Line-closed Subsets of Steiner Triple Systems and Classical Linear Spaces

Alan R. Camina and Alice Ann Miller

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Abstract

A proper non-empty subset $C$ of the points of a linear space $S = (P, L)$ is called line-closed if any two intersecting lines of $S$, each meeting $C$ at least twice, have their intersection in $C$. We show that when every line has $k$ points and every point lies on $r$ lines the maximum size for such sets is $r + k - 2$. In addition it is shown that this cannot happen for projective spaces $PG(n, q)$ unless $q = 2$, nor can it be obtained for affine spaces $AG(n, q)$ unless $n = 2$ and $q = 3$. However, for all odd values of $r$ there exist Steiner triple systems having such maximum line-closed subsets.

1 Introduction

A Linear Space $S = (P, L)$ is a set $P$ of points together with a set $L$ of distinguished subsets called lines, such that any two points lie on exactly one line. For all points $x, y \in P$ let $\{x, y\}$ denote the line on $x$ and $y$. A subspace $S'$ of $S$ is a set $P' \subset P$ of points together with a set $L' \subset L$ of lines, such that for all points $x, y \in P'$, $\{x, y\} \in L'$.

The number of points will be denoted by $v$ and the number of lines by $b$. A linear space is called regular when each line has the same number of points. In this case the number of points on a line is denoted by $k$ and the number of lines through a point by $r$.

A line-closed subset of a regular linear space will be a proper non-empty subset $C$ of $P$ such that for any pair of distinct lines $l_1$ and $l_2$ of $L$ such that

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(a) $|l_i \cap C| \geq 2$ for $i = 1, 2$ and
(b) $l_1$ and $l_2$ intersect,
then the intersection is a point of $C$.

We show that such a subset has at most $r + k - 2$ points and this paper addresses the problem of when this maximum can be achieved.

Line-closed subsets first arose when considering the fixed-point set of an automorphism of a regular linear space, see [4] and [6]. The maximum size that a fixed-point set can have is $r + k - 3$, (see [6]). Hartman and Hoffman in [7] consider the existence of involutions acting on Steiner triple systems. They give a complete analysis of the parameters of Steiner triple systems admitting involutions including when such involutions can have the maximal number of fixed points. No example of a linear space of line size greater than 3 admitting an automorphism with fixed point set of maximal size has been published. Similarly we have found no example of a linear space of line size greater than 3 exhibiting a maximum line-closed subset and the existence of such subsets is still an open question.

From the results in this paper it is clear that the situation is reasonably well understood for the linear spaces given by the classical projective and affine geometries. The following example illustrates this and shows that maximum line-closed subsets exist.

**Example 1.1** Let $S = (P, L)$ be the linear space formed by the points and lines of the projective space $PG(n, 2)$, $n \geq 2$, and let $C = \{p\} \cup H$ where $H$ is a hyperplane of $S$ and $p$ a point in $P \setminus H$. Then $C$ is a maximum line-closed subset of $S$.

It is straightforward to show that the above construction does indeed yield a maximum line-closed subset in this case.

### 2 Some Preliminary Results

In this section we prove some simple but useful results concerning line-closed subsets.

**Lemma 2.1** Let $C$ be a line-closed subset of $S$. Then $|C| \leq r + k - 2$.

**Proof** Choose any point, say $x$, not in $C$. There are $r$ lines through $x$ and at most one such line can contain more than 1 point of $C$. That line can contain at most $k - 1$ points from $C$. Thus we get that $|C| \leq r + k - 2$. 

We will call $C$ a maximum line-closed subset if $|C| = r + k - 2$. It is easy to see that if $C$ is a maximum line-closed subset of $S$ every line of $C$ has either $k$ or $k-1$ points. We shall show in Lemma 2.4 that in fact $C$ contains at least one line of each size. Our main objective is to decide for which parameters and which linear spaces such maximum line-closed subsets can occur. Clearly, if $k = 2$ then any $(v - 1)$-set of points of $S$ is a maximum line-closed subset. Therefore we shall assume henceforth that $k \geq 3$.

The argument in the proof of Lemma 2.1 immediately gives the following result:

**Lemma 2.2** Let $C$ be a subset of $P$. Then:

(i) $C$ is a line-closed subset of $S$ if and only if for each point $x \in P \setminus C$ there is at most one line on the pencil through $x$ which meets $C$ in more than one point.

(ii) $C$ is a maximum line-closed subset of $S$ if and only if for each point $x \in P \setminus C$ there is a unique line on $x$ that meets $C$ in more than one point.

