Signals & Systems Group: NUIM/SS/2002/01

A result on common quadratic Lyapunov functions

R. Shorten, K. Narendra and O. Mason

A result on common quadratic Lyapunov functions

Robert Shorten[*] Hamilton Institute NUI Maynooth Ireland

Kumpati S. Narendra Center for Systems Science Yale University U.S.A.

Oliver Mason Hamilton Institute NUI Maynooth Ireland

Abstract

In this paper we define strong and weak common quadratic Lyapunov functions (CQLF's) for sets of linear time-invariant (LTI) systems. We show that the simultaneous existence of a weak CQLF of a special form, and the non-existence of a strong CQLF, for a pair of LTI systems, is characterised by easily verifiable algebraic conditions. These conditions are found to play an important role in proving the existence of strong CQLF's for general LTI systems.

1 Introduction

The existence or non-existence of common quadratic Lyapunov functions (CQLF's) for two or more stable LTI systems is closely connected to recent work on the design and stability of switching systems. In this context numerous papers have appeared in the literature [1, 2, 3, 4, 5] in which sufficient conditions have been derived under which two stable dynamical systems

$$\Sigma_{A_i} : \dot{x} = A_i x, \quad A_i \in \mathbb{R}^{n \times n},\ i \in \{1, 2\}$$

have a CQLF. If the matrix $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$, simultaneously satisfies the Lyapunov equations $A_i^T P + PA_i = -Q_i$, $i \in \{1, 2\}$, where $Q_i > 0$, then $V(x) = x^T Px$ is said to be a strong

[*]Robert.Shorten@may.ie

CQLF for Σ_{A_1} and Σ_{A_2}. If $Q_i \geq 0$ for $i \in \{1,2\}$ then $V(x)$ is said to be a weak CQLF. This technical note considers pairs of stable LTI systems for which a strong CQLF does not exist, but for which a weak CQLF exists where $-Q_1$ and $-Q_2$ are both negative semi-definite and of rank $n-1$. We derive a result that can be used to determine necessary and sufficient conditions for the existence of a strong CQLF for certain classes of stable LTI systems.

2 Mathematical Preliminaries

In this section we present some results and definitions that are useful in proving the principal result of this note. Throughout, the following notation is adopted: $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively; $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space; $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices with real entries; x_i denotes the i^{th} component of the vector x in $\mathbb{R}^n$; a_{ij} denotes the entry in the (i,j) position of the matrix A in $\mathbb{R}^{n\times n}$.

Where appropriate, the proofs of individual lemmas are presented in the Appendix.

(i) **Stong and weak common quadratic Lyapunov functions :** Consider the set of LTI systems

$$\Sigma_{A_i} : \dot{x} \quad = \quad A_i x, \; i \in \{1,2,...M\}. \tag{1}$$

where M is finite and the A_i, $i \in \{1,2,...M\}$, are constant Hurwitz matrices in $\mathbb{R}^{n\times n}$ (i.e. the eigenvalues of A_i lie in the open left half of the complex plane and hence the Σ_{A_i} are stable LTI systems). Let the matrix $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$, be a simultaneous solution to the Lyapunov equations

$$A_i^T P + P A_i \quad = \quad -Q_i, \; i \in \{1,2,...M\}. \tag{2}$$

Then, $V(x) = x^T P x$ is a strong quadratic Lyapunov function for the LTI system Σ_{A_i} if $Q_i > 0$, and is said to be a *strong CQLF* for the set of LTI systems Σ_{A_i}, $i \in \{1,...,M\}$, if $Q_i > 0$ for all i. Similarly, $V(x)$ is a weak quadratic Lyapunov function for the LTI system Σ_{A_i} if $Q_i \geq 0$, and is said to be a *weak CQLF* for the set of LTI systems Σ_{A_i}, $i \in \{1, ..., M\}$, if $Q_i \geq 0$ for all i.

