Triangle-free Linear Spaces on 13 points with line sizes 2, 3 and 4

Alice Miller

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Abstract

In this document we consider linear spaces with lines of size 2, 3 and 4 for which there is no triangular set of pairs of points \( \{(a,b), (a,c), (b,c)\} \) in the lines of size 2 and 3 which are not all contained in a single line of size 3. We call these structures triangle-free linear spaces with lines of size 2, 3 and 4 (\( \{2,3,4\}\)-TFLSs). We show that the only \( \{2,3,4\}\)-TFLSs on 13 points are the 13 - 4 - 1 design and a unique structure consisting of 10 lines of size 4 and 4 of size 3 (and six lines of size 2). We show how, in the latter case, the lines can be manipulated to produce a 13 - 4 - 1 design.

The structures we investigate in this document arise during an attempt to construct a (13 - 4 - 1) design using a hill-walking algorithm based on Stinson’s algorithm [3]. The method (implemented by Richard Moore [4]) involves selecting pairs of triples from a set of triples \( T \) which overlap in two points. Initially the set \( T \) consists of all ordered triples from the set of points \( V = 1, 2, \ldots, 13 \). When a triple \( t \) is removed from \( T \) to make a block, all blocks intersecting \( t \) in 2 points are also removed from \( T \). The set \( T \) is the set of weight zero triples (see below). We investigate the structure obtained when the basic algorithm fails. That is when there are no pairs of weight zero triples which intersect in two points remaining, but less than 13 blocks have been constructed. This structure is a 2, 3, 4-TFLS. Note that we do not give details of how the hill-walking algorithm uses triples to make blocks here, for details see [4]).

1
1 Preliminaries

Definition 1 Let $V$ be a set of points and $K$ a set of distinct positive integers greater than 1 and at most $|V|$. A $K$ linear space ($K$-LS) on the set of points $V$ is a set of subsets (lines) from $V$ with sizes from $K$ in which every pair from $V$ appears exactly once.

Definition 2 A $\{2, 3, 4\}$-LS is said to be triangle free if there is no triangular set of pairs $(a,b)$, $(a,c)$ and $(b,c)$ in the lines of size 2 and 3 which are not all contained in a single line of size 3. We call these structures triangle-free linear spaces with lines of size 2, 3 and 4 ($\{2, 3, 4\}$-TFLSs). For historical reasons (see above) we refer to the lines of size 4 as blocks and those of size 3 as weight zero triples, or $w$-triples. The lines of size 2 are called $w$-pairs.

Definition 3 A $(v, k, \lambda)$ block design is a set $V$ and a set $B$ of subsets of $V$ such that $|V| = v$, every subset (block) in $B$ has size $k$, and every pair of points from $V$ appears in exactly $\lambda$ blocks.

Theorem 1 If $N$ is a $\{2, 3, 4\}$-TFLS then it is either the unique $(13, 4, 1)$ design, or is a unique structure containing 8 blocks, 4 non-intersecting $w$-triples and 6 non-intersecting $w$-pairs. In the latter case the blocks, $w$-triples and $w$-pairs can be manipulated to form a $(13 - 4 - 1)$ design.

We prove Theorem 1 in Section 3. In Section 2 below we give some more definitions and a construction for the case where there are less than 13 blocks.

2 A Construction

We derive a construction for a $\{2, 3, 4\}$-TFLS on 13 points when there are less than 13 blocks. But first, a definition:

Definition 4 If $N$ is a $\{2, 3, 4\}$-TFLS with point set $V$, $S \subseteq V$ and $x, y \in V \setminus S$, we say that $x$ and $y$ cover $S$ if there is no point in $S$ that is not in a block with either $x$ or $y$.

We give two preliminary lemmas:

Lemma 1 If $N$ is a $\{2, 3, 4\}$-TFLS on 13 points and there are no $w$-pairs, then $N$ is a $(13 - 4 - 1)$ design.
Proof If there are no \( w \)-pairs \( N \) is a proper linear space (PLS). There are only 3 PLSs with line size at most 4, namely the \((13 - 4 - 1)\) design and PLS(13a) and PLS(13b) given in Tables 1 and 2 (from [1]). Note that the blocks and triples should be read vertically.

Both PLS(13a) and PLS(13b) contain a triangular set of pairs that do not belong to a single \( w \)-triple (namely \( \{(b, f), (b, m), (f, m)\} \) and \( \{(a, h), (a, m), (h, m)\} \) respectively). Thus neither PSL(13a) or PSL(13b) are triangle free. It follows that \( N \) must be the \((13 - 4 - 1)\) design.