In the remainder of the paper, if $S$ is a linear space with maximum line-closed subset $C$ and $x \in P \setminus C$, let $\ell(x)$ denote the unique line on $x$ that intersects $C$ at $k - 1$ points.

Note that whenever we have a line-closed subset $C$ then $C$ can be given the structure of a linear space by defining the lines to be the intersection of the lines of $S$ with $C$. This structure is induced by that of $S$. We also note that $C$ is embedded in a regular linear space. Using these observations we can obtain some further restrictions by using the techniques in [5].

**Lemma 2.3** Let $d$ be the number of lines of size $k$ and $d'$ the number of lines of size $k - 1$ of a maximum line-closed subset $C$ of a linear space $S$ with line size $k$ and point regularity $r$. Then

$$d' = r(k - 2) - (k - 3)$$

and

$$k(k - 1)d = (r - 1)(r - k(k - 2)(k - 3)).$$

In particular $d$ and $d'$ depend only on the parameters of $S$ and not on $C$.

**Proof** For each point $x \in C$ let $r_i(x)$ denote the number of lines through $x$ which intersect $C$ in $k - i$ points.

$$(k - 1)r_0(x) + (k - 2)r_1(x) = r + k - 3$$ for all $x \in C$. 

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Further we get
\[ \sum_{x} r_0(x) = kd \text{ and } \sum_{x} r_1(x) = (k - 1)d'. \]

But if we recall that each point of \( P \) not in \( C \) lies on exactly one line which intersects \( C \) in \( k - 1 \) points, we get \( d' = v - r - k + 2 \). Using the formula for \( v \) in terms of \( r \) and \( k \) gives
\[ d' = r(k - 2) - (k - 3). \]

Combining this with the equations given above we get
\[ k(k - 1)d + (k - 2)(k - 1)(r(k - 2) - (k - 3)) = (r + k - 2)(r + k - 3). \]

Now some simple manipulation gives
\[ k(k - 1)d = (r - 1)(r - k(k - 2)(k - 3)). \]

The next corollary follows immediately from the proof of Lemma 2.3.

**Corollary 2.1** If \( S \) has a maximum line-closed subset \( C \) then
(i) \( (r - 1)(r - 2) \equiv 0 \ (\text{mod} \ (k - 1)) \),
(ii) \( r \geq k(k - 2)(k - 3) \) and
(iii) \( r(r - 1) \equiv 0 \ (\text{mod} \ k) \).

**Lemma 2.4** If \( C \) is a maximum line-closed subset of \( S \) then \( C \) has at least one line of length \( k - 1 \) and at least one line of length \( k \).

**Proof** We have already seen that all lines of \( C \) have length \( k \) or \( k - 1 \). Let \( x \) be any point in \( P \setminus C \). Then by Lemma 2.2, there is a line, \( l(x) \) that meets \( C \) in \( k - 1 \) points. Thus \( C \) contains at least one line of length \( k - 1 \). Suppose now that \( C \) contains no line of length \( k \). Then, by Lemma 2.3,
\[ (r - 1)(r - k(k - 2)(k - 3)) = 0. \]

Since \( r \neq 1 \) it follows that
\[ r = k(k - 2)(k - 3). \quad (1) \]

Hence \( r \equiv 0 \ (\text{mod} \ (k - 2)) \) and, since \( |C| = r + k - 2, |C| \equiv 0 \ (\text{mod} \ (k - 2)) \).

But, since all lines of \( C \) have length \( k - 1 \), \( C \) is a linear space with line regularity \( k - 1 \). Thus \( |C| - 1 \equiv 0 \ (\text{mod} \ (k - 2)) \), and so it follows that
\[ k = 3. \] But, from (1), if \( k = 3, r = 0 \) which is impossible. Thus \( C \) must contain at least one line of length \( k \).

Since a maximum line-closed subset must have a line of length \( k \), we note that maximum line-closed subsets are distinct from blocking sets as defined in [8].

**Lemma 2.5** Let \( S \) have a maximum line-closed subset \( C \). Then \( r > k(k - 2)(k - 3) \).

**Proof** By Corollary 2.1 \( r \geq k(k-2)(k-3) \). Assume that \( r = k(k-2)(k-3) \). Then, by Lemma 2.3, there are no lines of size \( k \) in \( C \). This contradicts Lemma 2.4.