(ii) **The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$:** The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is the parameterised family of matrices $\sigma_{\gamma[0,\infty)}[A_1, A_2] = A_1 + \gamma A_2,\ \gamma \in [0, \infty)$, where $A_1, A_2 \in \mathbb{R}^{n \times n}$. We say that the pencil is *non-singular* if $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is non-singular for all $\gamma \geq 0$. Otherwise the pencil is said to be *singular*. Further, a pencil is said to be *Hurwitz* if its eigenvalues are in the open left half of the complex plane for all $\gamma \geq 0$.

(iii) The following result provides a useful test for the singularity of a matrix pencil.

Lemma 2.1 [6]: Let $A_1,\ A_2 \in \mathbb{R}^{n \times n}$ with A_1 non-singular. A necessary and sufficient condition for singularity of the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is that the matrix product $A_1^{-1} A_2$ has a negative (real) eigenvalue. (If A_2 is also non-singular, then this is equivalent to $A_1 A_2^{-1}$ having a negative (real) eigenvalue.)

(iv) **The stability of Σ_A and $\Sigma_{A^{-1}}$:** The relationship between a matrix, its inverse, and a quadratic Lyapunov function will arise in our discussion. In this context we note the following fundamental result ([7]). Consider the LTI systems

$$\Sigma_A : \dot{x} = Ax,\ \Sigma_{A^{-1}} : \dot{x} = A^{-1}x,$$

where $A \in \mathbb{R}^{n \times n}$ is Hurwitz. Then, any quadratic Lyapunov function for Σ_A is also a quadratic Lyapunov function for $\Sigma_{A^{-1}}$.

Comment : Suppose that $V(x)$ is a strong CQLF for the stable LTI systems $\Sigma_{A_1},\ \Sigma_{A_2}$. It is easily verified that the same function $V(x)$ will be a strong quadratic Lyapunov function for the systems $\Sigma_{\sigma_{\gamma[0,\infty)}[A_1,A_2]}$ and $\Sigma_{\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]}$ for all $\gamma \in [0, \infty)$. Hence, $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are both necessarily Hurwitz for all $\gamma \in [0, \infty)$. Thus the non-singularity of these two pencils is a necessary condition for the existence of a CQLF for the systems $\Sigma_{A_1},\ \Sigma_{A_2}$.

(v) **Lemma 2.2 :** Let $u, v, x, y \in \mathbb{R}^n$ be any four non-zero vectors. There exists a non-singular $T \in \mathbb{R}^{n \times n}$ such that each component of the vectors Tu, Tv, Tx, Ty is non-zero.

(vi) **Lemma 2.3 :** Let x, y, u, v be 4 non-zero vectors in $\mathbb{R}^n$ such that for all Hermitian matrices $P \in \mathbb{R}^{n \times n}$, $x^T P y = -k u^T P v$ with $k > 0$. Then either

$$\begin{aligned} x &= \alpha u \text{ for some real scalar } \alpha, \text{ and } y = -(\frac{k}{\alpha})v \quad \text{or} \\ x &= \beta v \text{ for some real scalar } \beta \text{ and } y = -(\frac{k}{\beta})u. \end{aligned}$$

3 Main results

We consider pairs of stable LTI systems for which no strong CQLF exists, but for which a weak CQLF exists with Q_i, $i \in \{1, 2\}$, of rank $n-1$. Our principal result, Theorem 3.1, establishes a set of easily verifiable algebraic conditions, that are satisfied when such a weak CQLF exists.

Theorem 3.1 : Let A_1, A_2 be two Hurwitz matrices in $\mathbb{R}^{n \times n}$ such that a solution $P = P^T \geq 0$ exists to the non-strict Lyapunov Equations

$$A_i^T P + P A_i = -Q_i \leq 0, \; i \in \{1, 2\} \tag{3}$$

for some positive semi-definite matrices Q_1, Q_2 both of rank $n-1$. Furthermore suppose that no strong CQLF exists for Σ_{A_1} and Σ_{A_2}. Under these conditions, at least one of the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is singular. Equivalently, by lemma 2.1, at least one of the matrix products $A_1 A_2$ and $A_1 A_2^{-1}$ has a real negative eigenvalue.

