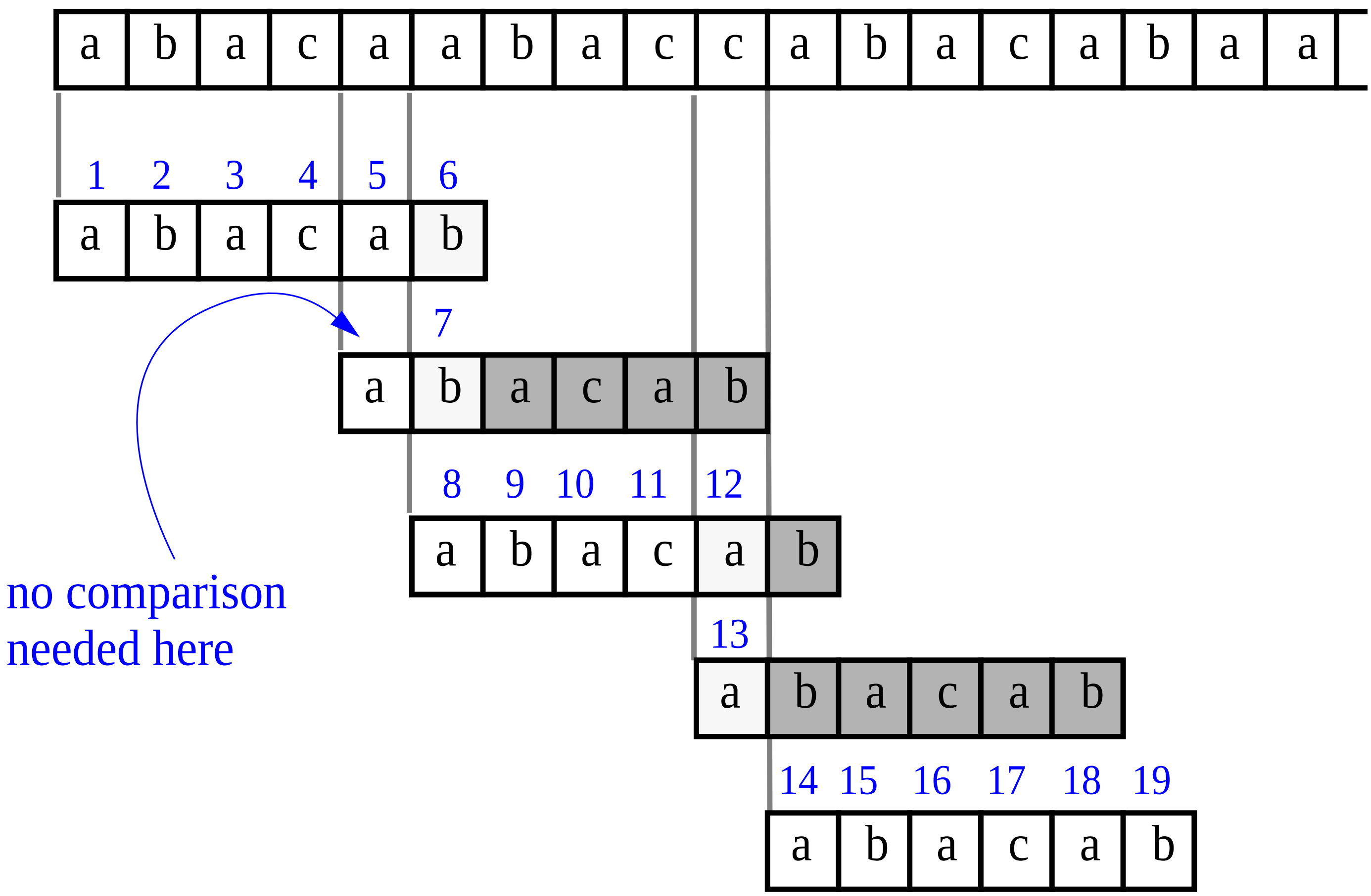
- The KMP failure function: Pseudo-Code

```
Algorithm KMPFailureFunction(P);
  Input: String P (pattern) with m characters
  Ouput: The faliure function f for P, which maps j to
     the length of the longest prefix of P that is a suffix
     of P[1,..,j]

  i ← 1
  j ← 0
  while i ≤ m-1 do
     if P[j] = T[j] then
        {we have matched j + 1 characters}
        f(i) ← j + 1
        i ← i + 1
        j ← j + 1
     else if j > 0 then
        {j indexes just after a prefix of P that matches}
        j ← f(j-1)
     else
        {there is no match}
        f(i) ← 0
        i ← i + 1
```

The KMP Algorithm (contd.)

- A graphical representation of the KMP string searching algorithm

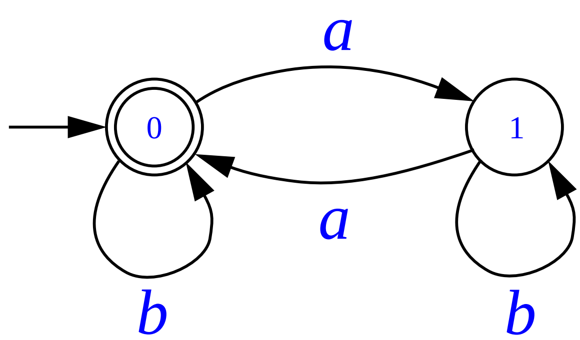


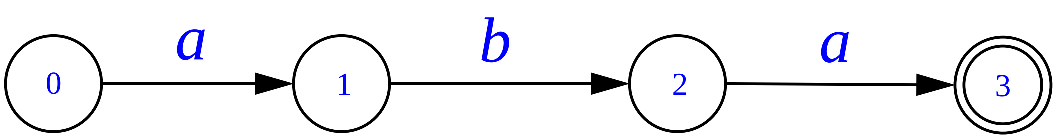
Regular Expressions

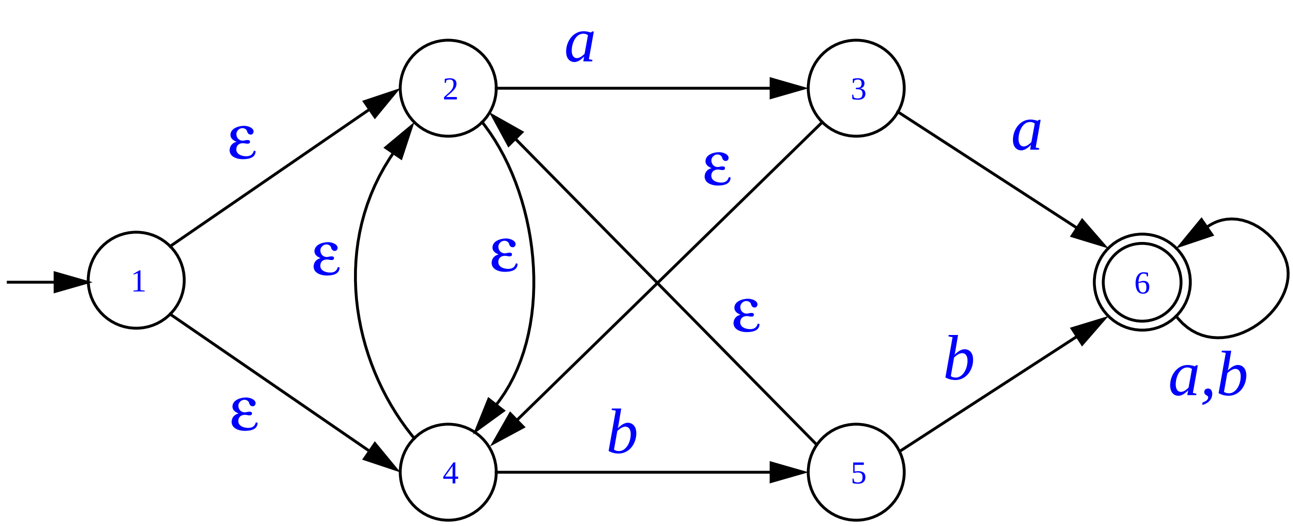
- notation for describing a set of strings, possibly of infinite size
- ε denotes the empty string
- **ab + c** denotes the set {ab, c}
- **a*** denotes the set {ε, a, aa, aaa, ...}
- Examples
  - **(a+b)*** all the strings from the alphabet {a,b}
  - **b*(ab*a)*b*** strings with an even number of a's
  - **(a+b)*sun(a+b)*** strings containing the pattern "sun"
  - **(a+b)(a+b)(a+b)a** 4-letter strings ending in a