**Lemma 2.6** Let \( S' = (P', L') \) be a subspace of \( S \) and \( C \) a maximum line-closed subset of \( S \). Then

(i) \( C \cap P' \) is a line-closed subset of \( S' \) or \( P' \subseteq C \),

(ii) If \( P' \not\subseteq C \) and for some point \( x \in P' \setminus C \) we have \( l(x) \in L' \) then, if \( S' \) is also regular, \( C \cap P' \) is a maximum line-closed subset of \( S' \).

**Proof** (i) Suppose that \( x, y \) and \( \hat{x}, \hat{y} \) are pairs of points in \( C \cap P' \) such that lines \( (x, y) \) and \( (\hat{x}, \hat{y}) \) intersect in \( P' \). As \( P' \subseteq P \), these lines intersect in \( P \) and so, since \( C \) is a line-closed subset of \( S \), they intersect in \( C \). Thus the two lines intersect in \( C \cap P' \) and so either \( P' \subseteq C \) or \( C \cap P' \) is a line-closed subset of \( S' \).

(ii) Suppose that \( P' \not\subseteq C \) and \( x \in P' \setminus C \) such that \( l(x) \in L' \). That is, \( l(x) \) meets \( C \cap P' \) in \( k - 1 \) points and, since \( S' \) is regular, \( S' \) has line regularity \( k \). Now in \( S \), every line on \( x \), except \( l(x) \), meets \( C \) in 1 point. Suppose that \( r'-1 \) of these lines meet \( C \cap P' \). It follows that \( |C \cap P'| = k + r' - 2 \) and that \( S' \) has point regularity \( r' \). Hence \( C \cap P' \) is a maximum line-closed subset of \( S' \).

### 3 Steiner Triple Systems

From Corollary 2.1 we see that a regular linear space can only have a maximum line-closed subset if \( r \geq k(k-1)(k-3) \). It is clear that there is no restriction in the case when \( k = 3 \). In addition it is easily shown that all values of \( r \) which satisfy the conditions for a Steiner triple system, namely that \( v \equiv 1 \) or \( 3 \) (mod 6), (see [9]), satisfy the remaining conditions required by Corollary 2.1 necessary for a maximum line-closed subset to exist. Let \( R \) denote the set of integers \( r \) for which there exists a Steiner triple system
with point regularity $r$. We shall consider the cases of $r \in R$ odd and $r \in R$ even separately.

### 3.1 The case when $r \in R$ is odd

We will first show how to construct Steiner triple systems with line-closed subsets of maximum size when $r \in R$ and $r$ is odd. Note that the maximum size in this case is $r + 1$. These conditions on $r$ imply that $(r-1)/2 \in R$. So there exist Steiner triple systems of size $r$. Choose two such systems $S_1$ and $S_2$ say, and one further point $\infty$. Let $\alpha$ be a bijection between the points of $S_1$ and the points of $S_2$. We now need to choose the lines of our $2 - (2r + 1, 3, 1)$ design, say $S$. We begin with all the lines of $S_1$ then we adjoin all the triples of the form \( \{x, \alpha(x), \infty\} \) where $x$ is a point of $S_1$. Finally let $x$ and $y$ be two points of $S_2$ and assume that $\{x, y, z\}$ is a line of $S_2$. We add the line $\{x, y, \alpha^{-1}(z)\}$ as a line of $S$. It is now straightforward to check that $S$ is a Steiner triple system and that the set $C$ consisting of the set of points of $S_1$ and $\{\infty\}$ is a line-closed subset of maximum size. Thus we have proved Theorem 3.1 below.

**Theorem 3.1** For all $r \in R$ with $r$ odd there exist Steiner triple systems with point regularity $r$ which contain a maximum line-closed set.

### 3.2 The case when $r \in R$ is even

This situation is less clear, but some considerable progress has been made towards constructing Steiner triple systems with line-closed subsets of maximum size where $r \equiv 0 \pmod{6}$. Consider the following problem:

**Problem 3.1** Let $r \equiv 0 \pmod{6}$. Arrange unordered pairs of distinct integers from the set $\{0, 1, \ldots, r-1\}$ into $r$ sets $L_0, L_1, \ldots, L_{r-1}$ in a way such that

1. Each set $L_i$, $0 \leq i \leq r-1$, contains the pair $(i, i+1) \pmod{r}$,
2. Each pair $(i, i+1)$ where $0 \leq i \leq r-1$ and $i \equiv 0$ or $1 \pmod{3}$ occurs once in set $L_i$ and once in precisely one other set $L_j$, $j \neq i$,
3. The pairs $(i, i+r/2) \pmod{r}$ where $i \equiv 1 \pmod{3}$ do not occur in any of the sets,
4. All other pairs occur exactly once and
5. Each set $L_i$, $0 \leq i \leq r-1$, contains exactly one pair containing the integer $k$, for all $k$ such that $0 \leq k \leq r-1$.