Lemma 2 If \( N \) is a \((2, 3, 4)\)-TFLS on 13 points and \( a \) a point that belongs to at least one \( w \)-triple, then

1. \( a \) is in 3 blocks and

2. no point is in more than 1 \( w \)-triple

Proof (1) Suppose that \((a, b, c)\) is a \( w \)-triple. If \( a \) is in 0 blocks then there are 10 points other than \( b \) and \( c \) that are not in a block with \( a \). Now \( b \) must be in a block with each of these points (otherwise there is a triangular set containing the pair \((a, b)\)) which is impossible, as \( b \) is in at most 3 blocks.

If \( a \) is in 1 block, then, since \( b \) must be in a block with all the points of \( V \setminus \{c\} \) not in a block with \( a \), \( b \) is in 3 blocks. Similarly \( c \) is in 3 blocks. If \( P = \{m, n, p\} \) is the set of points in a block with \( a \), the blocks on \( b \) miss one point from \( P \) (\( m \) say) and the blocks on \( c \) miss one other point (\( n \) say). Any
other block must contain one point from each block on \( b \) plus one other - so must contain \( m \). Similarly it must contain one point from each block on \( c \) plus one other - so must contain \( n \). This is not possible (as \( m \) and \( n \) already appear together in a block). So the only blocks are the blocks on \( a, b \) and \( c \).

Not all pairs from the set \( V' = V \setminus \{a, b, c, m, n, p\} \) can appear in a block together (there are 21 pairs and room for only 10 in the blocks). Now suppose that \( A \) and \( B \) are points in \( V' \) which do not appear in a block together. The blocks they are in must cover at least all but one of the points from \( V \setminus \{a, b, c\} \) (otherwise there are 2 \( u \)-triples containing \( A \) and \( B \)). That is, their blocks must cover 8 points. There are only 4 blocks containing \( A \) and \( B \) each of which contain \( b \) or \( c \), so there are only 8 spaces. It follows that the blocks on \( A \) and \( B \) must contain distinct points. Now consider the 4 points that appear in blocks on \( b \) which contain \( A \) or \( b \). At least 3 of these must appear in a block on \( c \) (only \( n \) appears in a block with \( b \) but not \( c \)). These three points must appear in the block on \( c \) that does not contain \( A \) or \( B \). But at least 2 of them have already appeared in the same block in the blocks on \( b \) so we have a contradiction.

Suppose that \( a \) is in 2 blocks, \( B_1 = (a, u_1, u_2, u_3) \) and \( B_2 = (a, v_1, v_2, v_3) \) say. Then there are 4 points, \( E = \{e_1, e_2, e_3, e_4\} \) that are in \( u \)-pairs with \( a \). Let \( D' \) be the set of points (not \( a \)) on \( B_1 \) or \( B_2 \). Points not in a block with \( b \) must belong to \( D' \), the same is true of \( c \). If \( H_5 \) and \( H_6 \) are the sets of points that are not in a block with \( b \) and \( c \) respectively, then \( H_5 \cap H_6 = \emptyset \) and \( H_5 \cup H_6 \subseteq D' \).

Now since both \( b \) and \( c \) are on at most 3 blocks, each are in a block with every element of \( E \) and \( H_5 \cap H_6 = \emptyset \) it follows that at least one of them is in 3 blocks (and both are in at least 2).

Now consider the pairs from \( E \). If there are blocks \( B \) and \( B' \) on \( b \) and \( c \) respectively, containing non-intersecting pairs from \( E \), then both \( b \) and \( c \) are in 3 blocks (since they both must be in 2 further blocks containing a single point from \( E \)). Hence if one of \( b \) and \( c \), \( b \) say, is in 2 blocks, we can assume the blocks on \( b \) and \( c \) are \( (b, e_1, e_2, *) \), \( (b, e_3, e_4, *) \), \( (c, e_1, e_3, *) \), \( (c, e_2, e_4, *) \) and \( (c, e_4, *, *) \), where \( * \in D' \).