Finite State Automaton

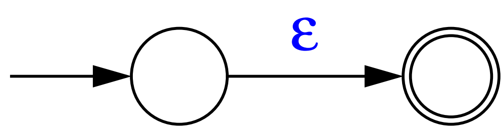
- “machine” for processing strings

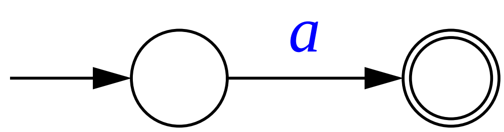


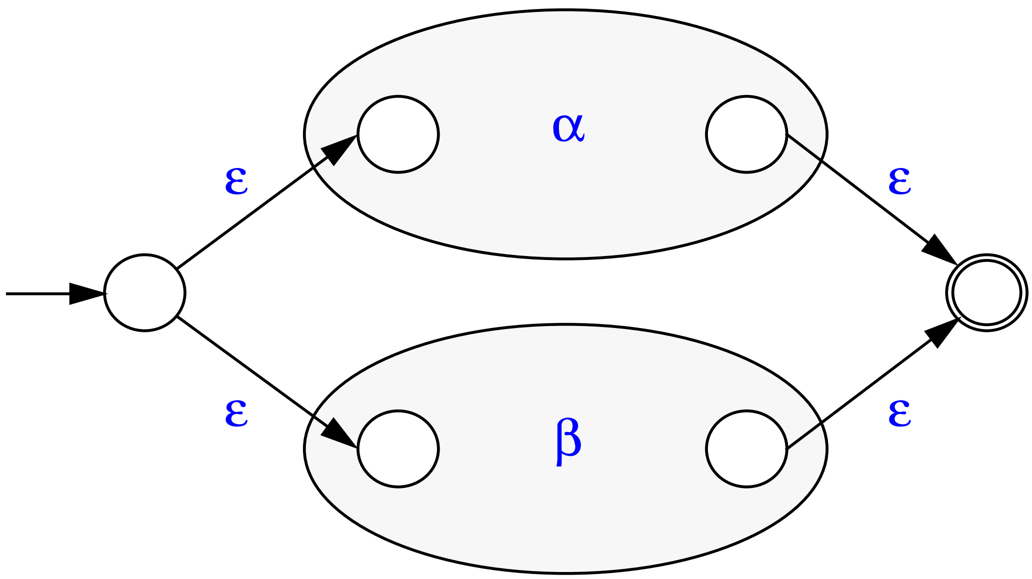


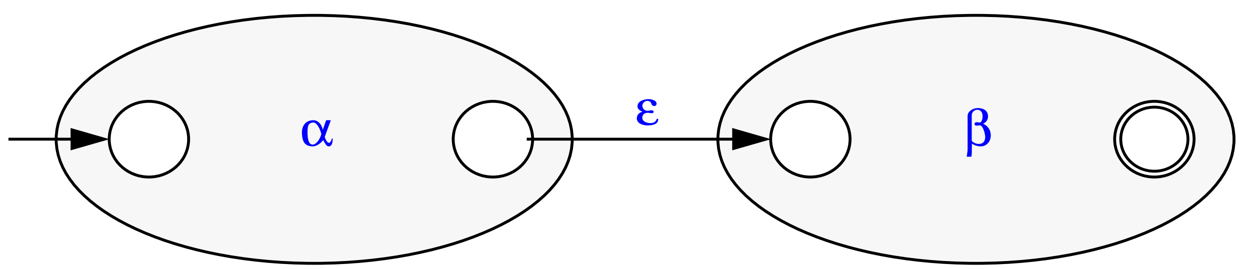


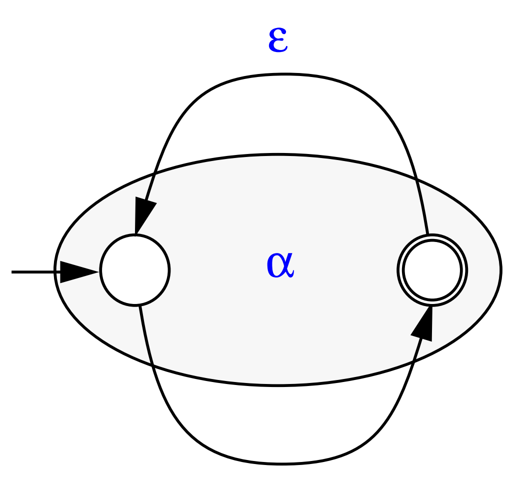
Composition of FSA's











Tries

- A trie is a tree-based date structure for storing strings in order to make pattern matching faster.
- Tries can be used to perform **prefix queries** for information retrieval. Prefix queries search for the longest prefix of a given string X that matches a prefix of some string in the trie.
- A trie supports the following operations on a set S of strings:

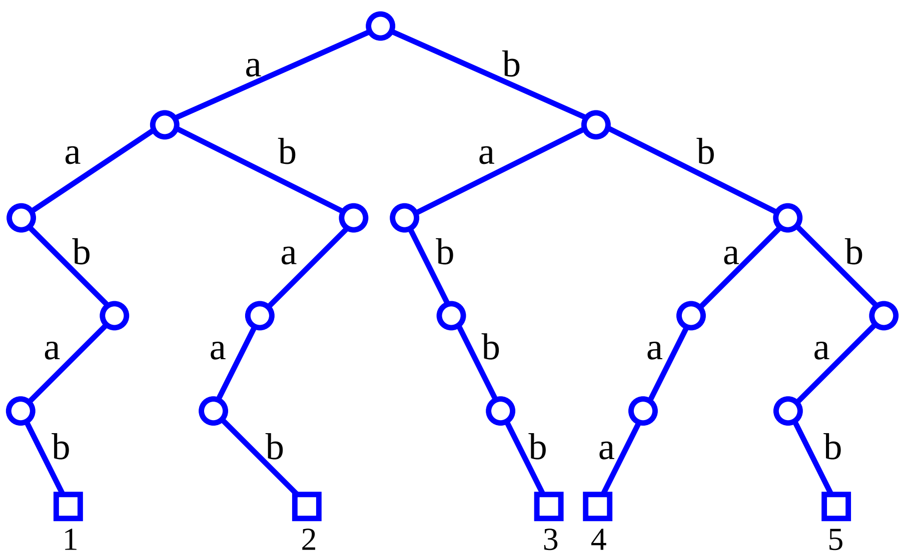
  insert(X): Insert the string X into S
  Input: String **Ouput**: None

  remove(X): Remove string X from S
  Input: String **Output**: None

  prefixes(X): Return all the strings in S that have a longest prefix of X
  Input: String **Output**: Enumeration of strings

Tries (cont.)

- Let S be a set of strings from the alphabet Σ such that no string in S is a prefix to another string. A standard trie for S is an ordered tree T that:
  - Each edge of T is labeled with a character from Σ
  - The ordering of edges out of an internal node is determined by the alphabet Σ
  - The path from the root of T to any node represents a prefix in Σ that is equal to the concantenation of the characters encountered while traversing the path.
- For example, the standard trie over the alphabet Σ = {a, b} for the set {aabab, abaab, babbb, bbaaa, bbab}



Tries (cont.)

- An internal node can have 1 to *d* children when d is the size of the alphabet. Our example is essentially a binary tree.
- A path from the root of *T* to an internal node *v* at depth *i* corresponds to an *i*-character prefix of a string of *S*.
- We can implement a trie with an ordered tree by storing the character associated with an edge at the child node below it.

Compressed Tries

- A compressed trie is like a standard trie but makes sure that each trie had a degree of at least 2. Single child nodes are compressed into an single edge.

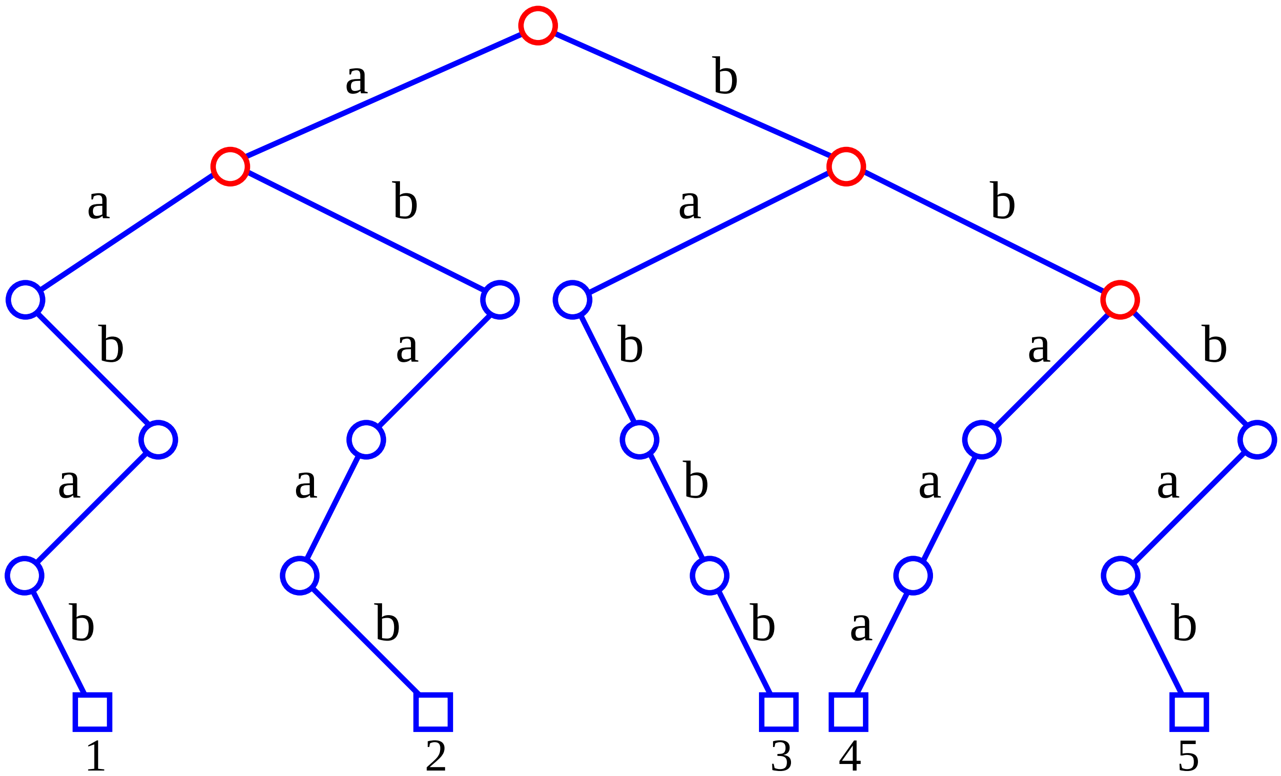
- A **critical node** is a node v such that v is labeled with a string from S, v has at least 2 children, or v is the root.

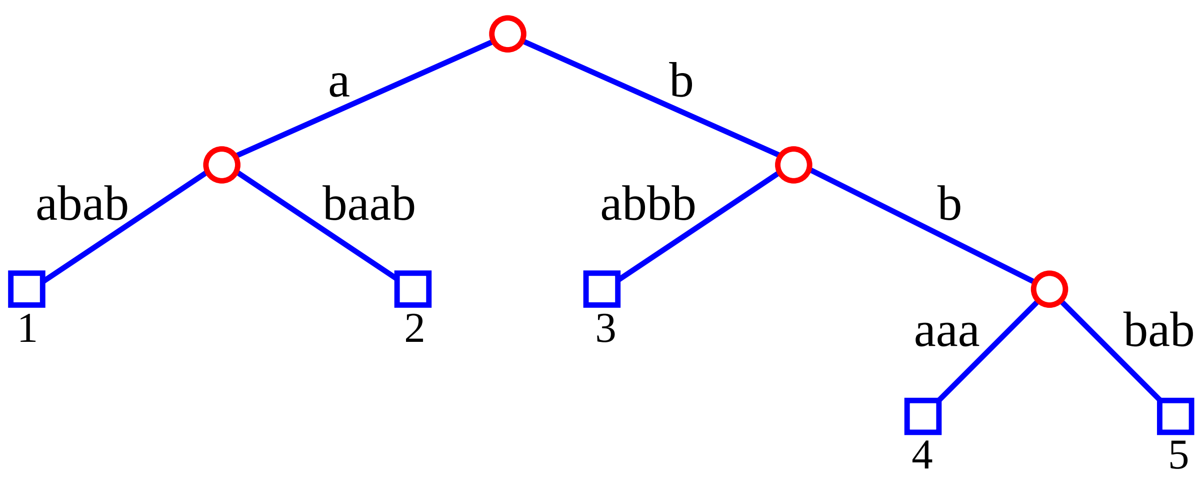
- To convert a standard trie to a compressed trie we replace an edge (v_0, v_1) each chain on nodes (v_0, v_1...v_k) for k 2 such that
  - v_0 and v_1 are critical but v_1 is critical for $0<i<k$
  - each v_1 has only one child

- Each internal node in a compressed tire has at least two children and each external is associated with a string. The compression reduces the total space for the trie from O(*m*) where *m* is the sum of the the lengths of strings in *S* to O(*n*) where *n* is the number of strings in *S*.

