Stable Matching with Couples
an Empirical Study

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Stable matching with couples – an empirical study

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Abstract

In practical applications, algorithms for the classical version of the Hospitals Residents problem (the many-one version of the Stable Marriage problem) may have to be extended to accommodate the needs of couples who wish to be allocated to (geographically) compatible places. Such an extension has been in operation in the NRMP matching scheme in the US for a number of years. In this setting, a stable matching need not exist, and it is an NP-complete problem to decide if one does. However, the only previous empirical study in this context (focused on the NRMP algorithm), together with information from NRMP, suggest that, in practice, stable matchings do exist and that an appropriate heuristic can be used to find such a matching.

The study presented here was motivated by the recent decision to accommodate couples in the Scottish Foundation Allocation Scheme (SFAS), the Scottish equivalent of the NRMP. Here, the problem is a special case, since hospital preferences are derived from a ‘master list’ of resident scores, but we show that the existence problem remains NP-complete in this case. We describe the algorithm used in SFAS, and contrast it with a version of the algorithm that forms the basis of the NRMP approach. We present an empirical study of the performance of a number of variants of these algorithms, and of a third simpler algorithm based on satisfying blocking pairs, using a range of data sets. The results indicate that, not surprisingly, increasing the ratio of couples to single applicants typically makes it harder to find a stable matching (and, by inference, less likely that a stable matching exists). However, the likelihood of the algorithm finding a stable matching is very high for realistic values of this ratio, and especially so for particular variants of the algorithms.

1 Introduction

Background

The Hospitals Residents problem (HR) is a well-known extension of the classical Stable Marriage problem, introduced (under the alternative name of the College Admissions problem) in the seminal paper of Gale and Shapley [2]. The terminology arises from the important application to matching schemes that assign applicants to positions in the medical domain. The best known of these schemes is the National Resident Matching Program [9] in the U.S., but there are many others, including the Scottish Foundation Allocation Scheme (SFAS) [15]. Our involvement with this latter scheme has been the main motivation for the study reported in this paper. It is well known that an instance of HR can be solved, i.e., a so-called stable matching can be found, in polynomial time, but a number of variants of the basic problem are more challenging. This includes the case where applicants may form couples, who submit joint, rather than individual, preferences.

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The Hospitals Residents problem with Couples (HRC) has been the subject of various studies, primarily motivated by developments in NRMP.

We consider a variant of HRC motivated by the decision to accommodate couples in SFAS with effect from 2009. This variant differs in some respects from those that have been studied in the literature, and from the version that is currently part of the NRMP – essentially it can be seen as a special case of these. So we first specify the problem, which we designate as Special HRC (or SHRC).

Statement of the problem

An instance of SHRC comprises a set of applicants (or residents), a set of programmes (or hospitals), and a set of couples. Each programme \( p \) offers a fixed number \( c(p) \) of places, the capacity of the programme. Each couple consists of a pair of distinct applicants, and no applicant can be in more than one couple. An applicant is either linked or single depending on whether or not he/she is a member of a couple. If applicants \( a \) and \( b \) form a couple then each of \( a \) and \( b \) is the partner of the other.

Each applicant, single or linked, has a strictly ordered preference list containing a subset of the programmes. Applicant \( a \) is said to prefer programme \( p \) to programme \( q \) if \( p \) precedes \( q \) in \( a \)'s preference list. A programme that appears on the preference list of an applicant is acceptable to that applicant. Each applicant \( a \) has a numerical score \( s(a) \). Applicant \( a \) is superior to applicant \( b \), and \( b \) is inferior to \( a \), if \( s(a) > s(b) \). Two applicants with the same score are said to be of equal rank. The preference list of a hospital is derived directly from the applicant scores, effectively giving a master preference list of applicants [3]. This contrasts with the classical versions of HR (and the NRMP context) in which each hospital has a preference list that is independent of the others. In practice, many applicants may have the same score, leading to the presence of ties in the master list and in the programmes’ preference lists derived from it, but we primarily consider the case where all of the scores are distinct (which can be realised by breaking all of the ties in some arbitrary way).

Each pair of programmes is designated as either compatible or not (primarily reflecting their geographical locations). It is assumed that a programme is compatible with itself. Each couple \((a, b)\) has a joint preference list that contains precisely the compatible pairs of programmes \((x, y)\) where \(x\) is acceptable to \(a\) and \(y\) to \(b\). The precise order of the pairs on this joint preference list is not crucial (although in the SFAS scheme, a couple’s joint preference list is constructed in a particular systematic and transparent way from the two individual preference lists – see [15] for details). A compatible pair that appears on the joint preference list of couple \((a, b)\) is said to be acceptable to that couple. A couple \((a, b)\) prefers a programme pair \((p, q)\) to a programme pair \((r, s)\) if \((p, q)\) precedes \((r, s)\) on \((a, b)\)'s joint preference list1. Again this represents a restriction of the general version of the problem, in which each couple has complete freedom to specify their own preference list of programme pairs. We comment further on the relationship between SHRC and the general HRC, and the implications of our work for the more general problem, at the end of Section 7.

A matching \(M\) is a set of applicant-programme pairs satisfying the following three conditions:

- each applicant \(a\) appears in at most one pair, and if \((a, p)\) is a pair in \(M\) then \(p\) is acceptable to \(a\);

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1Note that the SFAS scheme does not permit one member of a couple to be allocated to an acceptable programme and the other to be unallocated. However, in the algorithms that we study, this restriction can easily be relaxed by introducing a dummy programme with infinite capacity.
• if \((a, b)\) is a couple, then either \(\langle a, p \rangle\) and \(\langle b, q \rangle\) are in \(M\), where \((p, q)\) is acceptable to \((a, b)\), or there is no pair in \(M\) containing \(a\) or \(b\);

• the number of pairs in \(M\) containing the programme \(p\) is at most \(c(p)\).

