An Integer Programming Formulation for a Matching Problem

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Outline

1. Introduction
   - Matching Problems
   - Student-Project Allocation problem (SPA)
   - SPA with preferences over Projects (SPA-P)
   - The problem: MAX-SPA-P
2. An Integer Programming (IP) model for MAX-SPA-P
3. Experimental results
4. Discussions and Future work

Matching Problems

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- assigning conference papers to reviewers
- assigning students to projects

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SPA with preferences over Projects (SPA-P)

Students' preferences	Lecturers' preferences
s_1: p_3 p_2 p_1	l_1: p_1 p_2
s_2: p_1 p_2	l_2: p_3
s_3: p_3	

Project capacities: $c_1 = c_2 = c_3 = 1$.

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  - each student is not assigned more than one project
  - capacities of projects and lecturers are not exceeded

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however,

- s_2 would prefer to be assigned p_1
- this means l_1 also gets her most preferred project
- we call (s_2, p_1) a blocking pair

Definition: Blocking Pair

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   - (ii) $s_i \notin M(l_k)$ and l_k is undersubcribed, or
   - (iii) $s_i \notin M(l_k)$ and l_k prefers p_j to her worst non-empty project in $M(l_k)$.

Another matching..

Students' preferences				Lecturers' preferences		
s_1:	p_3	p_2	p_1	l_1:	p_1	p_2
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`Students' preferences`

s_1: p_3 p_2 p_1

s_2: p_1 p_2

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`Lecturers' preferences`

l_1: p_1 p_2

l_2: p_3

Project capacities: $c_1 = c_2 = c_3 = 1$.

Lecturer capacities: $d_1 = 2$, $d_2 = 1$.

- s_1 and s_2 would rather swap their assigned projects, in order to be better off
- we call $\{s_1, s_2\}$ a coalition

Definition: Coalition

Given a matching M, a *coalition* is a set of students $\{s_{i_0}, \ldots, s_{i_{r-1}}\}$, for some $r \geq 2$ such that each student s_{i_j} $(0 \leq j \leq r-1)$ is assigned in M and prefers $M(s_{i_{j+1}})$ to $M(s_{i_j})$, where addition is performed modulo r.

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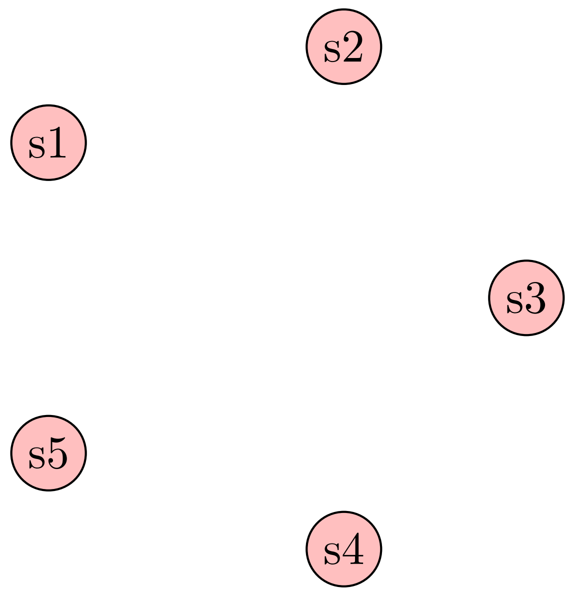

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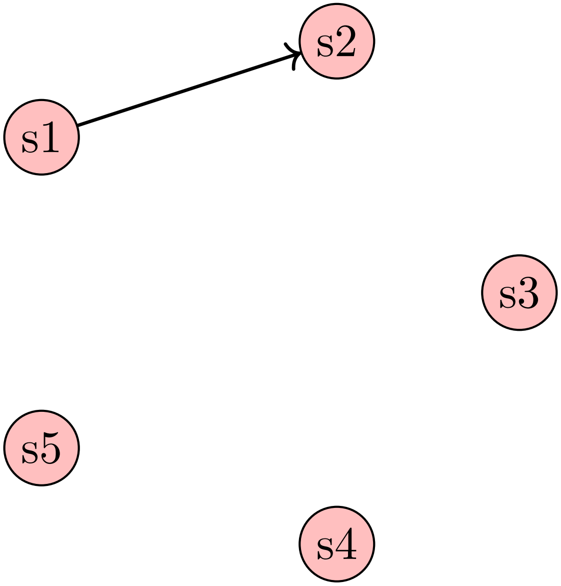