Compressed Tries (cont.)

- An example:





Prefix Queries on a Trie

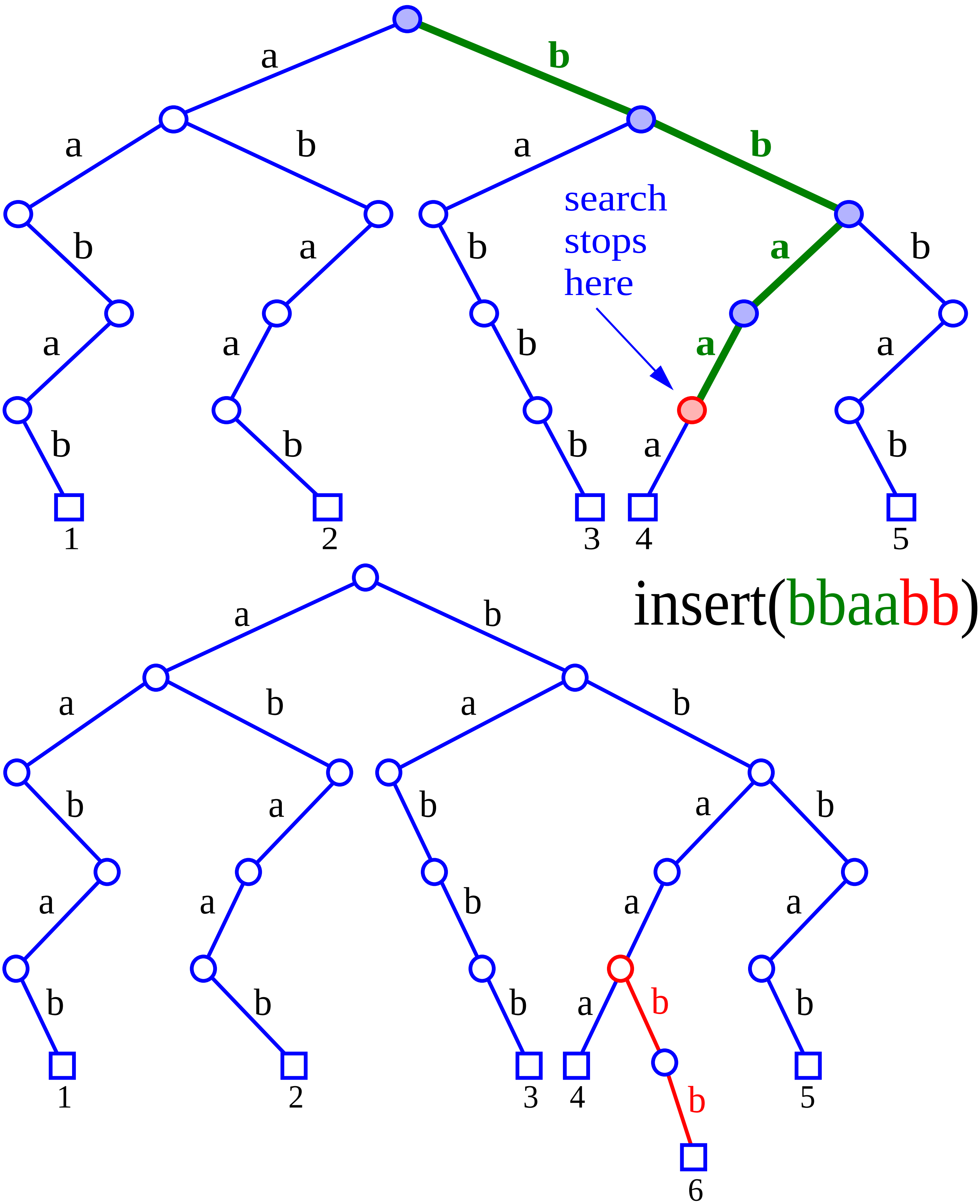
```
Algorithm prefixQuery(T, X):
  Input: Trie T for a set S of strings and a query string X
  Output: The node v of T such that the labeled nodes of
          the subtree of T rooted at v store the strings
          of S with a longest prefix in common with X
  v←T.root()
  i←0      {i is an index into the string X}
  repeat
    for each child w of v do
    let e be the edge (v,w)
    Y←string(e)  {Y is the substring associated with e}
    l←Y.length()  {l=1 if T is a standard trie}
    Z¨X.substring(i, i+l-1) {Z holds the next l charac
            ters of X}
    if Z = Y then
      v←w
      i←i+1{move to W, incrementing i past Z}
      break out of the for loop
    else if a proper prefix of Z matched a proper prefix
      of Y then
      v←w
      break out ot the repeat loop
until v is external or v≠w
return v
```

Insertion and Deletion

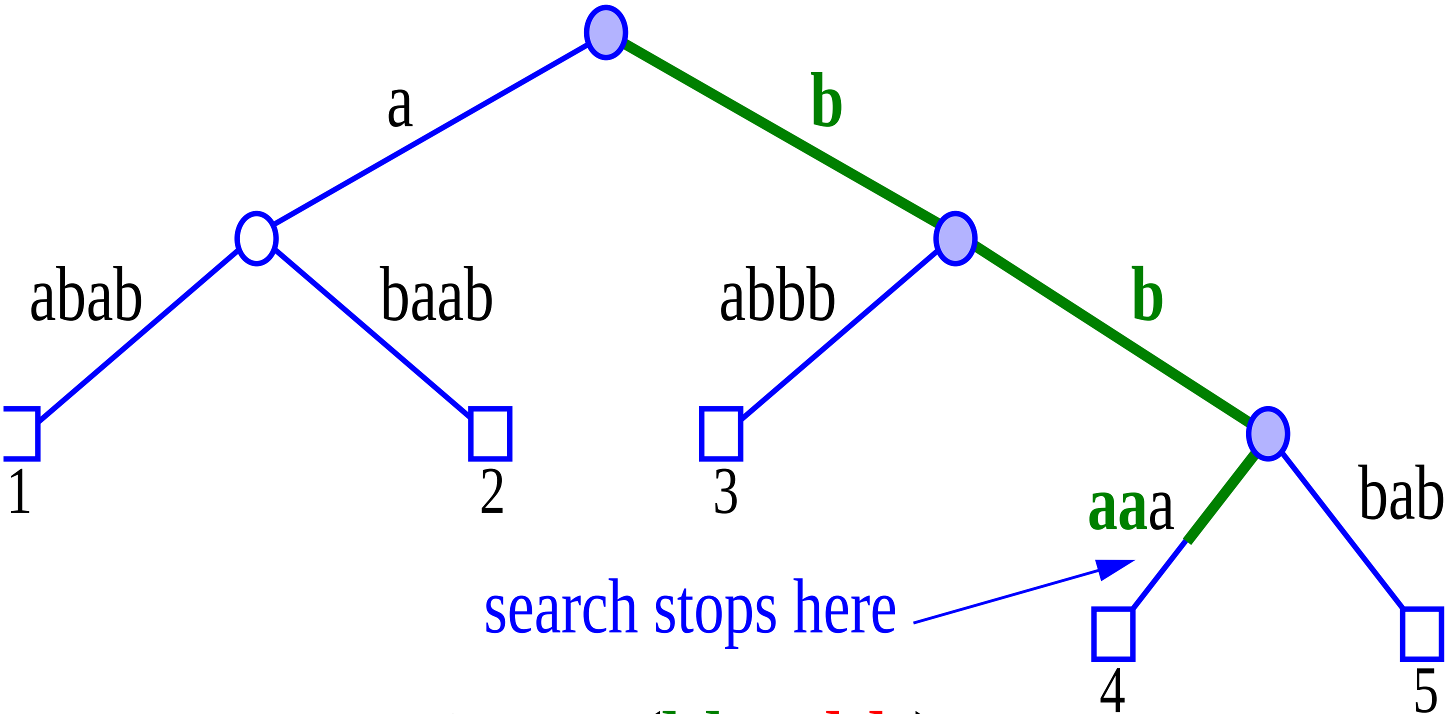
- Insertion: We first perform a prefix query for string X. Let us examine the ways a prefix query may end in terms of insertion.
  - The query terminates at node v. Let X_1 be the prefix of X that matched in the trie up to node v and X_2 be the rest of X. If X_2 is an empt string we label v with X and the end. Otherwise we creat a new external node w and label it with X.
  - The query terminates at an edge e=(v, w) because a prefix of X match prefix(v) and a proper prefix of string Y associated with e. Let Y_1 be the part of Y that X mathed to and Y_2 the rest of Y. Likewise for X_1 and X_2. Then $X=X_1+X_2$ = prefix(v) $+Y_1+X_2$. We create a new node u and split the edges(v, u) and (u, w). If X2 is empty then w label u with X. Otherwise we creat a node z which is external and label it X.

- Insertion is O(dn) when d is the size of the alphabet and n is the length of the string t insert.