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We now assume that $r \equiv 0 \pmod{6}$ and that the above problem is solvable for $r$. We shall construct a Steiner triple system with line-closed subset of maximum size in this case.

Since $r + 1 \equiv 1 \pmod{6}$ there exists a Steiner triple system of size $r + 1$. Choose one such system, $S = (P, L)$, say, and label the points of $S$ with the numbers $(0, 1, \ldots, r)$ in such a way that the lines on 0 are bof the form $(0, 2j - 1, 2j)$, $1 \leq j \leq r/2$. Take also the additional set of points $A = \{a_0, a_1, \ldots, a_{r-1}\}$. Now take the first $r/3$ of the lines on 0, namely $(0, 2j - 1, 2j)$, $1 \leq j \leq r/3$ and replace each with the 3 new lines $(0, 2j - 1, a_{3j-3}), (2j - 1, 2j, a_{3j-2})$ and $(0, 2j, a_{3j-1})$. Let us call these **type 1** lines. Leave all other lines within $S$, and call these **type 0** lines. In addition form the lines $(0, a_i, a_{i+r/2})$, where $i \equiv 1 \pmod{3}$. Call these **type 2** lines.

Now we use the sets of pairs $L_i, 0 \leq i \leq r - 1$ obtained from the solution of problem 3.1 to form the remaining lines. There are two types of lines remaining. Each line of type 3 involves points from $S \setminus \{0\}$ already on type 1 lines together with points of $A$, and each line of type 4 contains the remaining points of $S \setminus \{0\}$, together with points of $A$.

**Type 3** lines. All points in $S \setminus \{0\}$ that appear on type 1 lines already occur on 2 type 1 lines and $r/2 - 1$ lines within $S$. The remaining $r/2 - 1$ lines on these points are formed by combining these points with appropriate pairs from $A$. Thus type 3 lines have the form $(2s + m, a_{i_2s+m}, a_{j_2s+m})$ where $(i_2s+m, j_2s+m) \in L_{3s+m-1}$, $(i_2s+m, j_2s+m) \neq (3s+m-1, 3s+m)$, and $(i, j) \neq (v, u)$ if $i \neq v, \forall 0 \leq s \leq r/3 - 1, \forall 1 \leq m \leq 2$.

**Type 4** lines. All points in $S \setminus \{0\}$ that do not appear on a type 1 line, so far only lie on $r/2$ lines (within $S$) and do not lie on any lines with any points of $A$. The remaining $r/2$ lines on these points are thus formed by combining each of these points with all the pairs within one of the remaining sets of pairs, $L_j, j \equiv 2 \pmod{3}$. Thus type 4 lines have the form $((2r/3 + 1) + j, a_{i_1}, a_{i_2})$ where $(a_{i_1}, a_{i_2}) \in L_{3j+2}, 0 \leq j \leq r/3 - 1$.

It can be shown that the points of $P \cup A$ together with all lines of types 0, 1, 2, 3 and 4 form a Steiner triple system with maximum line-closed subset $P$. Thus we have proved Theorem 3.2 below

**Theorem 3.2** For all $r$ for which $r \equiv 0 \pmod{6}$ and for which problem 3.1 is solvable, there exists a Steiner triple system with point regularity $r$ which contains a maximum line-closed subset.

It is not known for which values of $r$ Problem 3.1 is solvable. However, the authors have solved Problem 3.1 for all $r \equiv 0 \pmod{6}$, $r \leq 36$, the results of which are included in the appendix of this paper. The authors
are optimistic that Problem 3.1 is solvable for all \( r \equiv 0 \pmod{6} \) and are currently investigating a generalisation for all such \( r \).

4 Projective and Affine Spaces

We first prove some lemmas that are necessary for the examination of projective and affine spaces.

**Theorem 4.1** Let \( S \) be the linear space obtained from the points and lines of a projective space \( PG(n, q) \), \( n \geq 2 \). Then \( S \) contains a maximum line-closed subset if and only if \( q = 2 \).