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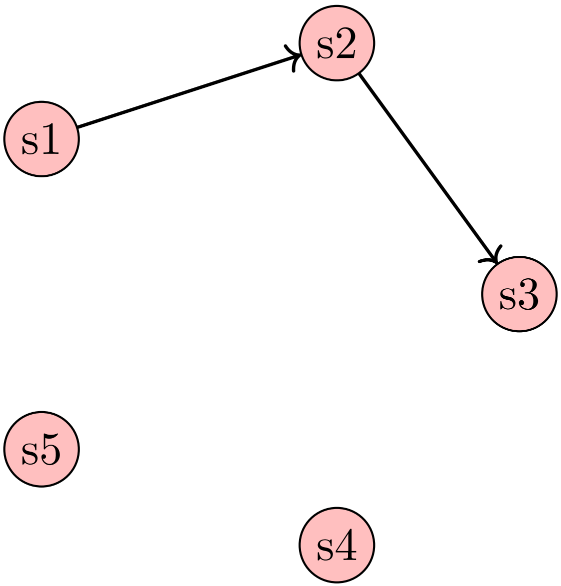

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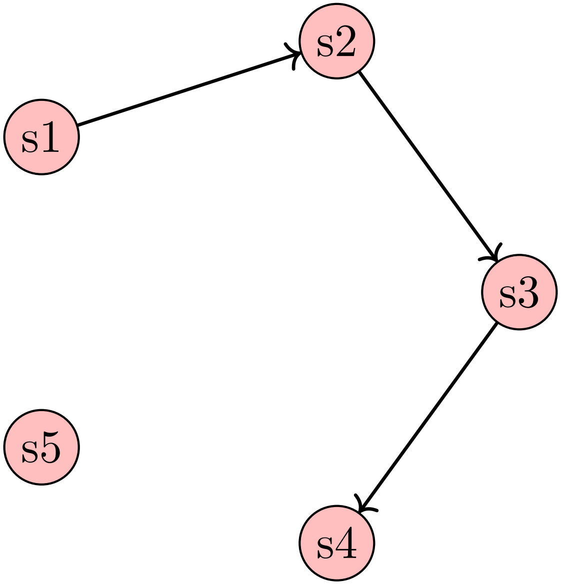

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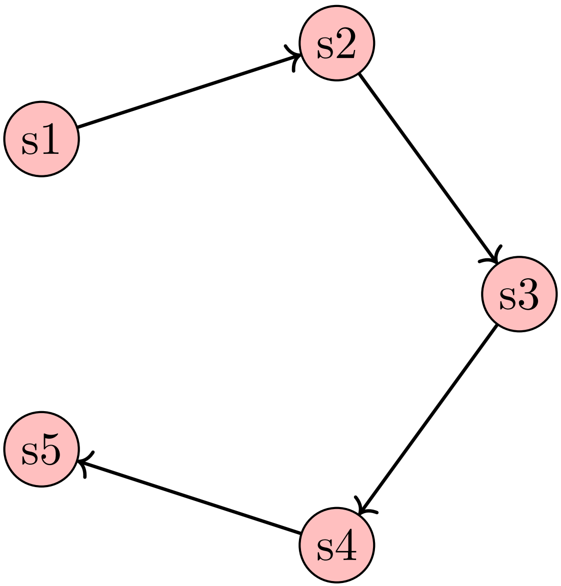

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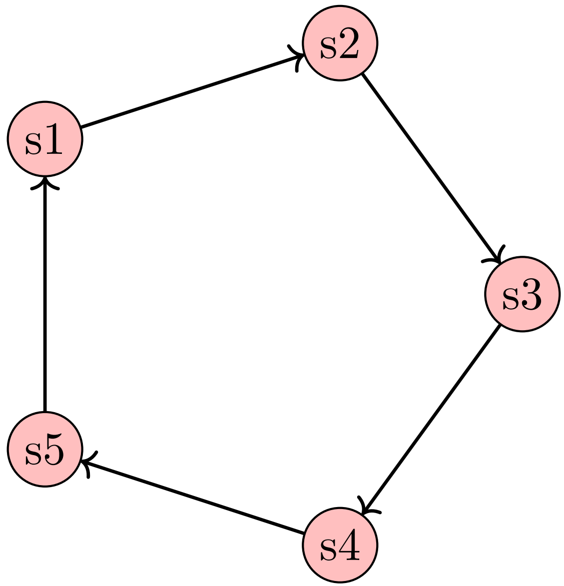

The type of matching we seek..