In a matching \(M\), an applicant \(a\) is \textit{matched} if there is a pair \(\langle a, p \rangle\) in \(M\) for some programme \(p\), and is otherwise \textit{unmatched}. A programme \(p\) is \textit{full} if there are exactly \(c(p)\) pairs of the form \(\langle a, p \rangle\) in \(M\), and is otherwise \textit{undersubscribed}. If applicant \(a\) is matched in \(M\), we denote by \(M(a)\) the programme \(p\) such that \(\langle a, p \rangle\) is in \(M\), i.e., \(a\)'s assigned programme in \(M\). If \(a\) is unmatched in \(M\) then \(M(a)\) is null. Likewise, for a programme \(p\), we denote by \(M(p)\) the set of applicants \(a\) such that \(\langle a, p \rangle\) is in \(M\), i.e., \(p\)'s assignees in \(M\).

\textbf{Stability}

The stability definition for this context is somewhat more complicated, and perhaps more contentious, than in the case where there are no couples. Crucially, in formulating such a definition, most previous authors appear to have overlooked the additional complication that arises because of the possibility that both members of the couple may be assigned to the same programme, or, as in [4], they have sidestepped the issue by forbidding couples from being assigned to the same programme. Only the recent papers of McDermid and Manlove [8] and Marx and Schlotter [7] have addressed this issue explicitly. We provide some detailed justification for our definition of stability, which differs slightly from that given in [8] and [7], but which we believe is appropriate for our context. We first give our definition, and then a detailed rationale for this choice.

A matching \(M\) is \textit{stable} if it is not \textit{blocked} by a pair \(\langle a, p \rangle\) consisting of a single applicant \(a\) and a programme \(p\), or by a pair \(\langle (a, b), (p, q) \rangle\) consisting of a couple \((a, b)\) and distinct programmes \(p\) and \(q\), or by a pair \(\langle (a, b), p \rangle\) consisting of a couple \((a, b)\) and a programme \(p\).

A single applicant \(a\) and a programme \(p\) \textit{block} \(M\) if

(a) \(a\) is unmatched, or prefers \(p\) to \(M(a)\); and

(b) \(p\) is undersubscribed, or \(a\) is superior to a member of \(M(p)\).

A couple \((a, b)\) and a compatible pair of distinct programmes \(p\) and \(q\) \textit{block} \(M\) if

(c) \(a\) and \(b\) are unmatched, or \((a, b)\) prefers \((p, q)\) to \((M(a), M(b))\); and

(d) \(p\) is undersubscribed, or \(p = M(a)\), or \(a\) is superior to a member of \(M(p)\); and

(e) \(q\) is undersubscribed, or \(q = M(b)\), or \(b\) is superior to a member of \(M(q)\).

These first two cases are intuitive, and coincide with the corresponding cases in the definitions given by earlier authors.

However the third case is less immediate. We say that a couple \((a, b)\) and a programme \(p\), acceptable to both \(a\) and \(b\), \textit{block} \(M\) if

(f) \(a\) and \(b\) are unmatched, or \((a, b)\) prefers \((p, p)\) to \((M(a), M(b))\); and

(g) either

- (i) \(p\) has at least two free places in \(M\); or
- (ii) \(p\) has one free place in \(M\), and \(p \in \{M(a), M(b)\}\) or both \(a\) and \(b\) are superior to a member of \(M(p)\); or
- (iii) \(p\) is full in \(M\) and
1. \( p \in \{ M(a), M(b) \} \) and both \( a \) and \( b \) are superior to a member of \( M(p) \); or
2. both \( a \) and \( b \) are superior to a member \( x \) of \( M(p) \), and \( x \) is a linked applicant whose partner is also in \( M(p) \); or
3. both \( a \) and \( b \) are superior to at least two members of \( M(p) \).

**Rationale**

The rationale for our definition is in terms of fairness to the applicants, and ease of justification, based on our practical experience of the SFAS matching scheme.

Once the outcome of the matching process is known, suppose that a single applicant \( a \) queries why he was not assigned to a particular preferred programme \( p \). Then we would like the appropriate response to be that programme \( p \) filled all of its places with applicants who are at least as good as \( a \), so there is no applicant whom \( p \) could reject in order to accommodate \( a \). This notion of stability corresponds exactly to the one that applies in the classical case where there are no couples.

In order to be able to provide an analogous guarantee to couples, a key requirement is to identify the circumstances in which a couple should take precedence over an applicant, and vice versa. We say that a couple \( c = (a, b) \) is superior to an applicant \( x \), and \( x \) is inferior to \( c \), if both \( a \) and \( b \) are superior to \( x \). An applicant \( x \) is superior to a couple \( c = (a, b) \), and \( c \) is inferior to \( x \), if \( x \) is superior to at least one of \( a \) and \( b \). On the face of it this definition may seem surprising, since it amounts to awarding a score to a couple on the basis of the weaker member. However, we can justify this in two different ways.

Firstly, consider a programme \( p \) with two places and three applicants \( a, b \) and \( x \), where \( a \) and \( b \) form a couple \( c \), \( x \) has a score intermediate between those of \( a \) and \( b \), and all three of these applicants have \( p \) as their first choice programme. If the two places were to be offered to \( a \) and \( b \) then it would be impossible to make the above response to \( x \), were he to query why he was not assigned to \( p \).

Secondly, if \( a \) and \( b \) were single applicants rather than a couple, then \( a \) and \( x \) would be assigned to \( p \)'s two places. If the places were given to \( a \) and \( b \), then applicant \( b \) would be seen to have gained an advantage by being part of a couple. Single applicants would have some justifiable cause for complaint if, in certain circumstances, the matching scheme were to bestow an advantage on one or more linked applicants – indeed applicants might be tempted to act strategically by forming “artificial” couples if this were the case.

Our precedence rule involving a couple and an applicant explains why in parts (ii) and (iii) of stability condition (g) we require that both members of a couple should satisfy a particular condition.