Comment : The following facts are established in Theorem 3.1.

(a) Vectors x_1, $x_2 \in \mathbb{R}^{n \times n}$ exist such that $Q_1 x_1 = 0$ and $Q_2 x_2 = 0$.

(b) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hyperplanes in the space of symmetric matrices defined by the following equations (in the free parameter H) :

$$\mathcal{H}_1 : x_1^T H A_1 x_1 = 0, \; \mathcal{H}_2 : x_2^T H A_2 x_2 = 0. \tag{4}$$

Then, $\mathcal{H}_1$ and $\mathcal{H}_2$ define the same plane.

(c) There is some real $\alpha_0 > 0$ with $x_1^T H A_1 x_1 = -\alpha_0 x_2^T H A_2 x_2$, for all $H = H^T$.

Proof of Theorem 3.1 : As Q_1 and Q_2 are of rank $n-1$, there are non-zero vectors x_1, x_2 such that

$$x_1^T Q_1 x_1 = 0, \;\; x_2^T Q_2 x_2 = 0. \tag{5}$$

The proof of Theorem 3.1 is split into two main stages.

<u>Stage 1 :</u> The first stage in the proof is to show that if there exists a Hermitian matrix $\overline{P}$ satisfying

$$x_1^T \overline{P} A_1 x_1 < 0, \;\; x_2^T \overline{P} A_2 x_2 < 0 \tag{6}$$

then a strong CQLF exists for Σ_{A_1} and Σ_{A_2}.

Note that as $x^T P A_1 x$ is a scalar for any x, we can write $x^T Q_1 x = 2x^T P A_1 x$. The same obviously holds for $x^T Q_2 x$.

Now assume that there is some $\overline{P}$ satisfying (6). We shall show that by choosing $\delta_1 > 0$ sufficiently small, it is possible to guarantee that $A_1^T(P+\delta_1\overline{P})+(P+\delta_1\overline{P})A_1$ is negative definite. Firstly, consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : \|x\| = 1 \text{ and } x^T \overline{P} A_1 x \geq 0\}.$$

Note that if the set Ω_1 was empty, then any positive constant $\delta_1 > 0$ would make $A_1^T(P+\delta_1\overline{P})+ (P+\delta_1\overline{P})A_1$ negative definite. Hence, we assume that Ω_1 is non-empty.

The function that takes x to $x^T \overline{P} A_1 x$ is continuous. Thus Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T P A_1 x$ is strictly negative on Ω_1.

Let M_1 be the maximum value of $x^T \overline{P} A_1 x$ on Ω_1, and let M_2 be the maximum value of $x^T P A_1 x$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant

$\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1} = C_1$$

and consider the Hermitian matrix

$$P + \delta_1 \overline{P}.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $\|x\| = 1$, it follows that for all non-zero vectors x of norm 1

$$x^T(A_1^T(P + \delta_1 \overline{P}) + (P + \delta_1 \overline{P})A_1)x < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$. Since the above inequality is unchanged if we scale x by any non-zero real number, it follows that $A_1^T(P + \delta_1 \overline{P}) + (P + \delta_1 \overline{P})A_1$ is negative definite. By a standard result of systems theory, this implies that the matrix $P + \delta_1 \overline{P}$ is positive definite.

The same argument can be used to show that there is some $C_2 > 0$ such that

$$x^T(A_2^T(P + \delta_1 \overline{P}) + (P + \delta_1 \overline{P})A_2)x < 0$$

for all non-zero x, for $0 < \delta_1 < C_2$. So, if we choose δ less than the minimum of C_1, C_2, we would have a positive definite matrix

$$P_1 = P + \delta \overline{P}$$

which defined a strong CQLF for Σ_{A_1} and Σ_{A_2}.

Stage 2 : So under our assumptions, no Hermitian solution $\overline{P}$ exists satisfying equations (6). We now show that such a solution $\overline{P}$ would exist unless one of the two pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ was singular.