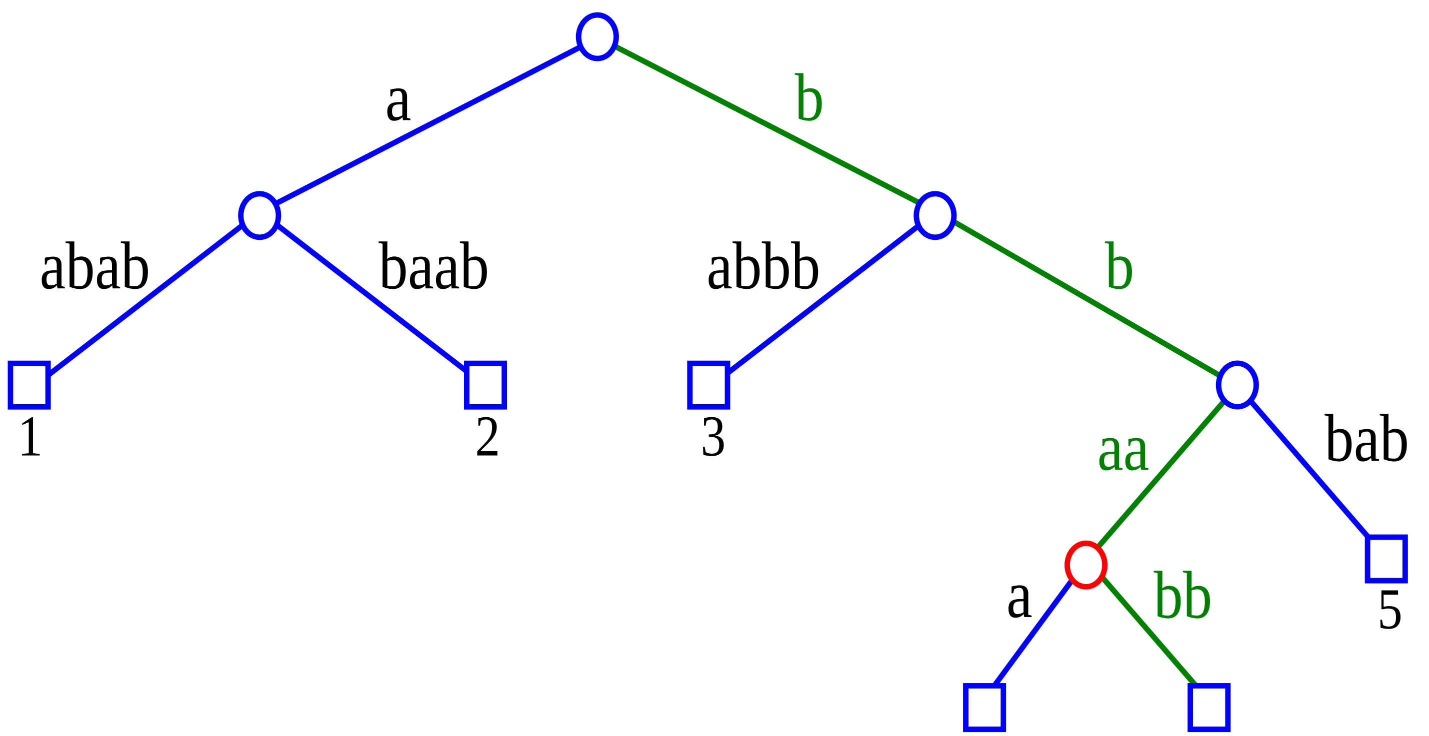
Insertion and Deletion (cont.)



Insertion and Deletion (cont.)



insert(bbaabb)



Lempel Ziv Encoding

- Constructing the trie:
  - Let phrase 0 be the null string.
  - Scan through the text
  - If you come across a letter you haven't seen before, add it to the top level of the trie.
  - If you come across a letter you've already seen, scan down the trie until you can't match any more chracters, add a node to the trie representing the new string.
  - Insert the pair (nodeIndex, lastChar) into the compressed string.
- Reconstructing the string:
  - Every time you see a '0' in the compressed string add the next character in the compressed string directly to the new string.
  - For each non-zero nodeIndex, put the substring corresponding to that node into the new string, followed by the next character in the compressed string.

Lempel Ziv Encoding (contd.)

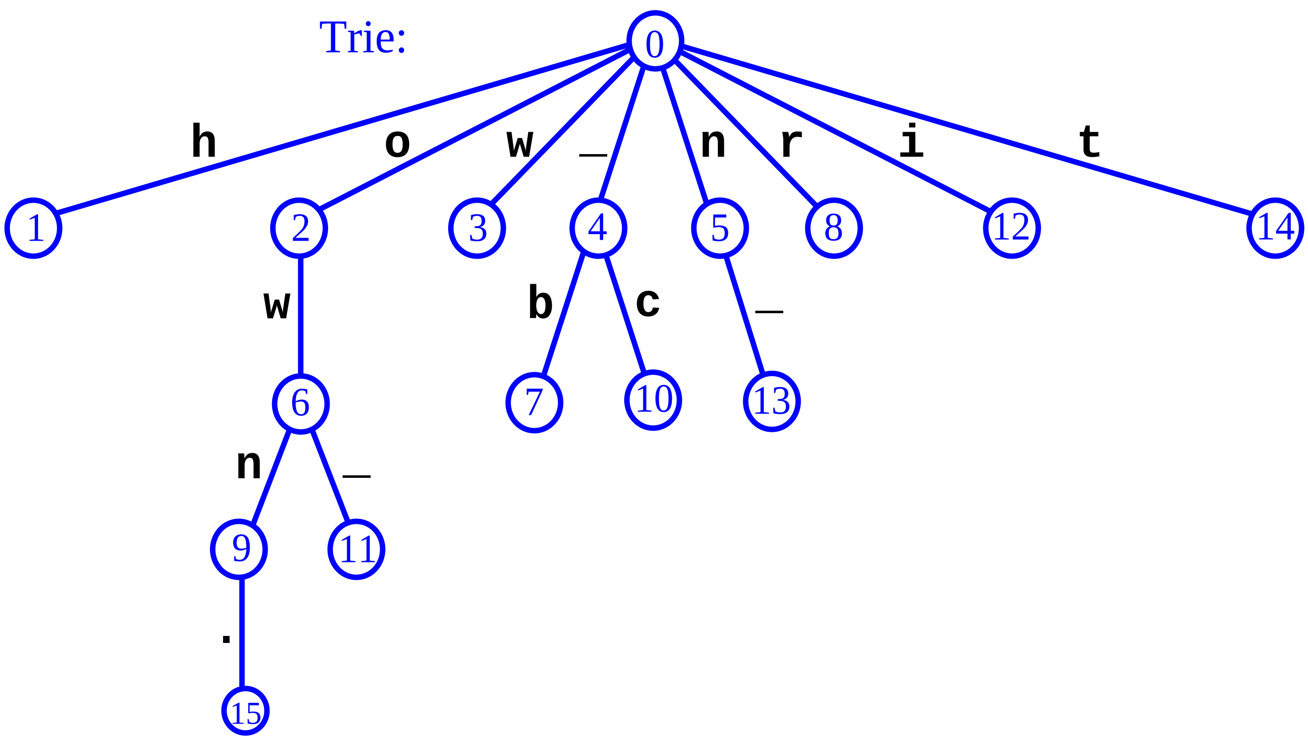
- A graphical example:

Uncompressed text: (nil) `how_now_brown_cow_in_town.`

phrases: 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Compressed text: `0h0o0w0_0n2w4b0r6n4c6_0i5_0t9.`

Trie:

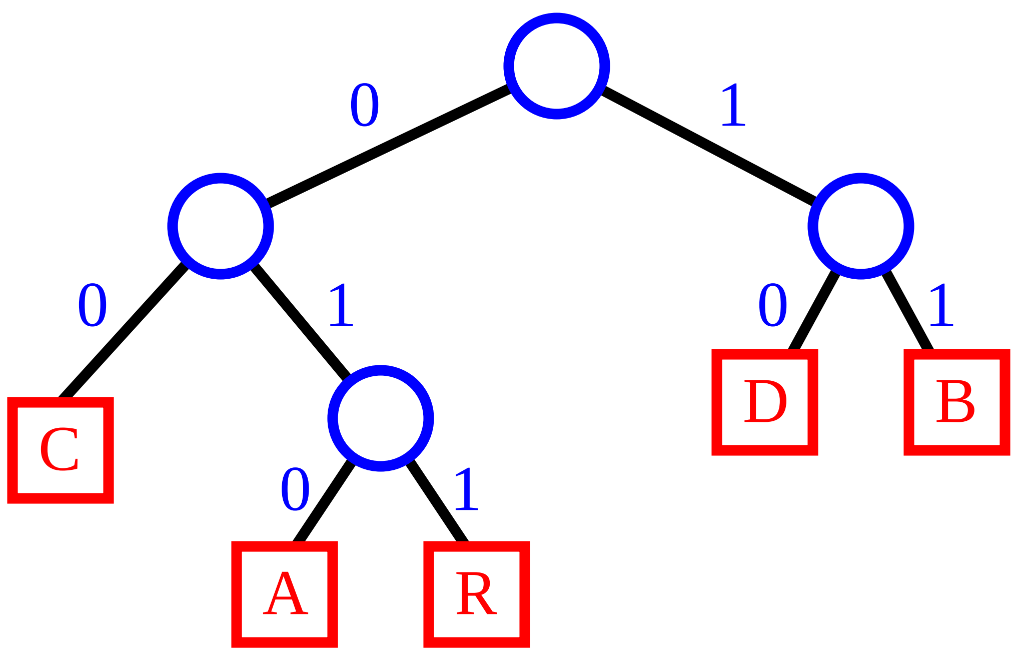


File Compression

- text files are usually stored by representing each character with an 8-bit ASCII code (type man ascii in a Unix shell to see the ASCII encoding)
- the ASCII encoding is an example of **fixed-length encoding**, where each character is represented with the same number of bits
- in order to reduce the space required to store a text file, we can exploit the fact that some characters are more likely to occur than others
- **variable-length encoding** uses binary codes of different lengths for different characters; thus, we can assign fewer bits to frequently used characters, and more bits to rarely used characters.
- Example:
  - text: java
  - encoding: a = “0”, j = “11”, v = “10”
  - encoded text: 110100 (6 bits)
- How to decode?
  - a = “0”, j = “01”, v = “00”
  - encoded text: 010000 (6 bits)
  - is this java, jvv, jaaaa ...