Next we extend the notions of superiority and inferiority to couples, as follows. Suppose for simplicity that a couple is written so that the first member is superior to the second member or of equal rank. Then couple \( (a, b) \) is superior to couple \( (a', b') \), and \( (a', b') \) inferior to \( (a, b) \), if (i) \( b \) is superior to \( b' \), or (ii) \( b \) and \( b' \) are of equal rank, and \( a \) is superior to \( a' \). Again, we are essentially awarding a score to a couple on the basis of the weaker member. However, we argue that this decision is a necessary consequence of the way we defined precedence between a couple and a single applicant. We now explain.

If there are two couples \( c = (a, b) \) and \( d = (a', b') \) and all of the individuals have unique scores, then, up to symmetry, there are three ways in which the members of the couples may be ranked, namely

1. \( a \ b \ a' \ b' \)
2. \( a \ a' \ b \ b' \)
3. \( a' \ a \ b \ b' \)

In the first two cases, there seems no doubt that we should regard couple \( c \) as being superior to couple \( d \), but the third case seems much less clear cut. However, suppose there
is a programme $p$ with two places, and that all four of these applicants, and an additional single applicant $x$, rank $p$ first among their preferences. Suppose further that the rank ordering of the five applicants is:

$$a' \ a \ b \ x \ b'.$$

Then it follows that $p$ prefers couple $c$ to $x$, and, as a consequence of our earlier decision, prefers $x$ to couple $d$. If preferences are to be transitive, which seems a natural and desirable property, then $p$ must prefer $c$ to $d$. In the given scenario, the only stable possibility is that $p$’s two places are filled by $a$ and $b$.

We note that this interpretation of precedence between couples is reflected in part (ii) of stability condition (g).

In addition to the above form of response to a query from a dissatisfied single applicant, we can now formulate analogous responses to queries from couples. Suppose that a couple $(a, b)$ question why they were not assigned to a preferred compatible pair of distinct programmes $(p, q)$. Then the appropriate response would be that either $p$ filled all of its places with applicants who are at least as good as $a$, or $q$ filled all of its places with applicants who are at least as good as $b$. So there are no two applicants who can be rejected, one by $p$ and one by $q$, in order to accommodate $a$ and $b$.

Finally, suppose that a couple $c = (a, b)$ question why they were not both assigned to a programme $p$. Then the appropriate response depends on whether one of them, say $a$, or neither of them, is actually assigned to $p$. In the first case, the response would be that $p$ is full and has no assignee who is inferior to both $a$ and $b$. In the second case, it would be either that $p$ has one free place but no assignee who is inferior to both $a$ and $b$, or that $p$ is full but has no assigned couple inferior to $c$ and no two assignees who are both inferior to $a$ and $b$.

Example 1, essentially the same as that given by Roth [13] and accredited by him to Klaus and Klijn, illustrates that, as in other variants of the problem, an instance of SHRC need not admit a stable matching.

**Example 1.** There are three applicants, comprising one single applicant $a_2$ and one couple $(a_1, a_3)$, and two programmes, each with just one place. The applicants are numbered in decreasing order of score ($a_1$ highest, $a_3$ lowest), and the preference lists are as shown in Figure 1.

$$a_1 : p_1$$

$$a_2 : p_1 \ p_2$$

$$a_3 : p_2$$

$$(a_1, a_3) : (p_1, p_2)$$

Figure 1: An SHRC instance with no stable matching

There are three non-empty matchings for this instance, $M_1 = \{(a_1, p_1), (a_3, p_2)\}$, $M_2 = \{(a_2, p_2)\}$ and $M_3 = \{(a_2, p_1)\}$. It may readily be verified that $M_1$ is blocked by $(a_2, p_2)$, $M_2$ by $(a_2, p_1)$, and $M_3$ by $((a_1, a_3), (p_1, p_2))$.

Example 2 illustrates an additional possibility that does not seem to have been pointed out before, namely that, even in a case where a stable matching does exist, some couple might wish to exchange their allocation, but doing so would violate stability.

**Example 2.** There are again three applicants, comprising one single applicant $a_2$ and one couple $(a_1, a_3)$, and two programmes, each with just one place. Again the applicants are numbered in decreasing order of score ($a_1$ highest, $a_3$ lowest). The preference lists are as shown in Figure 2.
The only stable matching for this instance is \( M = \{ \langle a_1, p_1 \rangle, \langle a_3, p_2 \rangle \} \) However, both members of the couple would prefer to exchange their positions.

**Related work**

Roth [11] first observed that a general instance of HRC need not admit a stable matching and Ronn [10] showed that the problem of deciding whether it does is NP-complete, even if all of the programme capacities are equal to one and there are no single applicants. Of course, in the general HRC problem, each programme has its own individual preference list, and the notion of stability is defined in terms of these preferences, rather than in terms of the global ‘superiority’ concept. As observed above, an instance of SHRC need not admit a stable matching, but it appears that Ronn’s original proof of NP-completeness for the general problem cannot be adapted, at least in a straightforward way, to this special case. Aldershof and Carducci [1] show that, in the HRC context, there is no concept analogous to the resident and hospital optimal stable matchings that are known to exist for any HR instance, and also that stable matchings, when they do exist, can have different sizes.

Roth and Peranson [12] describe the couples algorithm implemented by NRMP, and report on empirical studies, using real NRMP data, undertaken to investigate the effect of varying certain aspects of the implementation. A variant of that algorithm, which is actually very similar to Algorithm C of Section 3, is outlined by Klaus et al. [5], who showed, among other things, that, even in cases where a stable matching exists, there may be no possible execution of the algorithm that finds it.