Now the remaining pairs on \( E \) are \( (e_1, e_4), (e_2, e_3) \) and \( (e_2, e_4) \). If any of these pairs lie in blocks, they do so with 2 elements of \( D' \). If \( (e_1, e_4) \) is in such a block, with \( P \) and \( Q \) from \( D' \) say, then neither \( P \) or \( Q \) can be in a block with \( b \) (as the blocks on \( b \) each contain one of \( e_1 \) or \( e_4 \)). Since \( b \) and \( c \) cover \( D' \), \( P \) and \( Q \) must be in a block on \( c \). But there is only one such block that they can appear in (for similar reasons to the above). Hence they must
appear together in 2 blocks, which is a contradiction. Similarly neither of the other two pairs from $E$ belong to any block. But then there is at least one pair of $w$-triples that intersect in 2 points, namely $(a,e_1,e_4)$ and $(a,e_2,e_4)$.

Hence $b$ and $c$ are both in 3 blocks. Each of $b$ and $c$ are in one block containing a pair from $E$, and 2 blocks containing a single point from $E$. Suppose the pairs from $E$ contained in these blocks overlap. We can assume they are $(e_1,e_2)$ and $(e_1,e_3)$. Then there are 3 pairs from $E$ remaining that contain $e_4$. At most 1 of these can belong to a block, otherwise $e_4$ is in blocks with more than 6 elements of $D'$ (with 2 each in blocks on $b$ and $c$ and 2 in every additional block containing $e_4$). So there are at least 2 unused pairs containing $e_4$ resulting in 2 weight zero triples containing $a$ and $e_4$.

So the pairs from $E$ contained in the blocks on $b$ do not overlap. We can assume the blocks on $b$ and $c$ are $(b,e_1,e_2,\ast)$, $(b,e_3,\ast,\ast)$, $(b,e_4,\ast,\ast)$, $(c,e_3,e_4,\ast)$, $(c,e_1,\ast,\ast)$ and $(c,e_2,\ast,\ast)$, where $\ast \in D'$.

Now every element of $E$ can be in at most one additional block (as otherwise they will be in blocks with more than 6 elements of $D'$). It follows that two non-overlapping pairs from the remaining pairs from $E$ do not appear in blocks, $(e_1,e_2)$ and $(e_3,e_4)$ say.

Now consider the pair of points from $D'$ that are in the additional block containing pair $(e_1,e_4)$, $P$ and $Q$ say. The only block on $b$ that can contain either of these points is the middle block on $b$ (as the other blocks contain $e_1$ and $e_4$ respectively), and it can contain only one of $P$ and $Q$, $P$ say. Similarly there is an element from the additional block containing pair $(e_2,e_3)$ that does not appear in a block with $b$. Since $b$ is in a block with all but one of $D'$ it follows that the blocks containing pairs $(e_1,e_4)$ and $(e_2,e_3)$ contain a common point, $Q$ say. But then $Q$ can not appear in any of the blocks on $b$ and $c$, which is a contradiction, as $b$ and $c$ cover $D'$.

Hence $a$ is in at least 3 blocks. Since $a$ is in a $w$-triple, $a$ is in at most 3 blocks. Hence $a$ is in exactly 3 blocks.

(2) Since every point $a$ appearing in a $w$-triple must be in 3 blocks, there are only 3 points that are not in a block with $a$. So $a$ is in exactly one $w$-triple, and one $w$-pair.

**Lemma 3** If there is at least one $w$-triple then there are at least 3 $w$-triples.
Proof Suppose that there is a $w$-triple $t = (a, b, c)$. By Lemma 2, $a$, $b$ and $c$ are each in 3 blocks. Let the blocks on $a$ be 

\[
\begin{pmatrix}
a \\
X \\
Y \\
Z \end{pmatrix}
, \quad 
\begin{pmatrix}
a \\
P \\
Q \\
R \end{pmatrix}
, \quad 
\begin{pmatrix}
a \\
S \\
T \\
V \end{pmatrix}
\]

and let $e_1$ be the point not in a block or $w$-triple with $a$, and $D'$ the set of points (not $a$) in a block with $a$. Now $e_1$ must be in a block with both $b$ and $c$ (but not at the same time), and $b$ and $c$ must cover $D'$.

Now, since $e_1$ is not in a block with $a$, it is in at most 3 blocks, one with $b$ and one with $c$, and possibly one other. If we call a block a type i block if it contains $i$ points from $D'$. Both $b$ and $c$ are on 1 type 2 block and 2 type 3 blocks. There are no type i blocks, for $i \geq 3$, for otherwise it would contain at least 2 elements from the same block on $a$. So there are either 9 blocks (when $e_1$ is not in an additional block) or 10 blocks in total.