Encoding Trie

- to prevent ambiguities in decoding, we require that the encoding satisfies the **prefix rule**, that is, no code is a prefix of another code
  - a = "0", j = "11", v = "10" satisfies the prefix rule
  - a = "0", j = "01", v= "00" does **not** satisfy the prefix rule (the code of a is a prefix of the codes of j and v)
- we use an **encoding trie** to define an encoding that satisfies the prefix rule
  - the characters stored at the external nodes
  - a left edge means 0
  - a right edge means 1



A = 010

B = 11

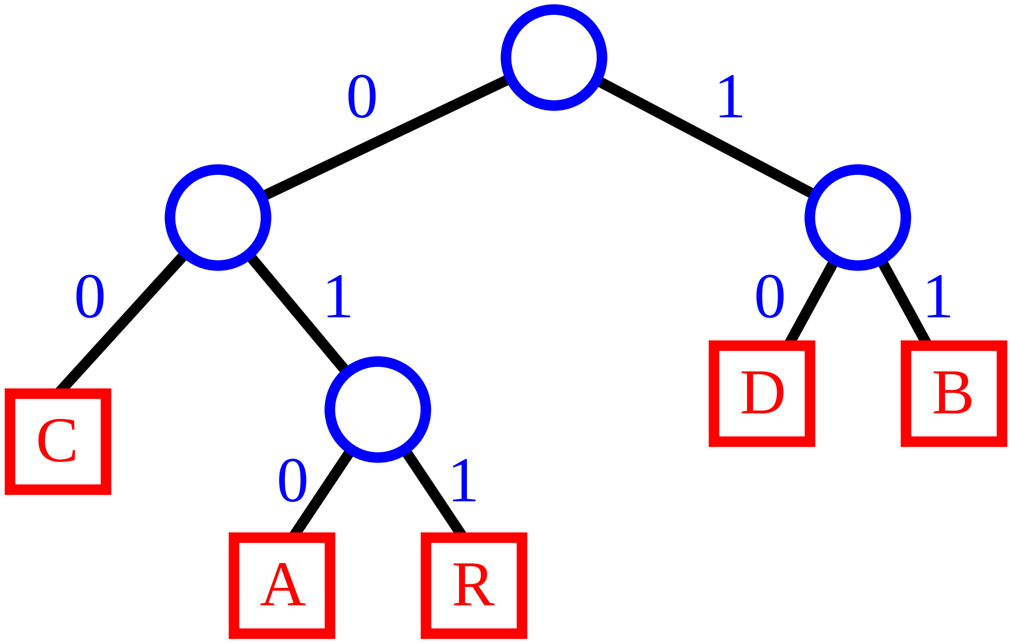
C = 00

D = 10

R = 011

Example of Decoding

- trie:



A = 010

B = 11

C = 00

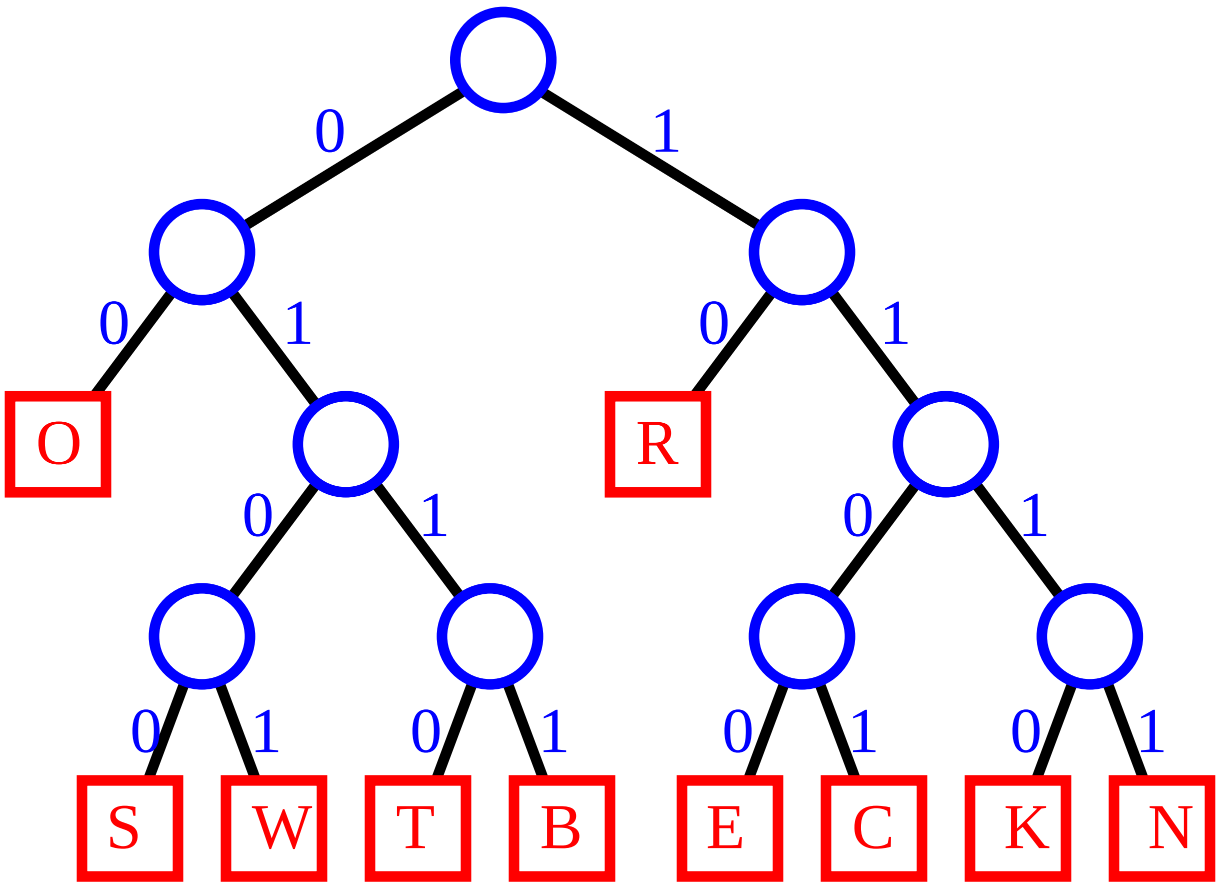
D = 10

R = 011

- encoded text:
  01011011010000101001011011010

- text:

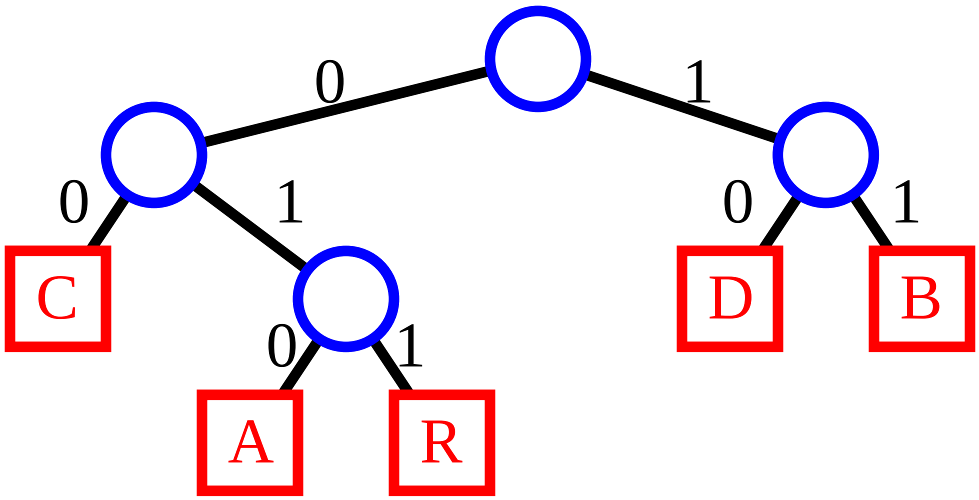
Trie this!



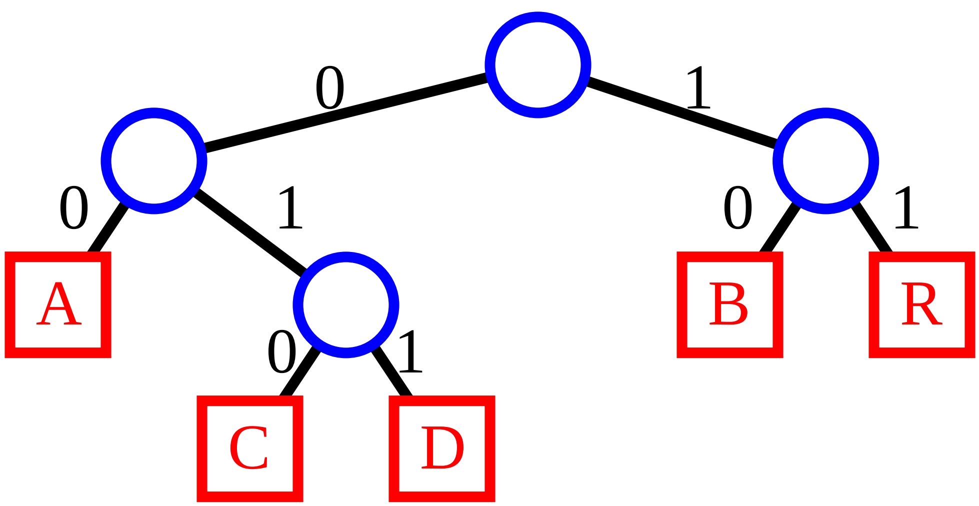
10000111110010011000111011110001010100110100

Optimal Compression

- An issue with encoding tries is to insure that the encoded text is as short as possible:

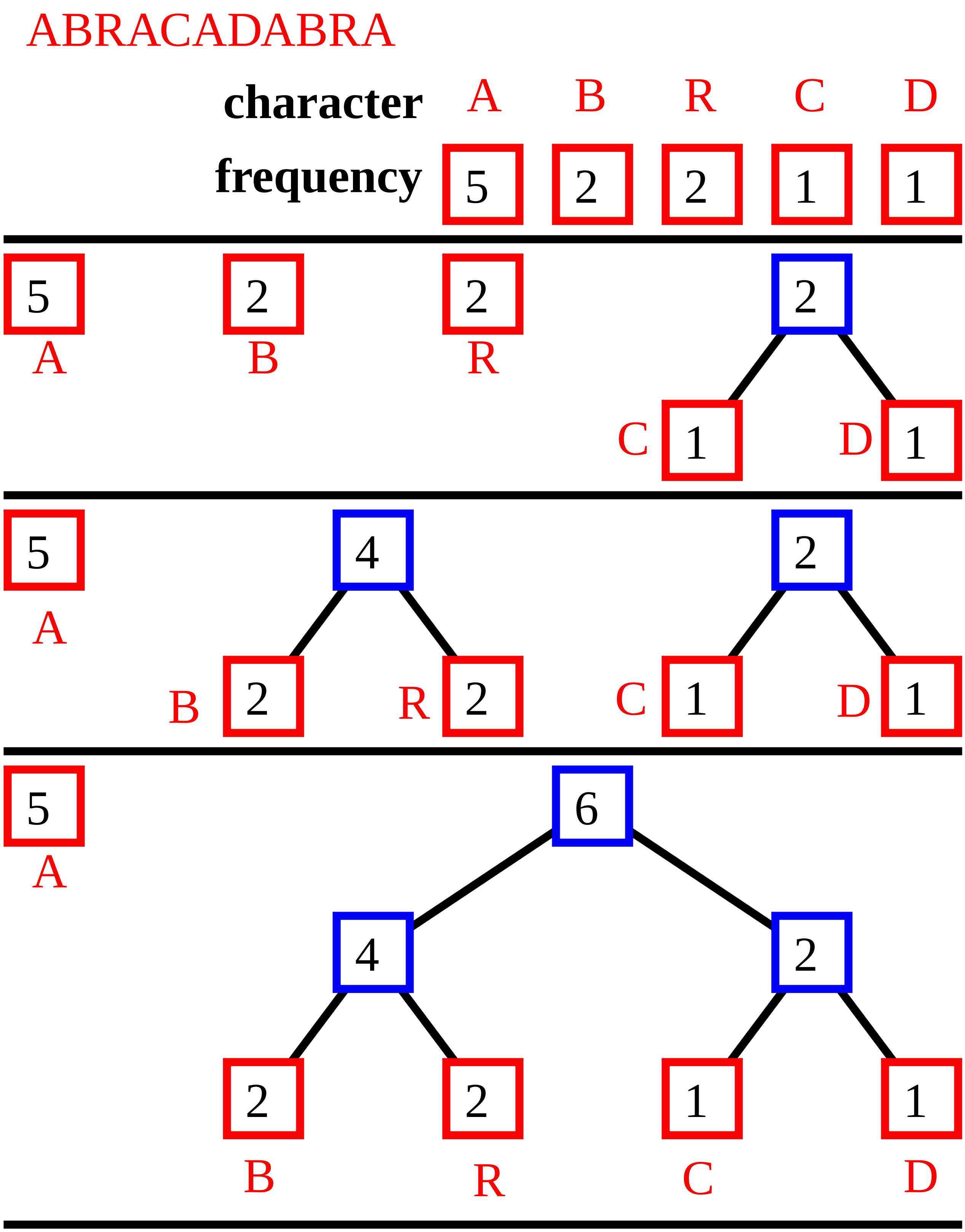


ABRACADABRA
01011011010000101001011011010
29 bits

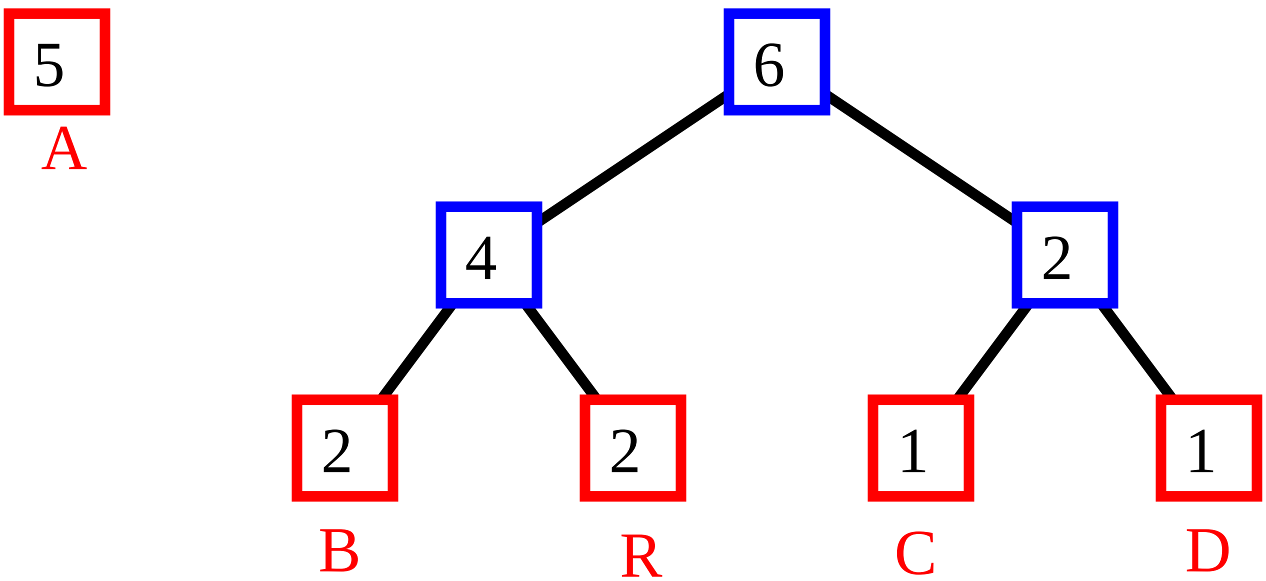


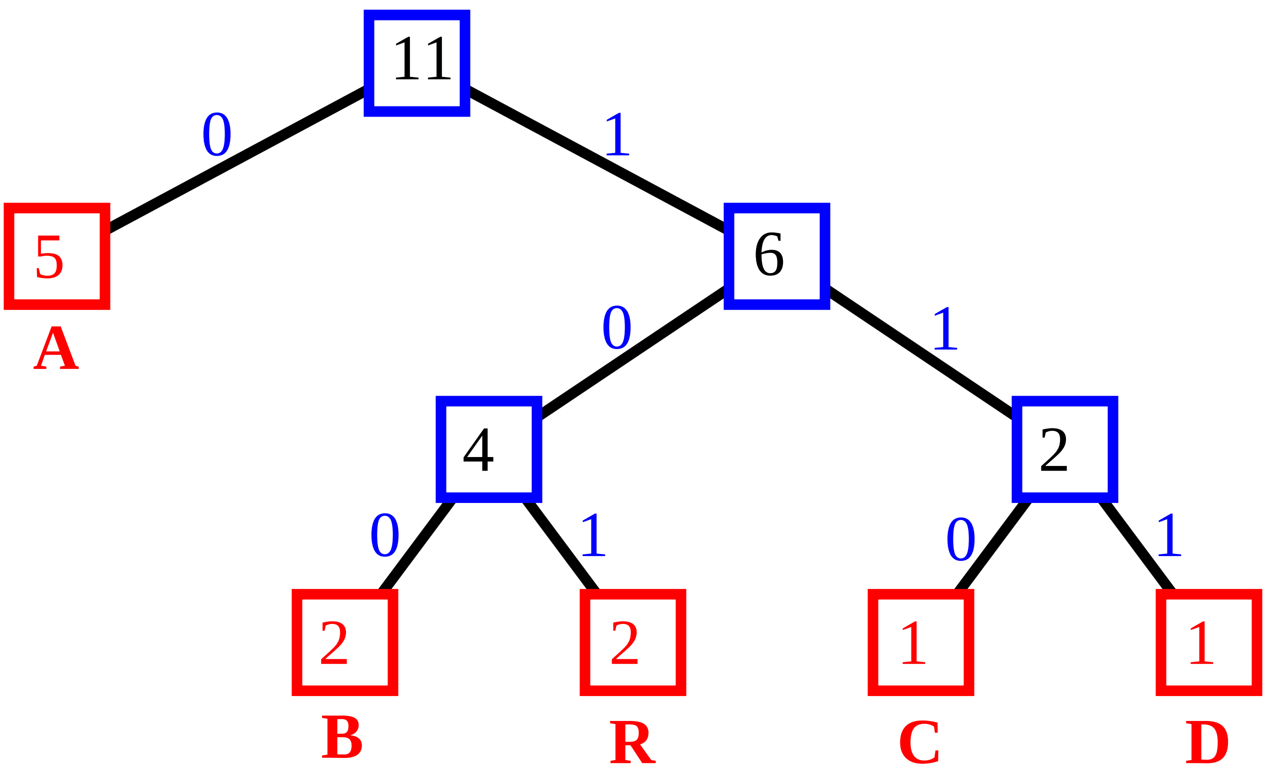
ABRACADABRA
001011000100001100101100
24 bits

Huffman Encoding Trie

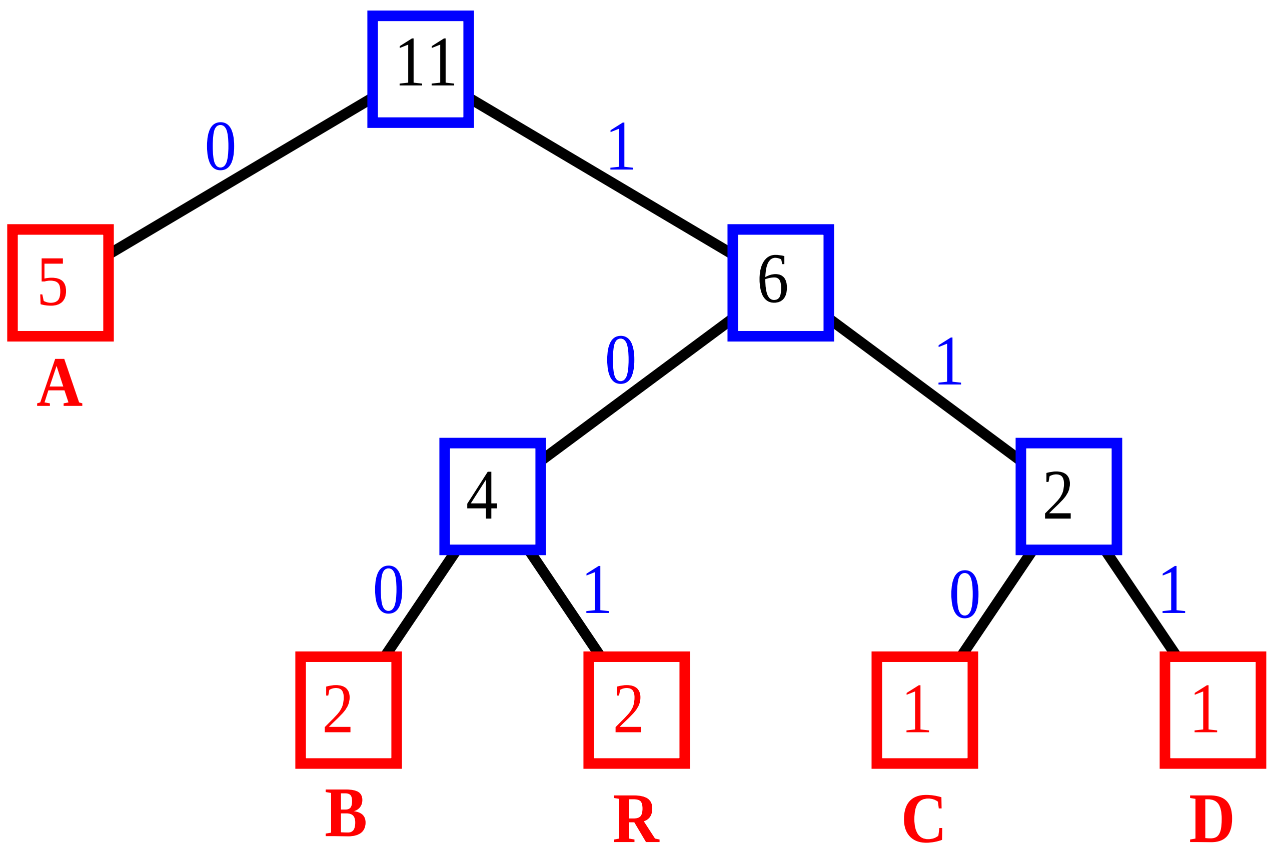


Huffman Encoding Trie (contd.)