Klaus and Klijn [4] study a restricted version of HRC where the couples’ preferences are ‘weakly responsive’; this means that they are derived in a logical way from their individual preferences, much as in our context, but crucially there are no incompatible programmes. In this context they show that a stable matching is bound to exist, but Kojima et al. [6] observe that such an assumption would be unrealistic in practice. McDermid and Manlove [8] consider a version of HRC in which couples’ preferences are derived in a similarly consistent way from individual preferences, but where pairs of programmes may be incompatible, and show that the problem of deciding whether a stable matching exists is NP-complete in this case, even when applicants’ preference lists have length at most three and programme capacities are at most two, and also even in the very special case when couples are required to be matched to the same hospital. On the other hand, they give a linear-time algorithm that determines, in this context, whether there is a matching that is stable in the classical (Gale-Shapley) sense, and in which assigned couples have compatible programmes. Marx and Schlotter study the HRC problem in the context of parameterized complexity, and show, amongst other things, that the existence problem is W[1]-hard when parameterized by the number of couples. Note, however, that in [4] members of a couple are explicitly forbidden from being assigned to the same hospital, while in [8] and [7], the definition of a blocking pair comprising a couple and a hospital differs slightly from ours, as discussed above. Sethuraman et al [14] discuss a model related to ours, in which each member of a couple submits an individual preference list, and the couple decides on the compatibility of programmes based on a partition into ‘regions’. They show that linear
programming can be used to determine in polynomial time whether there is a matching that is stable in the classical sense, i.e., with respect to the preferences of individuals, and in which the members of each couple are assigned to compatible programmes.

Recently, Kojima et al [6] have shown that, under certain conditions, including a tight bound on the ratio of couples to single applicants, a stable matching exists with high probability in HRC instances, and they present supporting empirical evidence based on several years data from the US market for clinical psychologists.

**The contribution of this paper**

In this paper, we first establish that the SHRC problem is NP-complete, even under quite severe restrictions. This is not a consequence of the known hardness results for more general versions of the problem. We then describe an algorithm for the problem, similar to that of Klaus et al [5], and indicate how certain implementation choices lead to a range of variants, including the one (Algorithm C-RAN described in Section 6) that currently forms the basis of the SFAS matching scheme. This algorithm is contrasted with the algorithm described by Roth and Peranson [12], and then a third, conceptually simpler, algorithm, based on satisfying blocking pairs, is described. Again, for each of the alternative algorithms, several possible variants are identified. The second part of the paper describes an empirical study designed to investigate the likelihood that a stable matching can be found in various circumstances, depending particularly on the ratio of couples to single applicants, and to compare the performance of a number of variants of the three algorithms. The final section summarises the results of this empirical study, and draws a number of conclusions regarding the relative merits of the algorithms and their variants, the likelihood of solving instances of SHRC, and the relevance of these results for more general versions of the problem.

**2 SHRC is NP-complete**

To justify our empirical study of heuristics for the SHRC problem, we need to establish that this special case, based on a ‘master list’ of applicants, remains NP-complete.

**Theorem 2.1.** The problem of determining whether a stable matching exists for an instance of SHRC is NP-complete, even if there is a strict master list on both sides and each hospital has capacity one.

**Proof** The problem is in NP, obviously. We transform from COMPLETE SMTI-2ML, that is the problem of finding a complete stable matching for an instance of the stable marriage problem with incomplete lists, ties and master lists on both sides. This problem is NP-complete ([3], Theorem 3.2.) even under the following restrictions: there are ties in the master list of women only, they are of length 2, each tie appears in only one individual list and it forms the whole of that individual list. Let $I$ be such an instance. We create an instance $I'$ of Hospitals / Residents problem with couples under the restrictions listed above, as follows.

First we construct the so-called proper part of $I'$. Let $U$ and $W$ be the set of men and women in $I$, respectively. Further, let $U_T \subseteq U$ denote the set of men such that each $m_i \in U_T$ has a single tie in his list, i.e., $m_i : (w_{i,1}, w_{i,2})$. The men and the women of $I$ will correspond to the applicants and the programmes in $I'$, respectively. Each programme in $I'$ has unit quota. Initially, let each man with a strict preference list have the same preference list in $I'$ as in $I$ by keeping also the two master lists. Now, for each $m_i \in U_T$ let us create two couples, $(a_{i,1}, a_{i,4})$ and $(a_{i,2}, a_{i,3})$ in $I'$ together with three new programmes, $p_{i,1}$, $p_{i,2}$ and $p_{i,3}$, with the following individual preference lists.
We replace $m_i$ with $a_{i,1}$, $a_{i,2}$, $a_{i,3}$ and $a_{i,4}$ in the master list of the applicants (in this order), whilst the tie $(w_{i,1}, w_{i,2})$ is replaced with $p_{i,1}$, $p_{i,2}$, $w_{i,3}$, $w_{i,1}$ and $w_{i,2}$ in the master list of the programmes (in this order). Furthermore, we suppose that $p_{i,1}$ and $p_{i,2}$ are geographically close to each other, whilst $p_{i,3}$, $w_{i,1}$ and $w_{i,2}$ are also geographically close to each other (but far from $p_{i,1}$ and $p_{i,2}$), therefore the following joint preference lists will be constructed:

$$(a_{i,1}, a_{i,4}) : (p_{i,1}, p_{i,2}) (p_{i,3}, w_{i,1})$$

$$(a_{i,2}, a_{i,3}) : (p_{i,1}, p_{i,2}) (p_{i,3}, w_{i,2})$$

This completes the construction of the proper part of $I'$. We shall verify that we have the following one-to-one correspondence between the stable matchings of $I$ and the stable matchings of the proper part of $I'$.

- $\{m_i, w_j\} \in M$ for some $m_i \in U \setminus U_T \iff \{m_i, w_j\} \in M'$
- $\{m_i, w_{i,1}\} \in M$ for some $m_i \in U_T \iff \{\langle a_{i,1}, p_{i,1} \rangle, \langle a_{i,4}, w_{i,1} \rangle, \langle a_{i,2}, p_{i,1} \rangle, \langle a_{i,3}, p_{i,2} \rangle \} \subseteq M'$
- $\{m_i, w_{i,2}\} \in M$ for some $m_i \in U_T \iff \{\langle a_{i,2}, p_{i,3} \rangle, \langle a_{i,3}, w_{i,2} \rangle, \langle a_{i,1}, p_{i,1} \rangle, \langle a_{i,4}, p_{i,2} \rangle \} \subseteq M'$

Finally, a man $m_i \in U_T$ is unmatched in $M$ if and only if either

- $\{\langle a_{i,1}, p_{i,1} \rangle, \langle a_{i,4}, p_{i,2} \rangle \} \subseteq M'$ and the couple $(a_{i,2}, a_{i,3})$ is unmatched in $M'$, or
- $\{\langle a_{i,2}, p_{i,1} \rangle, \langle a_{i,3}, p_{i,2} \rangle \} \subseteq M'$ and the couple $(a_{i,1}, a_{i,4})$ is unmatched in $M'$.