As there is no Hermitian solution to (6), any Hermitian H that makes the expression $x_1^T H A_1 x_1$ negative will make the expression $x_2^T H A_2 x_2$ positive. More formally

$$x_1^T H A_1 x_1 < 0 \iff x_2^T H A_2 x_2 > 0 \tag{7}$$

for Hermitian H. It follows from this that

$$x_1^T H A_1 x_1 = 0 \iff x_2^T H A_2 x_2 = 0.$$

The expressions $x_1^T HA_1x_1$, $x_2^T HA_2x_2$, viewed as functions of H, define linear functionals on the space of Hermitian matrices. Moreover, we have seen that the null sets of these functionals are identical. So they must be scalar multiples of each other. Furthermore, (7) implies that they are negative multiples of each other. That is,

$$x_1^T HA_1x_1 = -kx_2^T HA_2x_2 \tag{8}$$

with $k > 0$, for all Hermitian matrices H.

Now Lemma 2.3 implies that either $x_1 = \alpha x_2$ and $A_1x_1 = -(\frac{k}{\alpha})A_2x_2$ or $x_1 = \beta A_2x_2$ and $A_1x_1 = -(\frac{k}{\beta})x_2$. Consider the former situation to begin with. Then we have

$$\begin{aligned} A_1(\alpha x_2) &= -(\frac{k}{\alpha})A_2x_2 \\ \Rightarrow (A_1 + (\frac{k}{\alpha^2})A_2)x_2 &= 0 \end{aligned}$$

and thus the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is singular. It follows from Lemma 2.1 that the matrix $A_1A_2^{-1}$ has a negative eigenvalue.

On the other hand, in the latter situation, we have that

$$x_2 = \frac{1}{\beta}A_2^{-1}x_1$$

Thus

$$\begin{aligned} A_1x_1 &= -(\frac{k}{\beta^2})A_2^{-1}x_1 \\ \Rightarrow (A_1 + (\frac{k}{\beta^2})A_2^{-1})x_1 &= 0 \end{aligned}$$

Thus, in this case the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is singular. It follows from Lemma 2.1 that the matrix A_1A_2 has a negative eigenvalue. This completes the proof of Theorem 3.1.

Comment: A crucial point in the proof of theorem 3.1 is that there is a unique hyperplane containing the matrix P which separates the sets $\{\overline{P} : A_1^T\overline{P} + \overline{P}A_1 < 0\}$ and $\{\overline{P} > 0 : A_2^T\overline{P} + \overline{P}A_2 < 0\}$. For the question of CQLF existence for three or more LTI systems, such a hyperplane need not exist and alternative methods would need to be considered.

4 Application of main result

In this section we present an example to illustrate the use of Theorem 3.1.

Example (Second order systems) : Let Σ_{A_1} and Σ_{A_2} be stable LTI systems with $A_1, A_2 \in \mathbb{R}^{2\times 2}$. We note the following easily verifiable facts.

(a) If a strong CQLF exists for Σ_{A_1} and Σ_{A_2} then the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are necessarily Hurwitz.

(b) If A_1 and A_2 satisfy the non-strict Lyapunov equations (3) then the matrices Q_1 and Q_2 are both rank 1 (rank $n-1$).

(c) If a strong CQLF does not exist for Σ_{A_1} and Σ_{A_2} then a positive constant d exists such that a strong CQLF exists for Σ_{A_1-dI} and Σ_{A_2}. By continuity a non-negative $d_1 < d$ exists such that $A_1 - d_1 I$ and A_2 satisfy Theorem 3.1 and one of the pencils $\sigma_{\gamma[0,\infty)}[A_1 - d_1 I, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1 - d_1 I, A_2^{-1}]$ is necessarily singular. Hence, it follows that one of the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is <u>not Hurwitz</u>.