**Proof** From the example given in the introduction \( S \) contains a maximum line-closed subset if \( q = 2 \). Suppose then that \( q > 2 \), that is \( k \geq 4 \). If the dimension is 2 we see easily that \( r = k > k(k - 2)(k - 3) \) is impossible if \( k > 3 \) (see Lemma 2.5). When the dimension is greater than 2 it is natural to use the axioms of Veblen and Young for a projective incidence space (see for example [3], p.572), as follows:

(a) any two points are on exactly one line;
(b) if \( v, x, y, z \) are points such that the line through \( v \) and \( x \) intersects the line through \( y \) and \( z \) then the line through \( v \) and \( y \) intersects the line through \( x \) and \( z \);
(c) every line has at least three points and
(d) there are two disjoint lines.

Now if \( C \) were to be a maximum line-closed subset it would have to be a proper subspace or degenerate in some sense. By Lemma 2.4, since \( k > 3 \), \( C \) has lines of two different sizes both greater than two. Suppose that there are two disjoint lines. Then \( C \) satisfies the axioms above and is therefore isomorphic to a projective space \( PG(m, p) \) where \( m > 2 \) and \( p \) is prime (see for example [3], p.573). This is a contradiction.

Assume then that any two lines of \( C \) intersect. Let \( l_1 \) and \( l_2 \) be two lines of \( C \) of different sizes. Let \( x \) be a point on neither line, such a point exists since \( |C| > 2k - 1 \). But every line through \( x \) goes through both \( l_1 \) and \( l_2 \) and so \( |l_1| = |l_2| \) which is again a contradiction. Hence a projective space of order \( q \) contains a maximum line-closed subset if and only \( q = 2 \).

We now consider the case of Affine geometries. However before commencing on this we state a simple counting lemma.

**Lemma 4.1** Let \( S \) be a linear space with line size \( k \) and \( k - 1 \). Assume the following:
(a) there is at least one line of size $k$,
(b) there are at least two lines.
Then $v \geq k(k - 2) + 1$.

**Proof** This is Proposition 3.2.1 in [2].

In Theorem 4.2 below we show that if $AG(n, q)$ contains a maximum line-closed subset then $n = 2$ and $q = 3$. We prove this via the following two lemmas:

**Lemma 4.2** If $C$ is a maximum line-closed subset of $AG(2, 3)$ then $C$ is the union of two intersecting lines.

**Proof** Let $S = AG(2, 3)$ and let $C$ be a maximum line-closed subset of $S$. Then $|C| = 5$ and, by Lemma 2.4, $C$ contains at least one line of length $3$, $l$ say. Let $x \in S \setminus C$. Then there is a line on $x$, namely $l(x)$, that intersects $C$ in 2 points. Suppose that $l$ and $l(x)$ do not intersect. Then all points of $C$ lie on $l$ or $l(x)$ and there are 6 lines from $l$ to $l(x)$ which all meet $S \setminus C$ at different points. But $|S \setminus C| = 4$ and so we have a contradiction. Hence $l$ and $l(x)$ meet in $C$ and there is one point $p \in C$ such that $p$ does not lie on $l$ or $l(x)$. Now $p$ is on a line with all points of $l$ and all points of $l(x)$ and it follows by an argument similar to the above that $p$ is on a line $\hat{l}$ say, that meets $l$ and $l(x)$. Thus $l$ and $\hat{l}$ are two intersecting lines of length 3 in $C$ and, since all points of $C$ are adjacent to all other points of $C$, it follows that $C$ is the union of two intersecting lines.

**Lemma 4.3** Let $S$ be the linear space obtained from the points and lines of the affine space $AG(n, 3)$. Then $S$ contains a maximum line-closed subset if and only if $n = 2$.

**Proof** If $n = 2$ then clearly, by Lemma 4.2, $S$ contains a maximum line-closed subset. Let us assume then that $n > 2$. Suppose that $S$ has a maximum line-closed subset $C$. Then $|C| > 5$. Let $l = \{a, b, c\}$ be any line of $C$ of length 3 and let $x$ be any point in $S \setminus C$. Then $l$ and $x$ generate an affine plane of $S$, $S' = (P', L')$ say, such that $P' \not\subseteq C$. By Lemma 2.6, $P' \cap C$ is a line-closed subset of $S'$, and so $|P' \cap C| \leq 5$.