To prove this, first let $M$ be a stable matching in $I$ and let $M'$ be the corresponding matching in $I'$ as described above. Suppose for a contradiction that $M'$ is not stable. If $M'$ is blocked by a single applicant $m_i$ and a programme $w_j$ then this pair, $\{m_i, w_j\}$ would be blocking for $M$ as well. Suppose now that $M'$ is blocked by a couple $(a_{i,1}, a_{i,4})$. This couple cannot be matched to programmes $p_{i,1}$ and $p_{i,2}$, respectively, since this is their first choice, and therefore, according to our construction of $M'$, these two programmes must be occupied by the other possible couple, $(a_{i,2}, a_{i,3})$. In this case, $(a_{i,1}, a_{i,4})$ is not blocking with $(p_{i,1}, p_{i,2})$, so $(a_{i,1}, a_{i,4})$ must block with the second (and last) pair of programmes in their joint list, $(p_{i,3}, w_{i,1})$, but then $\{m_i, w_{i,1}\}$ would also be blocking for $M$, a contradiction. Similarly, we get a contradiction if we suppose that couple $(a_{i,2}, a_{i,3})$ is blocking for $M'$.

Now, let us suppose that $M'$ is a stable matching in $I'$. The stability of $M'$ implies that either $\{\langle a_{i,2}, p_{i,1} \rangle, \langle a_{i,3}, p_{i,2} \rangle \} \subseteq M'$ or $\{\langle a_{i,1}, p_{i,1} \rangle, \langle a_{i,4}, p_{i,2} \rangle \} \subseteq M'$ for each index $i$, where $m_i \in U_T$. Let $M$ be the corresponding matching in $I$ as described. Suppose for a contradiction that $M$ is not stable. If $M$ is blocked by $\langle m_i, w_j \rangle$ for some $m_i \in U \setminus U_T$ then the copy of this pair would block $M'$ too. Otherwise, if $M$ is blocked by $\langle m_i, w_{i,1} \rangle$ for some $m_i \in U_T$ then couple $(a_{i,1}, a_{i,4})$ would block $M'$ with the pair of programmes $(p_{i,3}, w_{i,1})$. We get a similar contradiction if $M$ is blocked by $(m_i, w_{i,2})$, so the proof of the statement (i.e. the one-to-one correspondence between the stable matchings of $I$ and the stable matchings of $I'$) is complete.

We refer to those involved in the proper part as proper programmes and proper applicants. Now we construct the additional part of $I'$. We extend the set of applicants with
seven applicants, \( \{a_i^* : 0 \leq i \leq 6\} \) by appending them to the end of the master list of the applicants \( (a_0^* \text{ highest, } a_6^* \text{ lowest}) \), we also add three new programmes \( \{p_i^* : 1 \leq i \leq 3\} \) appended to the end of the master list of the programmes in an arbitrary strict order. Let the applicants have the following individual preference lists.

\[
\begin{align*}
a_0^* &: \text{[all proper programmes]} \quad p_1^* \\
a_1^* &: p_1^* \\
a_2^* &: p_3^* \\
a_3^* &: p_3^* \\
a_4^* &: p_1^* \\
a_5^* &: p_2^* \\
a_6^* &: p_2^*
\end{align*}
\]

Moreover, six of the seven additional applicants form three couples with the following joint lists.

\[
\begin{align*}
(a_1^*, a_6^*) &: (p_1^*, p_2^*) \\
(a_2^*, a_4^*) &: (p_3^*, p_1^*) \\
(a_3^*, a_5^*) &: (p_3^*, p_2^*)
\end{align*}
\]

We show that \( I \) admits a complete stable matching if and only if \( I' \) admits a stable matching. Suppose first that \( M \) is a complete stable matching in \( I \). Let \( M' \) be the corresponding stable matching in the proper part of \( I' \) extended with \( \{(a_0^*, p_1^*), (a_3^*, p_3^*), (a_5^*, p_2^*)\} \). It is straightforward to show that this matching is stable. In the other direction, if \( M' \) is a stable matching then first we shall show that the proper programmes are completely filled with proper applicants. This is because \( a_0^* \) cannot be allocated to a proper programme, since otherwise it would not be possible to allocate the three additional couples to the three additional programmes in a stable way. But if \( a_0^* \) is not allocated to a proper programme then each proper programme must be filled by a proper applicant (since otherwise \( a_0^* \) would form a blocking pair with such an unallocated programme). This means that every applicant is matched to a proper programme in the restriction of \( M' \) to the proper part of \( I' \), therefore \( M \), the corresponding stable matching in \( I \), is complete. \( \square \)

Note that Theorem 2.1 obviously remains true for a version of the problem intermediate between SHRC and HRC, in which programmes’ preferences are derived from a master list, but couples have complete freedom to form their own joint preference lists. Also, the fact that the result holds when all programmes have capacity 1 means that NP-completeness does not depend on the precise formulation of the stability criterion for blocking pairs of the form \( \langle (a, b), p \rangle \).

### 3 The SFAS algorithm

The algorithm that forms the basis of SFAS, which we refer to as Algorithm C, consists of two phases.

**Phase 1 of Algorithm C**

In Phase 1, some initial simplification is undertaken, whereby single applicants can become (provisionally) assigned to the best available programme, and unattainable entries are deleted from preference lists.