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Stable matchings

- one with no blocking pair and no coalition

Image adapted from `https://bit.ly/2uBuuA0` (last accessed 28 March 2018).

Stable matchings..

A stable matching

Students' preferences				Lecturers' preferences		
s_1:	p_3	p_2	p_1	l_1:	p_1	p_2
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- 2 students are matched

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Another stable matching

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- 3 students are matched

Maximum cardinality stable matching

Another problem..

- finding a maximum cardinality stable matching (MAX-SPA-P)

Maximum cardinality stable matching

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Suppose the size of a maximum stable matching M is 12,

- 2-approximation algorithm[a], i.e., solution at least $\frac{1}{2}M = 6$

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- finding a maximum cardinality stable matching (MAX-SPA-P)
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Existing results for MAX-SPA-P

Suppose the size of a maximum stable matching M is 12,

- 2-approximation algorithm[a], i.e., solution at least $\frac{1}{2}M = 6$
- $\frac{3}{2}$-approximation algorithm[b], i.e., solution at least $\frac{2}{3}M = 8$
  - not approximable within $\frac{21}{19} - \epsilon$, for any $\epsilon > 0$, unless P = NP

[a] D.F. Manlove and G. O'Malley. Student project allocation with preferences over projects. Journal of Discrete Algorithms, 6:553–560, 2008

[b] K. Iwama, S. Miyazaki, and H. Yanagisawa. Improved approximation bounds for the student-project allocation problem with preferences over projects. Journal of Discrete Algorithms, 13:59–66, 2012.

An Integer Programming (IP) model for MAX-SPA-P

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  3. ensure matching does not admit a coalition;

An Integer Programming (IP) model for MAX-SPA-P

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- create binary-valued variables to represent the assignment of students to projects;
- enforce the following classes of constraints:
  1. find a matching;
  2. ensure matching does not admit a blocking pair;
  3. ensure matching does not admit a coalition;
- describe an objective function to maximise the size of the matching.

Encoding the binary-valued variables

Students' preferences		Lecturers' preferences	
s_1:	p_3 p_2 p_1	l_1:	p_1 p_2
s_2:	p_1 p_2	l_2:	p_3
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Project capacities: $c_1 = c_2 = c_3 = 1$.
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We encode each (s_i, p_j) as a variable $x_{i,j} \in \{0, 1\}$

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$x_{1,3}$ $\quad x_{1,2}$ $\quad x_{1,1}$

$\Downarrow$

$= 1$, then s_1 is assigned to p_3

Encoding the binary-valued variables

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s_1: p_3 p_2 p_1

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s_3: p_3

Lecturers' preferences

l_1: p_1 p_2

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$x_{3,3}$

Matching Constraints

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- capacities of projects are not exceeded

Matching Constraints

Students' preferences				Lecturers' preferences		
s_1:	p_3	p_2	p_1	l_1:	p_1	p_2
s_2:	p_1	p_2		l_2:	p_3	
s_3:	p_3					

Project capacities: $c_1 = c_2 = c_3 = 1$.
Lecturer capacities: $d_1 = 2,\ d_2 = 1$.

- each student is not assigned more than one project

$$\sum_{p_j \in A_i} x_{i,j} \leq 1 \quad (1 \leq i \leq n_1), \qquad \Longrightarrow \quad x_{1,3} + x_{1,2} + x_{1,1} \leq 1$$

- capacities of projects are not exceeded

$$\sum_{i=1}^{n_1} x_{i,j} \leq c_j, \quad (1 \leq j \leq n_2)$$

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Matching Constraints..