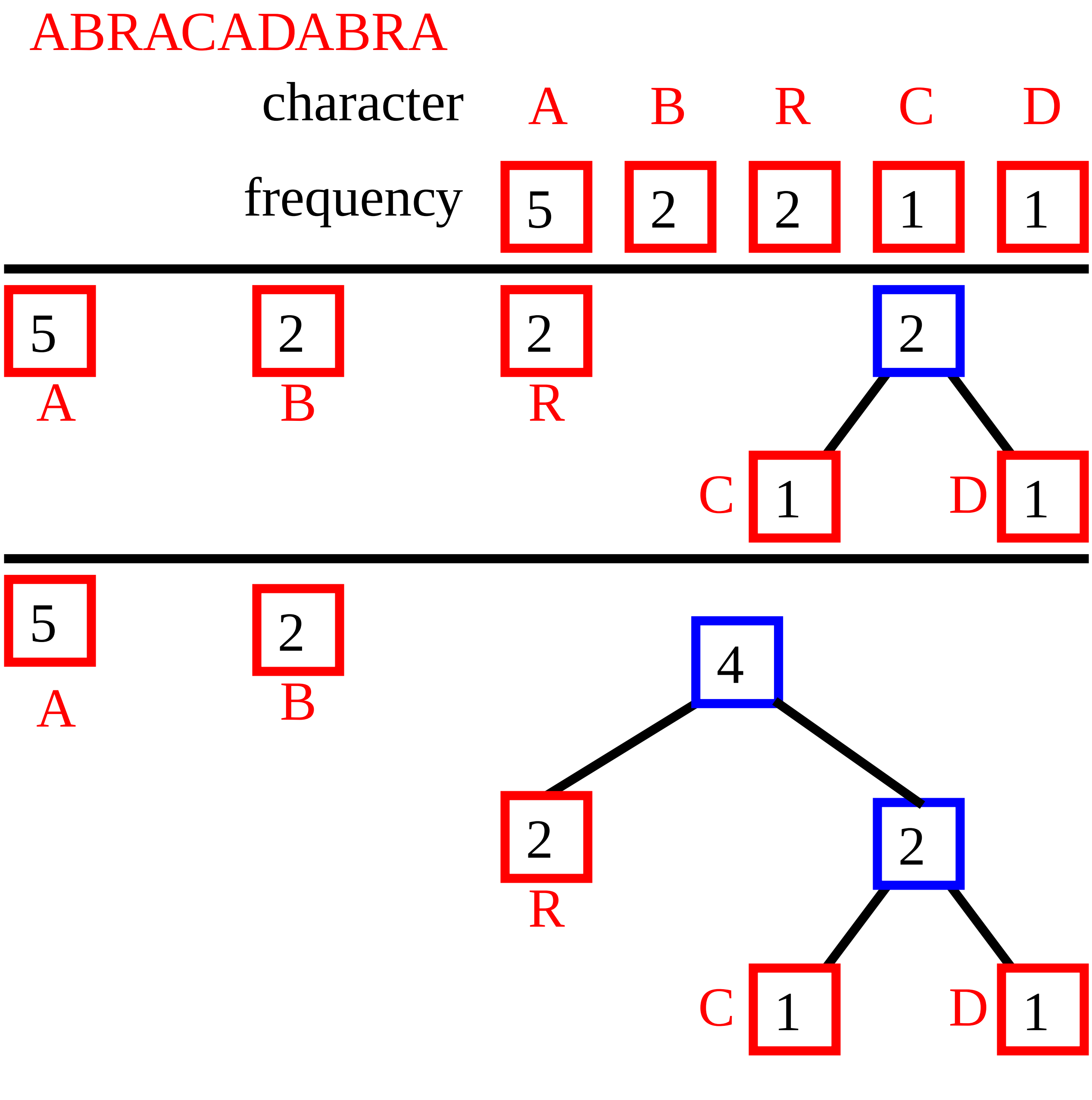
Final Huffman Encoding Trie



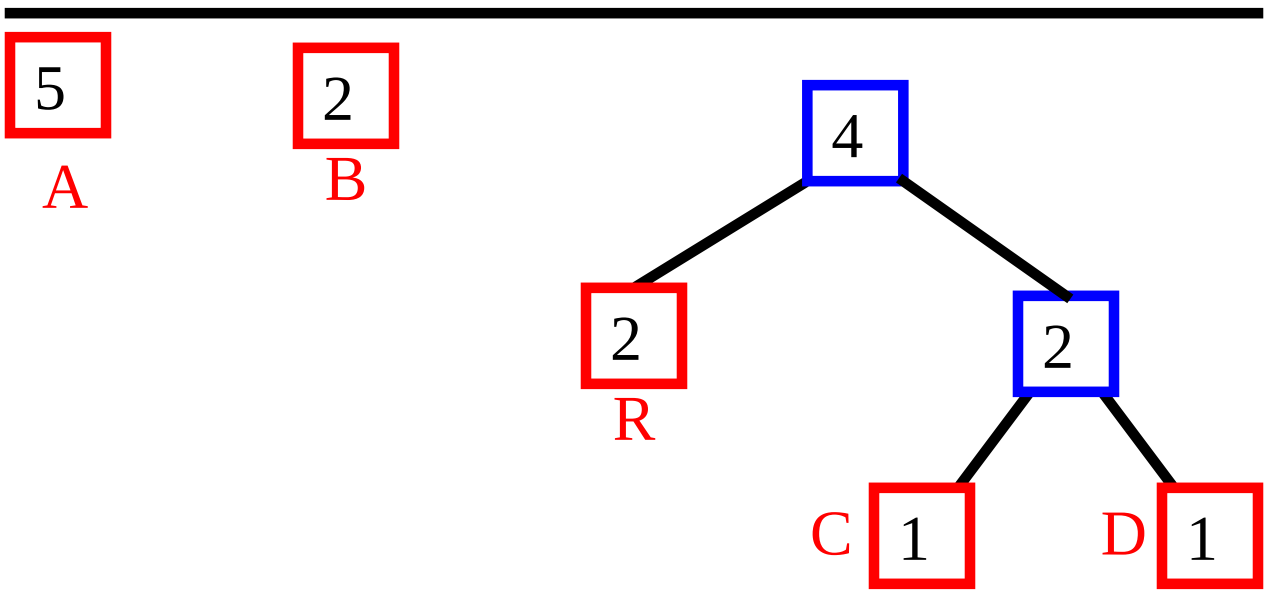
A	B	R	A	C	A	D	A	B	R	A
0	100	101	0	110	0	111	0	100	101	0