Ties consisting of applicants with identical scores are broken at random to produce a strictly ordered list of applicants. We refine the notion of superiority so that applicant \( a \) is now regarded as superior to applicant \( b \), and \( b \) inferior to \( a \), if \( a \) precedes \( b \) in this strictly
sort the applicants by decreasing score, breaking ties uniformly at random; 
for each applicant \( a \) in sorted order

  If \( a \) is a single applicant

    delete all full programmes \( x \) from \( a \)’s list;
    // \( a \) cannot be assigned to \( x \) because of superior single applicants

  if \( a \)’s list contains at least one programme

    assign \( a \) to the first programme on his preference list;

else // \( a \) is a member of a couple \( c \)

delete all entries \( (x, x) \) from \( c \)’s list where \( x \) has just one free place;
// couple \( c \) cannot be assigned to \( x \) because of superior single applicants

  if \( a \) is the superior member of \( c \)

    delete all entries \( (x, y) \) from \( c \)’s list where \( x \) is full;
    // \( a \) cannot be assigned to \( x \) because of superior single applicants

  else if \( a \) is the inferior member of \( c \)

    delete all entries \( (y, x) \) from \( c \)’s list where \( x \) is full;
    // \( a \) cannot be assigned to \( x \) for the same reason

Figure 3: Phase 1 of Algorithm C

ordered list. In this first phase, the applicants are processed in the order in which they appear in this strictly ordered list. Henceforth, a couple is always represented as an ordered pair \((a, b)\) such that \( a \) is superior to \( b \). (Of course, in general, breaking ties in different ways can be expected to lead to different outcomes. The entire algorithm, including the tie-breaking step, can be executed many times, and the ‘best’ solution returned, according to whatever optimality criterion may be appropriate.) A pseudocode version of Phase 1 of Algorithm C appears in Figure 3.

The outcome of Phase 1 is a reduced set of preference lists and an initial assignment of (a subset of) the single applicants to programmes.

Lemma 3.1. (i) If programme \( p \) is removed from the single applicant \( a \)’s preference list during Phase 1, then there is no stable matching in which \( a \) is assigned to \( p \).

(ii) If programme pair \((p, q)\) is removed from the couple \((a, b)\)’s preference list during Phase 1, then there is no stable matching in which \( a \) is assigned to \( p \) and \( b \) to \( q \).

Proof. (i) Suppose that, at step \( x \) of the algorithm, programme \( p \) is removed from applicant \( a \)’s preference list during Phase 1, and that there is a stable matching \( M \) in which \( a \) is assigned to \( p \). Suppose further that this was the first such removal. Then at step \( x \), \( p \) must have been full with applicants superior to \( a \). Hence at least one of these applicants, say \( b \), is not assigned to \( p \) in \( M \). But \( b \) cannot be assigned in \( M \) to a programme he prefers to \( p \), for such a programme would have to have been removed from his list prior to step \( x \), contrary to the assumption that the first such removal was at step \( x \). Hence \( b \) and \( p \) block \( M \), a contradiction.

(ii) The proof in this case is analogous to the proof of part (i). □

Phase 2 of Algorithm C

Define an agent to be either a single applicant or a couple. In Phase 2 of the algorithm, at any given stage, some agents are matched and some are not. Unmatched agents apply to the next entry in their preference list, where the next entry moves sequentially along the list, but may be reset to an earlier entry in the list in certain circumstances. Unmatched agents that have a next entry available are represented in a data structure, which we refer to as the waiting list. At the end of Phase 1, all the couples who have a non-empty preference list are added to the waiting list.
An application is accepted if it constitutes a blocking pair for the current matching, and is otherwise rejected. An accepted application may lead to the rejection of one or more weakest assignees to avoid programmes becoming over-subscribed. If one member of a matched couple is rejected the other member must withdraw from his assigned programme. Note that rejection of an agent advances the ‘next’ entry in the preference list of that agent (taking care to avoid the repetition of such a step when both members of a couple are simultaneously rejected.)

In addition, each programme that is, or has been, full, maintains a set of rejected applicants – its reserve list. So if some applicant withdraws from such a programme \( p \), because of their partner’s rejection, or because of the possibility of an improved assignment (see below), each applicant in \( p \)’s reserve list should, in due course, be allowed to re-apply to the programme if this might improve his assignment. This is achieved by withdrawing the applicant from any programme to which he is currently assigned (and likewise his partner, in the case of a linked applicant), conducting a ‘reset’ operation on the ‘next’ preference list position (of the applicant or the couple), and adding him (or the couple containing him) to the waiting list (if not already a member of it). Note that the ‘reset’ operation is conditional – it means moving the ‘next’ position to the one occupied by \( p \), unless this would imply a move forward in the list (perhaps the review of another programme already caused a reset). Reset for a couple means moving the ‘next’ position to the first entry containing \( p \) for the appropriate member (again, only if this represents a move to a position higher in the list).

A further data structure, which we refer to as the review list, holds the programmes that have experienced one or more withdrawals and that have a non-empty reserve list. Whenever a programme \( p \) is taken from this review list, each applicant on its reserve list whom \( p \) would now accept must be examined – this is referred to as reviewing the programme.

Consider first a single such applicant \( a \). The pair \( \langle a, p \rangle \) blocks the current matching, so \( a \) should withdraw from his currently assigned programme (if any), the next position in \( a \)’s preference list should be (conditionally) reset to the position occupied by \( p \), \( a \) should be added to the waiting list (if not already in that list) and should be removed from \( p \)’s reserve list.

The situation for a linked applicant is a little more subtle. If \( a \) is a linked applicant on \( p \)’s reserve list, say with partner \( b \), then entries of the form \( \langle p, q \rangle \) that precede the current assignment on \( (a, b) \)’s preference list must be examined in turn (potentially all such pairs if \( a \) and \( b \) are unassigned). The first such \( \langle p, q \rangle \) that blocks the current matching with \( (a, b) \) leads to actions similar to the previous case – \( a \) and \( b \) should withdraw from their currently assigned programmes (if any), the next position in \( (a, b) \)’s preference list should be (conditionally) reset to the position occupied by \( (p, q) \), \( (a, b) \) should be added to the waiting list, and \( a \) removed from \( p \)’s reserve list (but see the additional remark below). No further such pairs \( \langle p, q \rangle \) need then be considered. However, if, during the search for such a blocking pair, a pair \( \langle p, q \rangle \) is encountered that does not block the matching, this must be because \( q \) would reject \( b \). But \( b \) need not be on \( q \)’s reserve list, so must be added to it in that case (since a subsequent withdrawal from \( q \) might otherwise leave \( \langle (a, b), (p, q) \rangle \) as a blocking pair). A final subtlety arises if the pair \( \langle p, p \rangle \) is encountered on \( (a, b) \)’s preference list but does not block with \( (a, b) \). In this case \( p \) would accept \( a \) but not both \( a \) and \( b \). A subsequent withdrawal from \( p \) might change this, so we must ensure that one of these applicants remains on \( p \)’s reserve list, even if a blocking pair of the form \( \langle (a, b), (p, q) \rangle \) is subsequently found.