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Blocking pair constraints

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(i) $\theta_{i,j} + \alpha_j + \gamma_{i,j,k} \leq 2$;

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(i) $\theta_{i,j} + \alpha_j + \gamma_{i,j,k} \leq 2$; (ii) $\theta_{i,j} + \alpha_j + (1 - \beta_{i,k}) + \delta_k \leq 3$;

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Coalition constraints

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Envy graph

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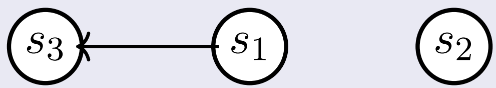

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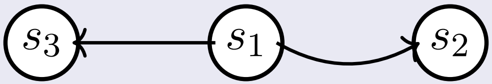

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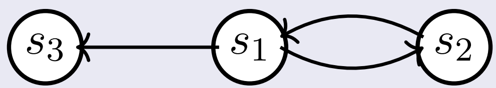

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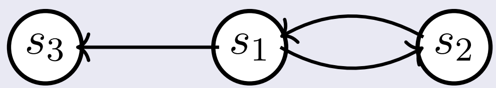

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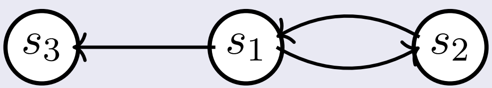

- admits topological ordering $\implies$ it is acyclic $\implies$ no coalition.
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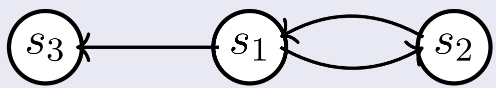

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  - $e_{i,i'} + 1 \geq x_{i,j} + x_{i',j'} \qquad i \neq i'$
- to hold the label of each vertex in the topological ordering, create an integer-valued variable v_i and enforce
  - $v_i < v_{i'} + n_1(1 - e_{i,i'}) \qquad n_1$ – number of students.

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$$\boxed{\max \sum_{i=1}^{n_1} \sum_{p_j \in A_i} x_{i,j}}$$

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Theorem

Given an instance I of SPA-P, there exists an IP formulation J of I such that an optimal solution in J corresponds to a maximum stable matching in I, and vice-versa.

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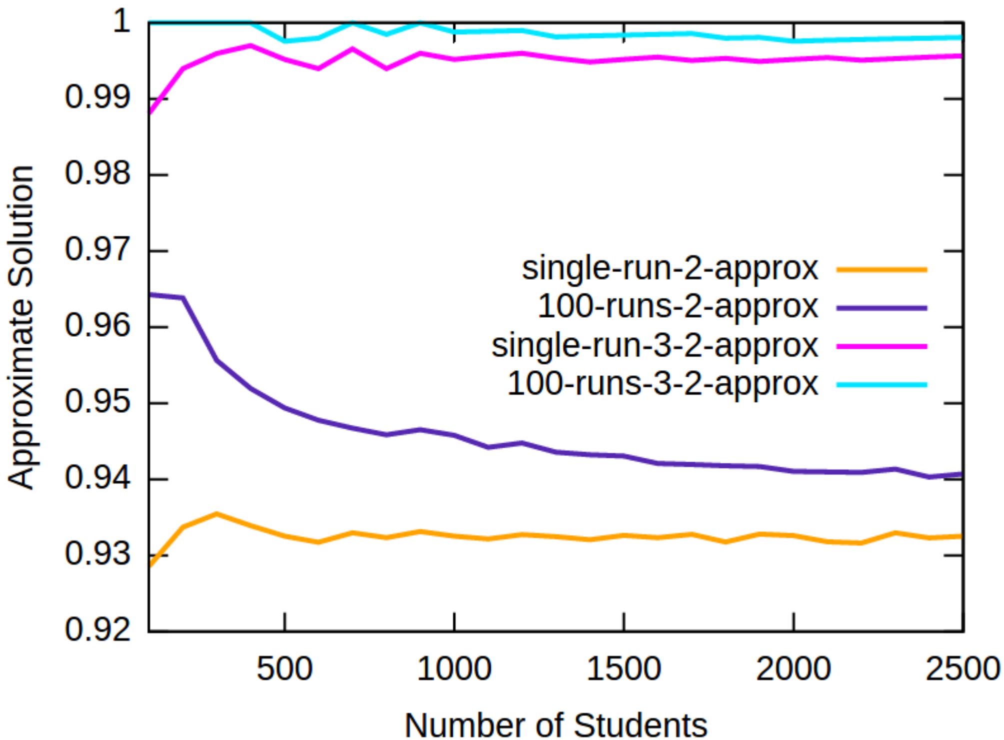
- for the purpose of this experiment, we removed the coalition constraints from our IP solver