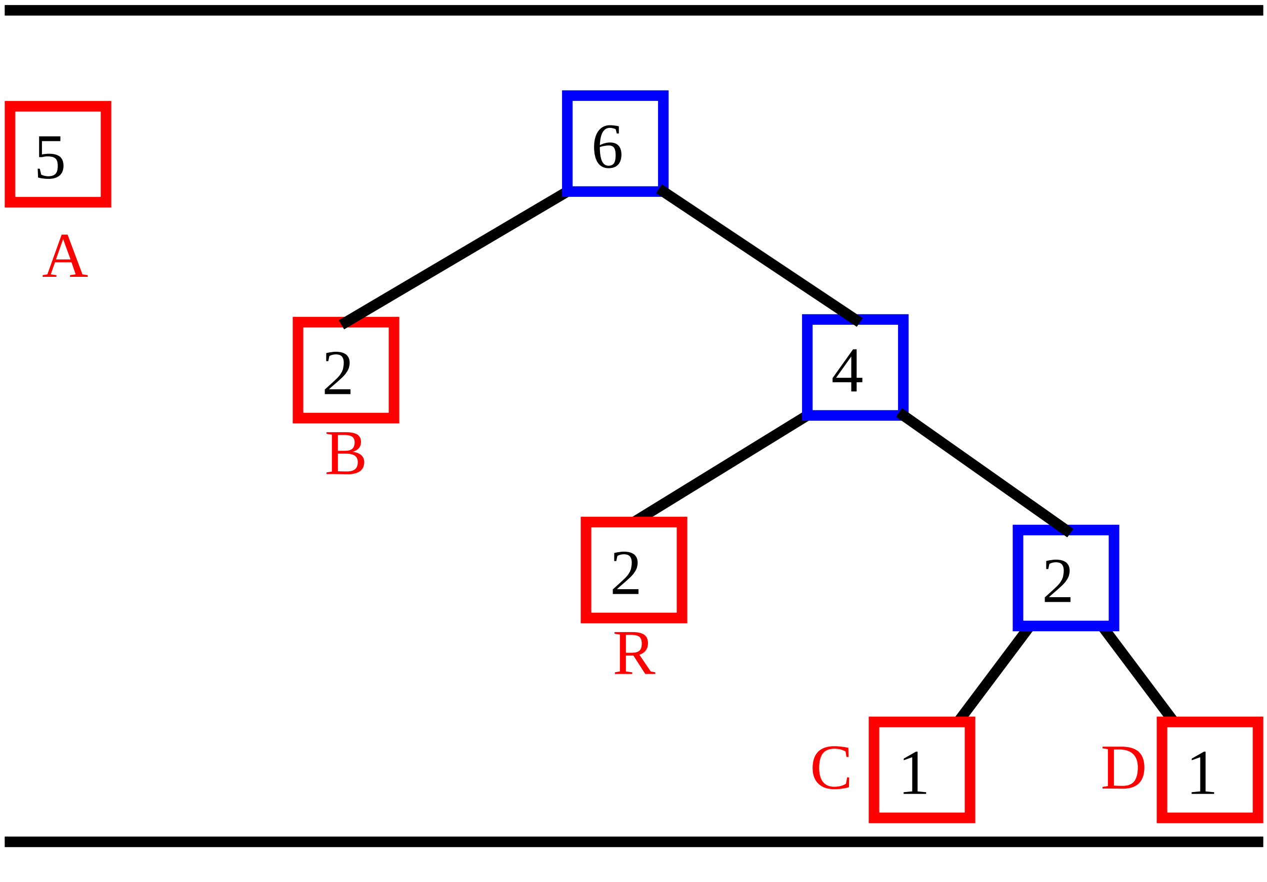
23 bits

Another Huffman Encoding Trie

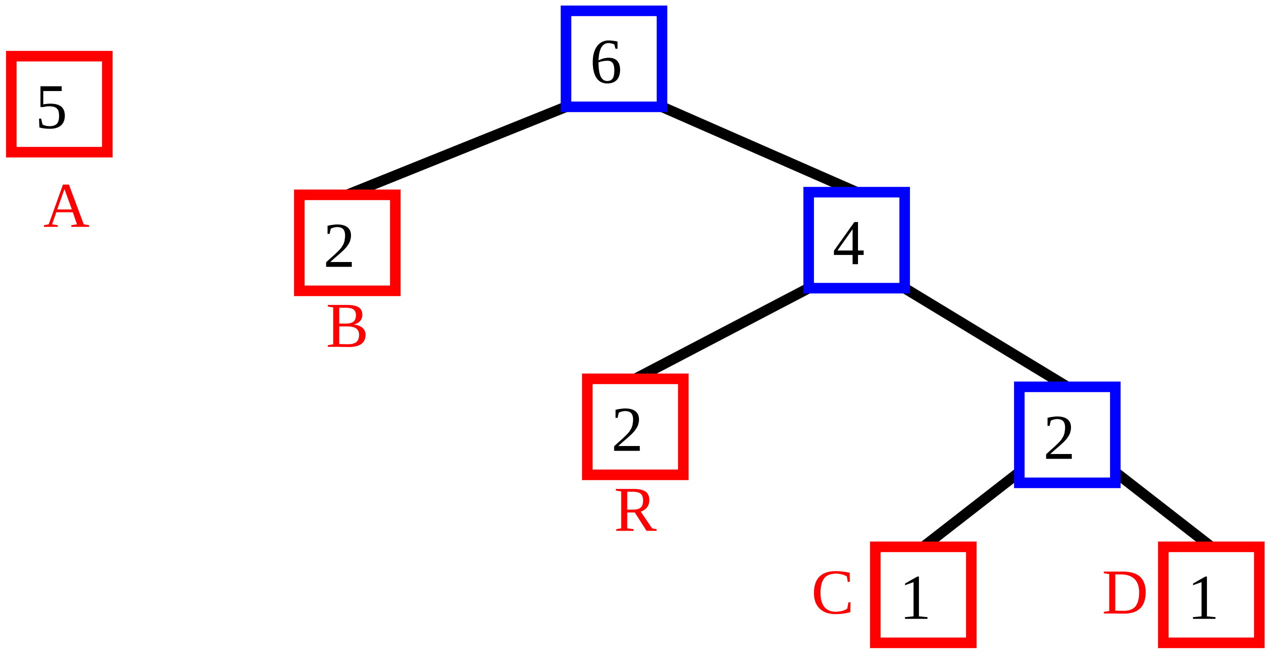


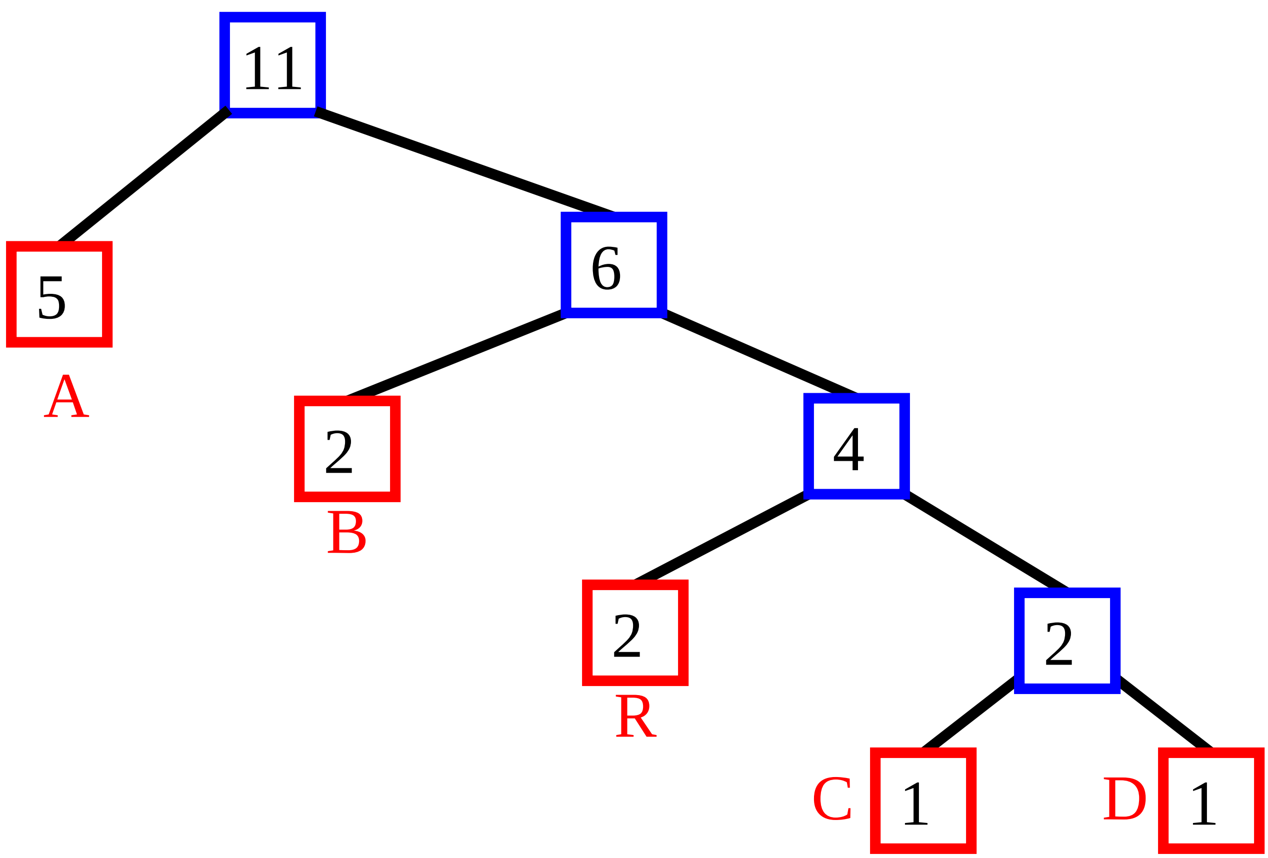
Another Huffman Encoding Trie



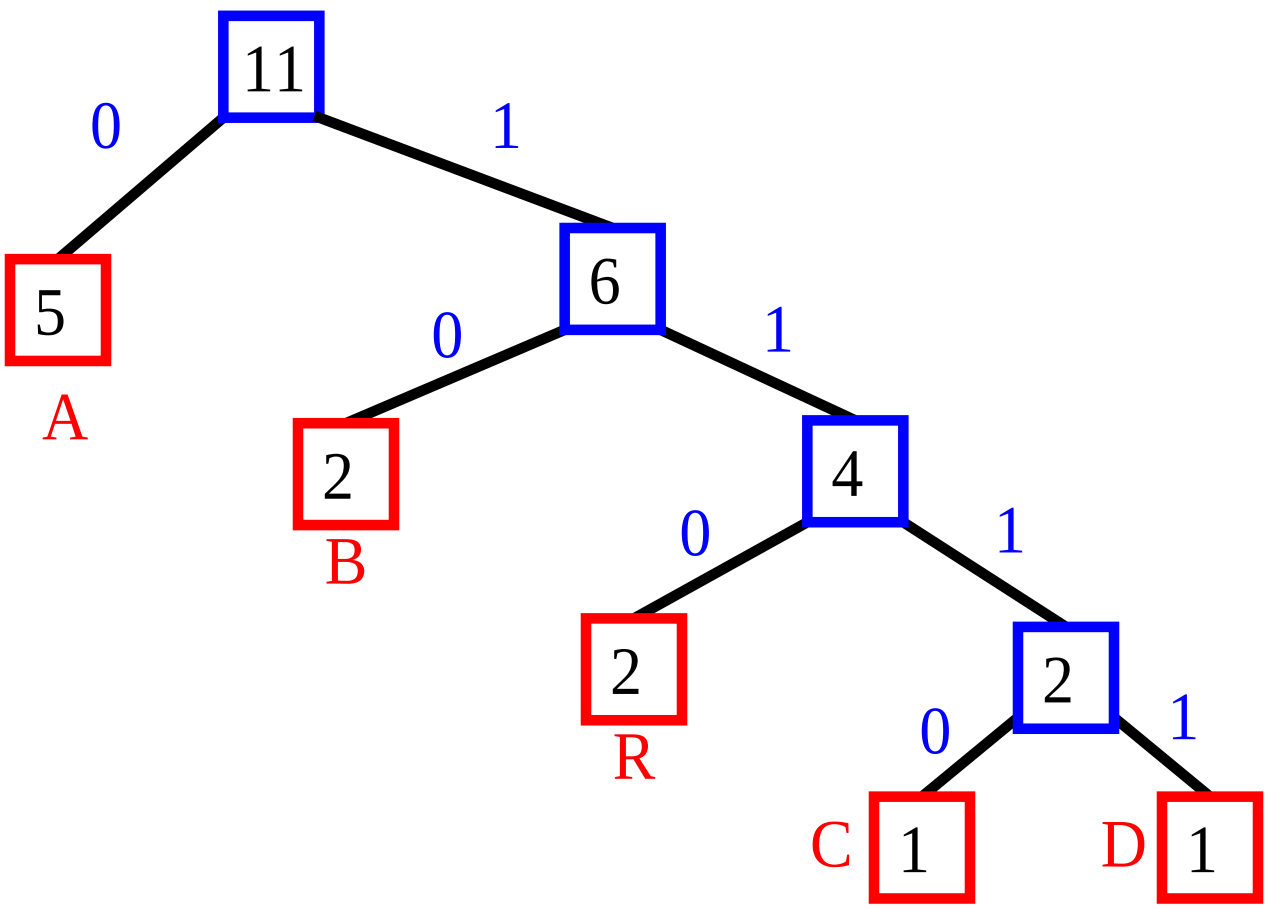


Another Huffman Encoding Trie





Another Huffman Encoding Trie



A B R A C A D A B R A

0 10 110 0 1100 0 1111 0 10 110 0

23 bits

Construction Algorithm

- with a Huffman encoding trie, the encoded text has minimal length

Algorithm Huffman(X):
Input: String X of length n
Output: Encoding trie for X

Compute the frequency $f(c)$ of each character c of X.
Initialize a priority queue Q.

for each character c in X **do**
  Create a single-node tree T storing c
  Q.insertItem($f(c)$, T)
while Q.size() > 1 **do**
  $f_1 \leftarrow Q$.minKey()
  $T_1 \leftarrow Q$.removeMinElement()
  $f_2 \leftarrow Q$.minKey()
  $T_2 \leftarrow Q$.removeMinElement()
  Create a new tree T with left subtree T_1 and right subtree T_2.
  Q.insertItem($f_1 + f_2$)
return tree Q.removeMinElement()

- runing time for a text of length n with k distinct characters: O(n + k log k)

Image Compression

- we can use Huffman encoding also for binary files (bitmaps, executables, etc.)
- common groups of bits are stored at the leaves
- Example of an encoding suitable for b/w bitmaps