Phase 2 terminates if the waiting list and review list both become empty.

A pseudocode description of a version of Phase 2 of Algorithm C appears in Figures
Recall that we are assuming that when a couple is represented as an ordered pair \((a, b)\), applicant \(a\) is superior to applicant \(b\).

Place each couple with a non-empty preference list on the waiting list;
set the review list to be empty;
while the waiting list \(W\) or the review list \(R\) is non-empty
  if \(W\) is non-empty
    remove agent \(x\) from \(W\);
    \(x\) applies to the next entry on its preference list;
  else remove programme \(p\) from \(R\);
    review programme \(p\).

Figure 4: Phase 2 of Algorithm C

**Theorem 3.1.** If Phase 2 of Algorithm C terminates then the final matching of applicants to programmes is stable.

**Proof** We first note some key consequences of the stability definition:

- if a programme \(p\) rejects an assignee \(a\) in favour of another single applicant, the new assignee is superior to \(a\);
- if \(p\) rejects one or two assignees (who do not themselves form a couple) in favour of a couple, both members of the new couple are superior to the rejected applicant(s);
- if \(p\) rejects an assigned couple in favour of a new couple, the weaker member of the rejected couple is inferior to both members of the new couple.

Let \(M\) be the matching produced by the algorithm on termination. Suppose first that \(M\) is blocked by the pair \(\langle a, p \rangle\). Then \(p\) must have rejected \(a\), possibly more than once. The last time that \(p\) rejected \(a\), say at step \(x\) in the execution of the algorithm, \(p\) must have been full with applicants superior to \(a\), and \(a\) must then have become a member of \(p\)'s reserve list. Denote by \(b\) the weakest assignee of \(p\) at that point. There could have been no subsequent withdrawals from \(p\), for this would have caused \(p\) to be added to the review list, and thereafter, when \(p\) was removed from this list, \(a\), as a member of its reserve list, would have had his preference list position reset, and would have to have applied again to \(p\) to finish up with a worse assignment than \(p\) (or no assignment at all). Hence, since \(p\) had no withdrawals after step \(x\), and since the rejection of an assignee after this step cannot give \(p\) an assignee inferior to \(b\), it follows that, on termination, \(p\) is full with assignees superior to \(a\), a contradiction.

Suppose now that \(M\) is blocked by the pair \(\langle (a_1, a_2), (p_1, p_2) \rangle\), where \(p_1\) and \(p_2\) are distinct programmes. Then \(p_1\) must have rejected \(a_1\) or \(p_2\) must have rejected \(a_2\), possibly more than once. Suppose, without loss of generality, that the last time this happened, \(p_1\) was full with applicants superior to \(a_1\), so that \(a_1\) became a member of \(p_1\)'s reserve list at that point. If there were no subsequent withdrawals from \(p_1\) then it is not possible that \((a_1, a_2)\) and \((p_1, p_2)\) block \(M\). On any subsequent withdrawal from \(p_1\), \(a_1\) would be retrieved from \(p_1\)'s reserve list. If \((a_1, a_2)\) and \((p_1, p_2)\) block the matching at that moment, then couple \((a_1, a_2)\) have their current position reset, and this ensures that they must apply again to \((p_1, p_2)\) (since they end up with a worse assignment than that), which is a contradiction. Otherwise, if \(p_1\) would not accept \(a_1\) at that point, then \(a_1\) will remain
// Single applicant $a$ applies to programme $p$
if $a$ and $p$ block the current matching
    assign $a$ to $p$;
    if $p$ is oversubscribed
        $p$ rejects its worst assignee;
else
    $p$ rejects $a$;

// Couple $(a, b)$ applies to the programme pair $(p, q)$ $(p \neq q)$
if $(a, b)$ and $(p, q)$ block the current matching
    assign $a$ to $p$ and $b$ to $q$;
    if $p$ is oversubscribed
        $p$ rejects its worst assignee;
    if $q$ is oversubscribed
        $q$ rejects its worst assignee;
else
    $p$ rejects $a$ and/or $q$ rejects $b$;

// Couple $(a, b)$ applies to the programme pair $(p, p)$
if $(a, b)$ and $p$ block the current matching
    assign $a$ and $b$ to $p$;
    if $p$ is oversubscribed
        $p$ rejects its worst assignee;
    if $p$ is still oversubscribed
        $p$ rejects its worst assignee;
else
    $p$ rejects $b$; // no need also to reject $a$

Figure 5: The application steps in Phase 2 of Algorithm C

// Programme $p$ rejects applicant $a$
if $a$ is a single applicant
    advance next position in $a$’s preference list;
    if $a$ has preferences remaining
        add $a$ to waiting list;
else  // $a$ is in a couple $c$
    advance next position in $c$’s preference list;
    if $c$ has preferences remaining
        add $c$ to waiting list (if not already in it);
add $a$ to $p$’s reserve list (if not already in it);
if $a$ is assigned to $p$
    unassign $a$ from $p$;
if $a$ is a linked applicant
    $a$’s partner withdraws from his assigned programme;

Figure 6: The rejection step in Phase 2 of Algorithm C
// Applicant a withdraws from programme p; if p has a non-empty reserve list add p to the review list; // if not already in it unassign a from p;