Experimental results: Randomly-generated SPA-P instances



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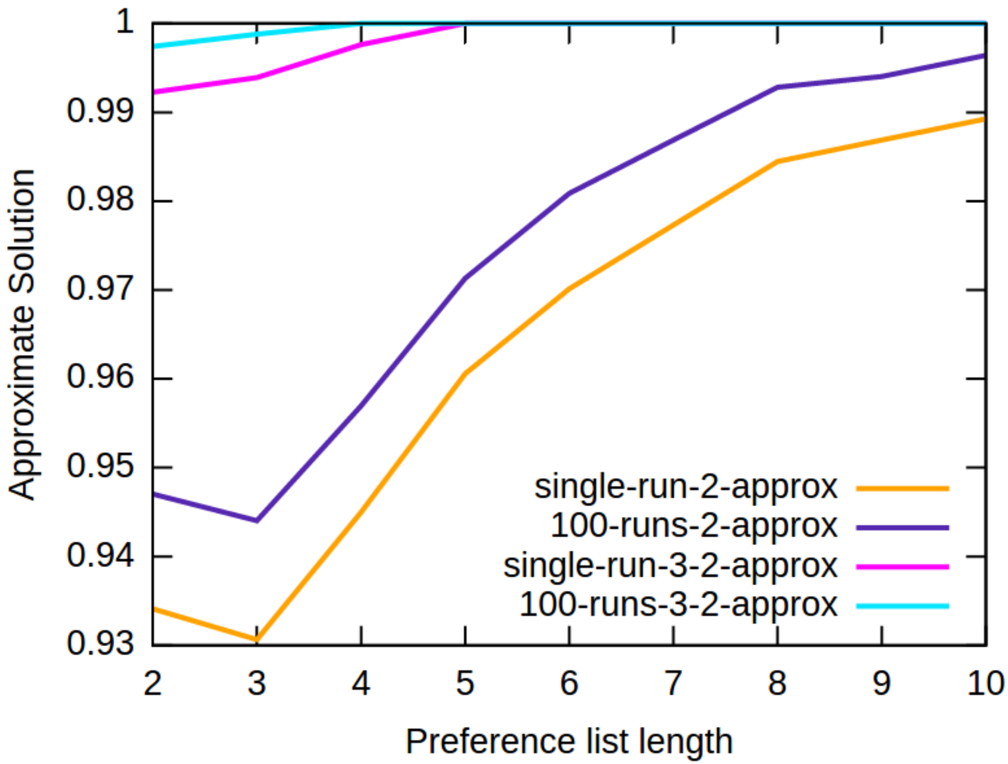

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					Random					Most popular					Least popular				
Year	n_1	n_2	n_3	l	A	B	C	D	E	A	B	C	D	E	A	B	C	D	E
2014	55	149	38	6	55	55	55	54	53	55	55	55	54	50	55	55	55	54	52
2015	76	197	46	6	76	76	76	76	72	76	76	76	76	72	76	76	76	76	75
2016	92	214	44	6	84	82	83	77	75	85	85	83	79	76	82	80	77	76	74
2017	90	289	59	4	89	87	85	80	76	90	89	86	81	79	88	85	84	80	77

Table 1: A, B, C, D and E denotes the solution obtained from the IP model, 100 runs of $\frac{3}{2}$-approximation algorithm, single run of $\frac{3}{2}$-approximation algorithm, 100 runs of 2-approximation algorithm, and single run of 2-approximation algorithm respectively. Also, n_1, n_2, n_3 and l is number of students, number of projects, number of lecturers and length of the students' preference lists respectively.

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- IP model can be employed in practice
- potential coalitions can subsequently be dealt with in polynomial-time

Future work

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  - what if there is a constant number of lecturer?

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  - more parameters yet to be explored..

Thank you for your attention

David Manlove[1], Duncan Milne and Sofiat Olaosebikan[2]. An Integer Programming Approach to the Student-Project Allocation Problem with Preferences over Projects. *To appear in proceedings of ISCO 2018: the 5th International Symposium on Combinatorial Optimisation, Lecture Notes in Computer Science, Springer, 2018.*

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