Now $x$ is on 4 lines in $S'$, each of which must intersect $C$, and so $|P' \cap C| \geq 4$. It follows that there is some line in $S'$ which intersects $P' \cap C$ at two points. Hence, by Lemma 2.6, $P' \cap C$ is a maximum line-closed subset of $S'$. By Lemma 4.2, $P' \cap C$ is therefore the union of two intersecting lines of length 3. Now since $P' \cap C$ is a maximum line-closed subset of $S'$ it is clear
that \( l(x) \setminus \{x\} \in S^I \cap C \). This assertion is true for each choice of \( x \) not in \( C \). So we see that every line of \( C \) of size 2 contains a point of \( l \).

This argument is valid for any line of \( C \) of size 3. Thus we can only have at most 2 lines of size 3. Since \( C \) has been shown to contain the union of two lines of length 3, it follows that \( C \) is the union of 2 lines of length 3, and thus \( |C| = 5 \), which is a contradiction. Hence \( S \) contains a maximum line-closed subset if and only if \( n = 2 \).

**Theorem 4.2** Let \( S \) be the linear space obtained from the points and lines of the affine space AG(\( n, q \)). Then \( S \) contains a maximum line-closed subset if and only if \( n = 2 \) and \( q = 3 \).

**Proof** If \( S \) is an affine plane then \( r = q + 1 \) and so \( r > q(q - 2)(q - 3) \) (see Lemma 2.5) implies that \( q = 3 \). By Lemma 4.2, \( S \) contains a maximum line-closed subset in this case. From now on we will assume that \( q > 3 \) and the dimension is greater than 2. Let \( C \) be a maximum line-closed subset of \( S \).

Let \( l \) be a line with \( q \) points in \( C \). Let \( x \) be a point of \( C \) not on \( l \). Consider the plane \( S^I = (P^I, L^I) \) containing \( l \) and \( x \). By Lemma 2.6, \( P^I \cap C \) is a line-closed subset of \( S^I \) or the whole of \( P^I \). Now \( S^I \cap C \) is a linear space with lines of size \( q \) and \( q - 1 \) and so, by Lemma 4.1, \( |P^I \cap C| > q(q - 2) + 1 \). However the maximum size of a line-closed subset of an affine plane is \( 2q - 1 \) and so we have a contradiction unless \( P^I \cap C = P^I \).

Thus we can deduce that in any plane containing a line of \( C \) of size \( q \), any line of \( C \) has size \( q \). Let \( m \) be a line of \( C \) of size \( q - 1 \). Again choosing a point \( x \) not on \( m \) we can consider the plane \( S'' = (P'' , L'') \) (of order \( q \)) containing \( m \) and \( x \). Clearly \( P'' \cap C \neq P'' \) and \( S'' \cap C \) contains no lines of size \( q \). Hence \( S'' \cap C \) is a regular linear space an has at least \( (q - 1)(q - 2) + 1 \) points. This is larger than \( 2q - 1 \) unless \( q = 4 \). But then we would have a projective plane of order 2 (namely \( S'' \cap C \)) embedded in an affine plane of order 4. However any such plane would have to be a Baer subplane of the projective space and does not lie in the affine subspace. Hence \( P'' \cap C = P'' \) which is a contradiction. Thus every line of \( C \) has size \( q \), which contradicts Lemma 2.4. Thus \( q = 3 \) and so, by Lemma 4.3, \( n = 2 \), which is a contradiction. Hence \( S \) contains a maximum line-closed subset if and only if \( n = 2 \) and \( q = 3 \).

This concludes the major results of this paper. Before giving the last result we would like to raise the question of the nature of the line-closed subsets of projective and affine spaces. It is clear that for projective spaces a point non-incident hyperplane is a line-closed subset not contained in any
other. However the question of what other line-closed subsets there can be is open to investigation.

In the statement of the next theorem the definition of dimension is taken from \[1\], page 9.

**Theorem 4.3** Suppose that \( S = (P, L) \) is a regular linear space such that \( v = q^3 + q^2 + q + 1 \), \( k = q + 1 \) and \( S \) has dimension greater than 2, then \( S \) is the linear space formed by the points and lines of the projective space \( PG(3, q) \).

**Proof** Let \( D \) be a subspace of \( S \) of dimension 2. Then \( D \) has at least as many points as a projective plane of order \( q \). Thus
\[
|D| \geq q^2 + q + 1. \tag{1}
\]
Since \( D \) has line regularity \( q + 1 \) \( D \) has point regularity \( \hat{r} \) say and
\[
|D| = \hat{r} q + 1. \tag{2}
\]
From (1) and (2) we see that \( \hat{r} \geq q + 1 \).