Let $T$ be the set of triples, one from each block on $a$. At least 9 triples from $T$ must be used in the $w$-triples, the type 1 blocks, or broken up for the blocks on $b$ and $c$ containing 2 elements of $D'$. Now every type 3 block involves an element from $T$, and the type 2 blocks contain pairs from triples which are broken up to provide the pairs (call these broken triples). Now if there is only one broken triple, say, both pairs in the type 2 blocks come from $t$ and must intersect at some point, $X$, say. But then $X$ is in two blocks with $e_1$, which is impossible. Hence there are 2 broken triples and at most 7 of the triples available have been used. But then there are at least 2 further triples available. Hence there are at least 3 $w$-triples.

By Lemmas 2 and 3 there are 3 cases to consider, when there are 0, 3 (distinct) and 4 (distinct) $w$-triples. We consider these separately in Sections 2.1, 2.2 and 2.3.

2.1 No $w$-triples

Let $r_{\min}$ denote the minimum number of blocks on a point. Clearly $r_{\min} < 4$ (otherwise we would have built a block design already). We consider the cases $r_{\min} = 0, 1, 2, 3$ separately.
2.1.1 \( r_{\text{min}} = 0 \)

Suppose that there is a point \( p \) which is on no block and let \( E = V \setminus \{p\} \) be \( \{e_1, e_2, \ldots, e_{12}\} \). Then every pair from \( E \) must be in a block somewhere (otherwise, if \((e_i, e_j)\) is not in a block, \((p, e_i, e_j)\) would be a \( w \)-triple, contradiction). In this case we have a complete set of blocks on 12 points, i.e. a \((13 - 4 - 1)\) design. But, since \( 15 \not\equiv 1 \) or \( 4 \pmod{12} \), this is impossible.

2.1.2 \( r_{\text{min}} = 1 \)

Let \( p \) be a point in 1 block only and assume that this block also contains points \( X, Y \) and \( Z \). Let \( W = \{X, Y, Z\} \). Each of the remaining points, \( E = \{e_1, e_2, \ldots, e_9\} \) say, must be in some block with each other point in \( E \). These blocks can contain at most one point of \( X \) and so their restriction to \( E \) forms a proper linear space of order 9. However, there is no proper linear space of order 9 apart from \( STS(9) \) [1]. In this case there are 12 type 1 blocks. However, since each of these blocks contain a point of \( W \), each of which can be in at most 3 type 1 blocks, this is impossible.

2.1.3 \( r_{\text{min}} = 2 \)

Let \( p \) be a point in 2 blocks only and assume that these blocks are:

\[
\begin{pmatrix}
  p \\
  X \\
  Y \\
  Z
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  p \\
  Q \\
  R \\
  S
\end{pmatrix}
\]

The remaining points, \( E = \{e_1, e_2, \ldots, e_9\} \) say, must be in some block with each other point in \( E \). There are 21 pairs from \( E \). The remaining blocks contain either 0, 1 or 2 points from \( W = \{X, Y, Z, Q, R, S\} \). Let a type \( i \) block be a block containing \( i \) points from \( W \), for \( i = 0, 1, 2 \) and \( A_i \) the number of blocks of type \( i \). Then

\[
6A_0 + 3A_1 + A_2 = 21
\]  \hspace{1cm} (1)

Since \( |E| = 6 \), either \( A_0 = 0 \) and \( A_1 \leq 3 \) or \( A_0 = 1 \) and \( A_1 \leq 1 \). In addition, since any type 2 block must contain one point from each of the blocks on \( p \), \( A_2 \leq 9 \). It follows that there is no solution to Equation 1.
2.1.4  \( r_{\text{min}} = 3 \)

Let \( p \) be a point in 3 blocks only and assume that these blocks are:

\[
\begin{pmatrix}
p \\
X \\
Y \\
Z \\
\end{pmatrix}, \quad \begin{pmatrix}
p \\
Q \\
R \\
S \\
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
p \\
T \\
U \\
V \\
\end{pmatrix}.
\]

There are two remaining points, \( e_1 \) and \( e_2 \) say, which must be in some block together. Using notation similar to the above, they must appear in a type 2 block together, and all other blocks are type 3 blocks, containing either \( e_1 \) or \( e_2 \). Now suppose, w.l.o.g. that \( X \) is in the type 2 block. Then \( X \) can not appear in any type 3 block (as it has already appeared with \( e_1 \) and \( e_2 \)), so is in only 2 blocks in total. This contradicts the fact that \( r_{\text{min}} = 3 \).

2.2  3 \( w \)-triples

By Lemma 2, none of the \( w \) triples intersect.