Items (a)-(c) establish the following facts. Given two stable second order LTI systems Σ_{A_1} and Σ_{A_2}, a necessary condition for the existence of a strong CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz. Conversely, a necessary condition for the non-existence of a strong CQLF is that one of the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is not Hurwitz. Together these conditions yield the following known result [5, 2]:

A necessary and sufficient condition for the LTI systems Σ_{A_1} and Σ_{A_2}, $A_1, A_2 \in \mathbb{R}^{2\times 2}$ both Hurwitz , to have a strong CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz.

or in the equivalent matrix product form [8]:

A necessary and sufficient condition for the LTI systems Σ_{A_1} and Σ_{A_2}, $A_1, A_2 \in \mathbb{R}^{2\times 2}$ both Hurwitz , to have a strong CQLF is that the matrix products $A_1 A_2$ and $A_1 A_2^{-1}$ have no negative real eigenvalues.

5 Concluding remarks

In this paper a result related to strong and weak CQLF's has been derived. It is shown that if a strong CQLF does not exist for a pair of stable LTI systems, but a weak CQLF of a specific form exists, then at least one of the matrix pencils $A_1 + \gamma A_2$, $A_1 + \lambda A_2^{-1}$ is singular for some positive γ $(or \lambda)$ (and at least one of the matrix products $A_1 A_2$ or $A_1 A_2^{-1}$ has a negative eigenvalue). It is possible to adapt the method of proof of theorem 3.1 to obtain corresponding results for discrete-time systems involving the *bilinear* or *Cayley* transform $C(A) = (A - I)(A + I)^{-1}$ ([9]).

Acknowledgements

This work was supported by the European Union funded research training network *Multi-Agent Control*, HPRN-CT-1999-00107[1] and by the Enterprise Ireland grant SC/2000/084/Y.

References

[1] T. Ando, "Sets of matrices with a common Lyapunov solution," *Archiv der Mathematik*, vol. 77, pp. 76–84, 2001.

[2] N. Cohen and I. Lewkowicz, "A necessary and sufficient criterion for the stability of a convex set of matrices," *IEEE Transactions on Automatic Control*, vol. 38, no. 4, pp. 611–615, 1993.

[3] Y. Mori, T. Mori, and Y. Kuroe, "On a class of linear constant systems which have a common quadratic Lyapunov function," in *Proceedings of 37th Conference on Decision and Control*, 1998.

[4] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE control systems magazine*, pp. 59–70, 1999.

[1]This work is the sole responsibility of the authors and does not reflect the European Union's opinion

[5] R. N. Shorten and K. S. Narendra, "Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function in two stable second order linear systems," in *Proceedings of American Control Conference*, 1999.

[6] D. Hershkowitz and H. Schneider, "On the inertia of intervals of matrices," *SIAM J. of Matrix Anal. Appl.*, vol. 715.

[7] R. Loewy, "On ranges of real Lyapunov transformations," *Linear Algebra and its Applications*, vol. 13, pp. 79–89, 1976.

[8] R. Shorten and K. Narendra, "Necessary and sufficient conditions for the existence of a common quadratic lyapunov functions for a finite number of stable second order linear time-invariant systems." To appear in International Journal of Adaptive Control and Signal Prcoessing.

[9] O. Mason and R. Shorten, "On common quadratic lyapunov functions for stable discrete-time LTI systems," tech. rep., Technical Report SS/2002/05, Signals and Systems Group, NUI Maynooth, 2002.

Appendix

Proof of Lemma 2.2 : Consider the norm $\|A\|_\infty = sup\{|a_{ij}| : 1 \leq i, j \leq n\}$ on $\mathbb{R}^{n\times n}$, and let z be any non-zero vector in $\mathbb{R}^n$. Then it is easy to see that the set $\{T \in \mathbb{R}^{n\times n} : det(T) \neq 0, (Tz)_i \neq 0, 1 \leq i \leq n\}$ is open. On the other hand, if $T \in \mathbb{R}^{n\times n}$ is such that $(Tz)_i = 0$ for some i, an arbitrarily small change in an appropriate element of the i^{th} row of T will result in a matrix T' such that $(T'z)_i \neq 0$. From this it follows that arbitrarily close to the original matrix T, there is some $T_1 \in \mathbb{R}^{n\times n}$ such that $T_1 z$ is non-zero component-wise.