Now \( D \) is a line-closed subset of \( S \), but not a maximum line-closed subset, since all lines of \( D \) have the same size. Thus \( |D| < r + (q + 1) - 2 = q^2 + 2q \). Hence \( \hat{r}q + 1 < q^2 + 2q \) and it follows that \( \hat{r} < q + 2 \). Hence \( q + 1 \leq \hat{r} < q + 2 \), and so \( \hat{r} = q + 1 \). Thus \( |D| = q^2 + q + 1 \) and it follows that \( D \) is a projective plane of order \( q \) (see for example \[3\], p.28). Since every 2-dimensional space of \( S \) is a projective plane of order \( q \), \( S \) is a projective space of order \( q \). Hence \( S \) is the linear space formed by the points and lines of the projective space \( PG(3, q) \).

We note that amongst the 80 Steiner triple systems on 15 points, only one has dimension 3.

5 Appendix

In Section 3 we claimed to have solved Problem 3.1 for all \( r \equiv 0 \pmod{6} \), \( r \leq 36 \). Here we give the solutions, explicitly in the case \( r = 6 \), and recursively in the larger cases. Throughout we use the notation \( L_i + a \) to denote the set of pairs of the form \(((x + a) \pmod{r}, (y + a) \pmod{r})\) for \((x, y) \in L_i\).

\( r=6 \)
\( L_0 = \{(0,1), (3,4), (2,5)\}, L_1 = \{(1,2), (3,5), (4,0)\}, L_2 = \{(2,3), (0,1), (4,5)\}, \)
\[ L_3 = \{(3,4),(0,2),(5,1)\}, \ L_4 = \{(4,5),(1,2),(0,3)\}, \ L_5 = \{(5,0),(1,3),(2,4)\}. \]

\[ r = 12 \]
\[ L_0 = \{(0,1),(3,8),(4,9),(5,10),(2,6),(7,11)\}, \]
\[ L_1 = \{(1,2),(0,3),(4,7),(5,8),(6,10),(9,11)\}, \]
\[ L_j = L_{j-3} + 3, \text{ for } j \equiv 0 \text{ or } 1 \pmod{3}, 3 \leq j \leq 10, \]
\[ L_2 = \{(2,3),(0,6),(7,9),(8,10),(11,1),(4,5)\}, \]
\[ L_5 = \{(5,6),(0,1),(7,8),(10,11),(2,4),(3,9)\}, \]
\[ L_8 = \{(8,9),(1,2),(3,4),(6,7),(5,11),(10,0)\}, \]
\[ L_{11} = \{(11,0),(1,3),(4,6),(5,7),(2,8),(9,10)\}. \]

\[ r = 18 \]
\[ L_0 = \{(0,1),(2,10),(3,11),(4,12),(7,13),(8,14),(9,15),(16,5),(17,6)\}, \]
\[ L_1 = \{(1,2),(0,5),(16,3),(17,4),(7,9),(8,10),(12,14),(6,13),(11,15)\}, \]
\[ L_j = L_{j-3} + 3, \text{ for } j \equiv 0 \text{ or } 1 \pmod{3}, 3 \leq j \leq 16, \]
\[ L_2 = \{(2,3),(0,4),(1,5),(9,13),(10,14),(6,7),(16,17),(8,11),(12,15)\}, \]
\[ L_8 = \{(8,9),(1,2),(3,4),(6,7),(5,11),(10,0)\}, \]
\[ L_{11} = \{(11,0),(1,3),(4,6),(5,7),(2,8),(9,10)\}. \]

\[ r = 24 \]
\[ L_0 = \{(0,1),(2,13),(7,18),(9,20),(4,14),(11,21),(12,22),(6,15),(8,17),(10,19),(16,23),(3,5}\}, \]
\[ L_1 = \{(1,2),(0,8),(11,19),(7,15),(6,12),(22,4),(23,5),(9,16),(20,3),(10,14),(17,21),(13,18)\}, \]
\[ L_j = L_{j-3} + 3, \text{ for } j \equiv 0 \text{ or } 1 \pmod{3}, 3 \leq j \leq 22, \]
\[ L_2 = \{(2,3),(6,9),(7,10),(8,11),(18,21),(19,22),(20,23),(0,1),(4,5),(13,15),(14,16),(12,17)\}, \]
\[ L_j = L_{j-3} + 3, \text{ for } j = 5,8,11, \]
\[ L_{14} = \{(14,15),(0,12),(5,17),(9,21),(3,7),(6,10),(18,22),(1,2),(19,20),(8,13),(23,4),(11,16)\}, \]
\[ L_{17} = \{(17,18),(0,4),(21,1),(9,13),(12,16),(2,7),(5,10),(6,11),(14,19),(3,15),(8,20),(22,23)\}, \]
\[ L_{20} = \{(20,21),(0,5),(17,22),(1,3),(4,6),(8,10),(7,9),(2,14),(11,23),(12,13),(15,16),(18,19)\}, \]
\[ L_{23} = \{(23,0),(2,4),(5,7),(10,12),(11,13),(3,8),(9,14),(6,18),(15,19)\}, \]

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$(20, 1), (16, 17), (21, 22)$.