Suppose that \( t_1 = (a, b, c) \) is a \( w \)-triple. Using an argument similar to that used to prove Lemma 3, there are 10 blocks, 3 each containing \( a, b \) and \( c \), and an additional block containing \( e_1 \), the point not in a \( w \)-triple or block with \( a \). The \( w \)-triples \( t_2 \) and \( t_3 \) consist of a point from each block on \( a \).

Let \( P = \{m, n, p, q\} \) be the set of 4 points that are not in the \( w \)-triples. These points need to fill 13 places in the blocks (as the other points all fill 3 places each). So at least one of these points, \( m \) say, is in 4 blocks. Note that \( m \) can not be \( e_1 \). The other elements of \( P \) must be in blocks with at least 2 points from each of the \( w \)-triples, as well as \( m \). So they must be in 3 blocks each. A pair from \( P \setminus \{m\} \) must belong to the blocks, otherwise \( (n, p, q) \) is a \( w \)-triple. So there are at least 4 pairs from \( P \) that must appear in the blocks.

Every block must contain at most one point from each \( w \)-triple, so must contain at least one point from \( P \). Suppose that 3 points of \( P \) appear in a block together, with \( a \) say. Then only one other block on \( a \) can contain a point from \( P \), which is a contradiction. Similarly two blocks on \( a \) can not contain pairs from \( P \). Since there are 4 pairs from \( P \) contained in the blocks it also follows that the blocks on \( a \) (similarly \( b \) and \( c \)) can not each contain a single element from \( P \). Hence the blocks on \( a \) contain a pair and two singles from \( P \) respectively (similarly for blocks on \( b \) and \( c \)), and \( e_1 \notin P \).
The final block, containing $e_1$ must contain a pair from $P$ (as there are at least 4 such pairs). Thus $e_1$ is in blocks with at least 5 points from $P$ (as the blocks on $a$, $b$ and $c$ containing $e_1$ each contain at least one point from $P$), which is a contradiction.

2.3 $4$ w-triples

Again there are 4 orthogonal w-triples and one point, $m$ say, that is in none of the w-triples.

Let $(a, b, c)$ be one of the w-triples. Since 12 points are in 3 blocks, either there is one point that is in 0 blocks, or there are 10 blocks and a point is in 4 blocks. In the first case, a point in 0 blocks would be in a w-triple with $a$ and $b$ say, which is a contradiction.

Hence there are 10 blocks, and one point is in 4 blocks.

This point is in none of the w-triples (as otherwise it would be in 3 blocks only) and so is $m$. So the point that is not in a block with $a$ belongs to the w-triples.

If we add a new point, $oo_1$ say, to the w-triples, we have a set of 14 blocks on 14 points, and every point is in a block with all but 1 points. So we have a $4-GDD$ of type $2^7$ [2]. Indeed, if we add a further point, $oo_2$ say, to each of the $w$-pairs, we have a $PBD$ with 15 points, 14 blocks of size 4 and 7 of size 3, where all the blocks of size 3 contain a common point. From [1] we see there is only 1 suitable $PBD$, namely that given in Table 3. Clearly $o$ is $oo_1$. W.l.o.g we can assume that $a$ is $oo_1$. Removing $o$ and $a$, we have the structure given in Table 4 with $w$-pairs: $(b, m), (c, i), (d, g), (e, k), (f, j)$ and $(h, l)$.

Note that $n$ is in no $w$-pair, we refer to $n$ as the pivot.

Theorem 2 If $N$ is a $\{2, 3, 4\}$-TFLS on 13 points and $N$ is not a $(13-4-1)$-design then:
Table 4: Structure 1

1. There are 4 non-intersecting w-triples and the set of blocks and w-triples are as for Structure 1. There is a single point, n that does not appear in the w-triples called the pivot.

2. For any w-triple t there is a unique block B containing n that does not contain any element of t, and any block that does not contain n intersects with B.

3. For each of the w-triples t, for 1 ≤ i ≤ 4, let C_i be the unique block which does not intersect with t_i. For 1 ≤ i ≤ 3, if we choose p_i ∈ C_i in such a way that each of the p_i comes from a different w-triple (from {t_1, t_2, t_3}) and each of the pairs (p_i, p_j), for i ≠ j, belongs to a different block, then by removing the blocks B_1, B_2 and B_3 containing pairs (p_1, p_2), (p_1, p_3) and (p_2, p_3) we can add new blocks (t_i, p_i), for 1 ≤ i ≤ 3, where (t_i, p_i) is the block formed by combining w-triple t_i with point p_i.