Now to prove the lemma, simply select a non-singular T_0 such that $T_0 x$ is non-zero component-wise. Suppose that some component of $T_0 y$ is zero. By the arguments in the previous paragraph, it is clear that we can select a non-singular $T_1 \in \mathbb{R}^{n\times n}$ such that each component of T_1x and T_1y is non-zero. Now it is simply a matter of repeating this step for the remaining vectors u and v to complete the proof of the lemma.

Proof of Lemma 2.3 : We can assume that all components of x, y, u, v are non-zero. To see why this is so, suppose that the result was proven for this case and we were given four arbitrary non-zero vectors x, y, u, v. We could transform them via a single non-singular transformation T such that each component of Tx, Ty, Tu, Tv was non-zero (Lemma 2.2). Then for all Hermitian matrices P we would have $(Tx)^TP(Ty) = x^T(T^TPT)y$, and hence, that $(Tx)^TP(Ty) = -k(Tu)^TP(Tv)$. Then $Tx = \alpha Tu$ and thus $x = \alpha u$ or $Tx = \beta Tv$ and $x = \beta v$. So we shall assume that all components of x, y, u, v are non-zero. Suppose that x is not a scalar multiple of u to begin with. Then for any index i with $1 \le i \le n$, there is some other index j and two non-zero real numbers c_i, c_j such that

$$x_i = c_iu_i,\ x_j = c_ju_j,\ c_i \neq c_j \tag{9}$$

Choose one such pair of indices i, j. Equating the coefficients of p_{ii}, p_{jj} and p_{ij} respectively in the identity $x^TPy = -ku^TPv$ yields the following equations.

$$\begin{aligned} x_iy_i &= -ku_iv_i && (10)\\ x_jy_j &= -ku_jv_j && (11)\\ (x_iy_j + x_jy_i) &= -k(u_iv_j + u_jv_i) && (12) \end{aligned}$$

If we combine (9) with (10) and (11), we find

$$\begin{aligned} y_i &= -\frac{k}{c_i}v_i && (13)\\ y_j &= -\frac{k}{c_j}v_j && (14) \end{aligned}$$

Using (10)-(14) we find $c_iu_iy_j + c_ju_jy_i = -k(u_iv_j + u_jv_i)$. Hence, $u_iv_j(\frac{c_j-c_i}{c_j}) = u_jv_i(\frac{c_j-c_i}{c_i})$. Recall that $c_i \neq c_j$ so we can divide by $c_j - c_i$ and rearrange terms to get

$$\frac{c_i}{c_j} = (\frac{v_i}{v_j})(\frac{u_j}{u_i}) \tag{15}$$

But using (9) we find

$$\frac{c_i}{c_j} = (\frac{x_i}{x_j})(\frac{u_j}{u_i}) \tag{16}$$

Combining (15) and (16) yields

$$\frac{v_i}{v_j} = \frac{x_i}{x_j} \tag{17}$$

Thus $x_i = cv_i$, $x_j = cv_j$ for some constant c. Now if we select any other index k with $1 \leq k \leq n$, and write $x_k = c_k u_k$ then c_k must be different to at least one of c_i, c_j. Without loss of generality, we may take it that $c_k \neq c_i$. Then the above argument can be repeated with the indices i and k in place of i and j to yield

$$x_i = cv_i,\, x_k = cv_k. \tag{18}$$

But this can be done for any index k so we conclude that $x = cv$ for a scalar c. So we have shown that if x is not a scalar multiple of u, then it is a scalar multiple of v.

To complete the proof, note that if $x = \beta v$ for a scalar β then by (10), $\beta v_i y_i = -k u_i v_i$ for all i. Thus $y = -(\frac{k}{\beta})u$ as claimed. The same argument will show that if $x = \alpha u$ for a scalar α, then $y = -(\frac{k}{\alpha})v$. **Q.E.D**