$r=30$

$L_0 = \{(0, 1), (7, 19), (8, 20), (12, 24), (4, 18), (9, 23), (2, 26), (6, 17), (10, 21), (11, 22), (13, 26), (14, 27), (15, 28), (25, 3), (29, 5)\}.$

$L_1 = \{(1, 2), (0, 8), (5, 13), (3, 9), (4, 10), (12, 19), (7, 14), (20, 27), (6, 16), (11, 21), (15, 24), (17, 26), (23, 28), (25, 29), (18, 22)\},$

$L_j = L_{j-3} + 3$, for $j \equiv 0 \pmod{3}$, $3 \leq j \leq 28$,

$L_2 = \{(2, 3), (1, 11), (16, 26), (10, 19), (25, 4), (18, 23), (17, 21), (12, 15), (27, 0), (6, 8), (5, 7), (20, 22), (13, 14), (28, 29), (9, 24)\},$

$L_{17} = \{(17, 18), (4, 7), (11, 14), (16, 19), (20, 23), (25, 28), (1, 6), (3, 8), (10, 15), (0, 2), (27, 29), (24, 26), (12, 13), (21, 22), (5, 9)\},$

$L_{20} = \{(20, 21), (0, 1), (3, 4), (6, 7), (9, 10), (15, 16), (27, 28), (2, 5), (8, 11), (22, 25), (23, 26), (12, 17), (13, 18), (19, 24), (14, 29)\},$

$L_{23} = \{(23, 24), (25, 0), (22, 27), (4, 9), (6, 11), (15, 20), (8, 12), (14, 18), (28, 1), (7, 10), (13, 16), (26, 29), (3, 5), (19, 21), (2, 17)\},$

$L_{26} = \{(26, 27), (7, 12), (9, 14), (16, 21), (11, 15), (5, 8), (10, 13), (17, 20), (19, 22), (29, 2), (28, 0), (1, 3), (4, 6), (16, 18), (21, 23), (24, 25)\},$

$L_{29} = \{(29, 0), (28, 3), (16, 21), (2, 6), (1, 4), (14, 17), (7, 9), (10, 12), (13, 15), (22, 24), (25, 27), (18, 19), (5, 20), (8, 23), (11, 26)\}.$

$r=36$

$L_0 = \{(0, 1), (2, 19), (3, 20), (7, 24), (10, 25), (11, 26), (12, 27), (4, 16), (5, 17), (6, 18), (13, 21), (14, 22), (15, 23), (28, 33), (29, 34), (30, 35), (31, 8), (32, 9)\}.$

$L_1 = \{(1, 2), (3, 19), (4, 20), (5, 21), (28, 6), (29, 7), (30, 8), (12, 22), (13, 23), (14, 24), (26, 32), (27, 33), (34, 9), (35, 10), (0, 11), (16, 25), (18, 31), (15, 17)\},$

$L_j = L_{j-3} + 3$, for $j \equiv 0 \pmod{3}$, $3 \leq j \leq 34$,

$L_2 = \{(2, 3), (5, 14), (6, 15), (23, 32), (24, 33), (7, 10), (8, 11), (9, 12), (25, 28), (26, 29), (27, 30), (13, 19), (31, 1), (0, 4), (18, 22), (16, 20), (17, 21), (34, 35)\},$

$L_j = L_{j-3} + 3$, for $j = 5, 8, 11, 14, 17$,

$L_{20} = \{(20, 21), (4, 11), (5, 12), (6, 13), (22, 29), (23, 30), (24, 31), (7, 9), (8, 10), (21, 22)\}. $
\[(25, 27), (26, 28), (14, 32), (15, 33), (16, 17), (18, 19), (0, 1), (34, 2), (35, 3)\],

\[L_j = L_{j-3} + 3, \text{ for } j = 23, 26, 29, 32, 35.\]

References


University of East Anglia
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
UK