4. Let t_4 be the remaining w-triple. Then the removed blocks B_1, B_2 and B_3 are:

\[
\begin{pmatrix}
  p_1 \\
  p_2 \\
  u_1 \\
  v_1
\end{pmatrix},
\begin{pmatrix}
  p_1 \\
  p_3 \\
  u_2 \\
  v_2
\end{pmatrix},
\begin{pmatrix}
  p_2 \\
  p_3 \\
  u_3 \\
  v_3
\end{pmatrix},
\]

where, for 1 ≤ i ≤ 3, u_i ∈ t_4 and v_i ∈ C_4, 1 ≤ i ≤ 3. Let B = \{B_1, B_2, B_3\}. There are 3 pairs (u_i, v_i), where 1 ≤ i, e_i ≤ 3, which do not appear in the blocks. We can add 3 further blocks, namely B'_4, B'_5,
and $\mathcal{B}'_0$ defined as follows:

$$
B_4 = \begin{pmatrix}
  u_1 \\
  v_{u_1} \\
  u_{u_1} \\
  p_{r_1}
\end{pmatrix}, \\
B_5 = \begin{pmatrix}
  u_2 \\
  v_{u_2} \\
  u_{u_2} \\
  p_{r_2}
\end{pmatrix}, \\
B_6 = \begin{pmatrix}
  u_3 \\
  v_{u_3} \\
  u_{u_3} \\
  p_{r_3}
\end{pmatrix},
$$

where, for $1 \leq i \leq 3$, $u_{u_i}$ appears with $v_{u_i}$ in $\mathcal{B}$, and $p_{r_i}$ appears in $\mathcal{B}$ with $u_i$ and $u_{u_i}$ (and $v_{u_i}$), to make a $(13,4,1)$ design.

Proof (1) and (2) follow by examining the structure above.

(3) For $1 \leq i \leq 3$, as $p_i$ is in $C_i$ which does not contain any element of $t_i$, it is in only two blocks containing elements of $t_i$, which belong to $\mathcal{B}$. Since these blocks have been removed, no pair consisting of an element from $t_i$ and $p_i$ still belong in the blocks. Hence we can add block $(t_i,p_i)$.

(4) Since every block that does not contain $n$ consists of an element from each $w$-triple, clearly $u_i \in t_4$ for $1 \leq i \leq 3$. Similarly every block must contain an element from each of the $C_i$, $1 \leq i \leq 3$, hence $u_i \in C_i$.

Now each of the $v_i$ are in one further block. Since all remaining blocks must contain an element of $C_4$, all of the remaining blocks contain one of the $v_i$. Hence there are only 3 remaining blocks. Since each of the $u_i$ are also in one more block, it follows that the 3 remaining blocks each contain a unique pair $(u,v)$ where $u \in U = \{u_i : 1 \leq i \leq 3\}$ and $v \in V = \{v_i : 1 \leq i \leq 3\}$. Hence there are three remaining distinct pairs, $(u_1,v_{u_1})$, $(u_2,v_{u_2})$, and $(u_3,v_{u_3})$, which do not appear in the blocks. Since, for $1 \leq i \leq 3$, $u_i$ and $v_{u_i}$ are in different blocks from $\mathcal{B}$, there is some point from $\{p_1,p_2,p_3\}$ which is in both of the blocks from $\mathcal{B}$ containing $u_i$ and $v_{u_i}$. Call this point $p_{r_i}$. Now, for $1 \leq i \leq 3$, if we take $u_{u_i}$ to be the point from $U$ in $\mathcal{B}$ with $v_{u_i}$, then removing $\mathcal{B}$ will release pairs pairs $(u_i,p_{r_i})$, $(v_{u_i},p_{r_i})$, $(u_{u_i},p_{r_i})$ and $(v_{u_i},u_{u_i})$. Since pairs $(u_i,u_{u_i})$ and $(u_i,v_{u_i})$ are $w$-pairs, it is safe to add new blocks $B_4$, $B_5$ and $B_6$.

Since we have removed 3 blocks and added 6, we have a set of 13 blocks, and so we have a $(13,4,1)$ design, as required.

3 Proof of Theorem 1

If there are 13 blocks then every pair from $V$ appears in the blocks and we have a $(13,4,1)$ block design. Otherwise, if there are fewer than 13 blocks
then, by Theorem 2 we have the structure of Construction 1 which can be manipulated to form a $(13 - 4 - 1)$ design.

References