Figure 7: The withdrawal step in Phase 2 of Algorithm C

// Review programme p;
for each applicant a in p’s reserve list whom p would now accept if a is a single applicant if ⟨a, p⟩ blocks the current matching a withdraws from assigned programme (if any); conditionally reset a; // to position occupied by p add a to waiting list; // if not already in it remove a from p’s reserve list; else if a is the superior member of a couple (a, b) for each pair of the form ⟨p, q⟩ in (a, b)’s preference list, in order if ⟨(a, b), (p, q)⟩ is in the current matching break; else if ⟨(a, b), (p, q)⟩ blocks the current matching for some q a and b withdraw from assigned programmes (if any); conditionally reset (a, b); // to position occupied by (p, q) add (a, b) to waiting list; // if not already in it remove a from p’s reserve list unless (a, b) prefers (p, p) to (p, q); (A) break; else if q would reject b add b to q’s reserve list; (B) else deal analogously with the case where a is the inferior member of a couple (b, a)

Figure 8: The review step in Phase 2 of Algorithm C
on $p_1$'s reserve list awaiting a possible further withdrawal. Finally, if $(a_1, a_2)$ and $(p_1, p_2)$ fail to block the current matching because $p_2$ is full of applicants superior to $a_2$, it may happen that a future withdrawal from $p_2$ causes $(a_1, a_2)$ and $(p_1, p_2)$ to block the matching at that point. However, in this case $a_2$ will have been placed on the reserve list of $p_2$ (at the step labeled (A) in Figure 8), and this will ensure that $(a_1, a_2)$ once again apply to $(p_1, p_2)$, a contradiction.

Finally suppose that $M$ is blocked by the pair $\langle (a_1, a_2), p \rangle$. Then again $p$ must have rejected $(a_1, a_2)$, and recalling our assumption that $a_1$ is superior to $a_2$, then the last time this happened, $p$ must have had at least $c(p) - 1$ assignees, excluding $a_1$, who are superior to $a_2$. Applicant $a_2$ would have been placed on $p$'s reserve list at that point. If there is no subsequent withdrawal from $p$, then $\langle (a_1, a_2), p \rangle$ cannot block $M$. If, when such a withdrawal takes place, the resulting matching is blocked by $\langle (a_1, a_2), p \rangle$, then provided $a_2$ is on $p$'s reserve list at that point, $(a_1, a_2)$ would have their preference list position reset to ensure a further application to $p$. So how can this fail to happen? It may be that, following a withdrawal from $p$, the resulting matching is not blocked by $\langle (a_1, a_2), p \rangle$ but is blocked by $\langle (a_1, a_2), (r, p) \rangle$ where $r$ is some other programme. In this case we must be careful not to remove $a_2$ from $p$'s reserve list if the resulting reset is to a point lower in the list than $(p, p)$. This is ensured by the step labeled (B) in Figure 8. □

However, Phase 2 of the algorithm may not terminate; certainly if the problem instance admits no stable matching, this will be the case. Furthermore, even if a stable matching does exist, it may not be found by the algorithm, as is illustrated by Example 3.

**Example 3.** There are eight applicants, comprising three couples $(a_1, a_5)$, $(a_2, a_4)$ and $(a_6, a_8)$ together with two single applicants $a_3$ and $a_7$. There are eight programmes, $p_1, \ldots, p_8$, each with just one place. The applicants are numbered in decreasing order of score ($a_1$ highest, $a_8$ lowest), and the individual and joint preference lists are as shown in Figure 9.

\[
\begin{align*}
    a_1 : & & p_1 & & p_3 \\
    a_2 : & & p_4 & & p_1 & & p_3 \\
    a_3 : & & p_1 & & p_5 \\
    a_4 : & & p_5 & & p_2 & & p_7 \\
    a_5 : & & p_2 & & p_6 \\
    a_6 : & & p_6 \\
    a_7 : & & p_6 & & p_8 \\
    a_8 : & & p_8 \\
    (a_1, a_5) : & & (p_1, p_2) & & (p_3, p_6) \\
    (a_2, a_4) : & & (p_4, p_5) & & (p_1, p_2) & & (p_3, p_7) \\
    (a_6, a_8) : & & (p_6, p_8)
\end{align*}
\]

**Figure 9:** An awkward SHRC instance for Algorithm C

There is a unique stable matching
\[
M = \{ \langle a_1, p_3 \rangle, \langle a_2, p_1 \rangle, \langle a_3, p_5 \rangle, \langle a_4, p_2 \rangle, \langle a_5, p_6 \rangle, \langle a_7, p_8 \rangle \}
\]
for this instance. However, the algorithm fails to converge to this matching; it will reach the matching $M' = \{ \langle a_1, p_1 \rangle, \langle a_2, p_3 \rangle, \langle a_3, p_5 \rangle, \langle a_4, p_2 \rangle \langle a_5, p_2 \rangle \}$, and will then cycle for the unsolvable sub-instance induced by $\{a_6, a_7, a_8\}$.

**Variants of Algorithm C**

A number of variants of Algorithm C arise as a result of possible implementation choices, such as
• the organisation of the waiting list: agents can be removed from this list randomly, or the list can be handled in some more restricted way, say as a stack;
• the relative priority of single applicants and couples: these can be treated equally, or we might choose to prioritise one or the other when choosing an agent from the waiting list;
• the review list can be given priority over the waiting list, so that at each stage a programme from the former list is reviewed, and only when the review list is empty is an agent taken from the waiting list.

4 An algorithm based on the Roth-Peranson approach

The couples algorithm described by Roth and Peranson [12] as the basis for the NRMP approach, when adapted to our context, has much in common with Algorithm C. The main distinction is that agents, i.e., single applicants or couples, are introduced into the market one at a time, and after each such introduction, the resulting sequence of applications, rejections, withdrawals, etc. is allowed to continue until stability is achieved before the next agent is introduced. Our implementation of the Algorithm RP is, of course, adapted to the context of applicant scores, and incorporates the stability definition given in Section 1. (It is not clear how Roth and Peranson define a blocking pair comprising a couple and one hospital.) We refer to this adaptation of the Roth-Peranson approach as Algorithm RP. In implementing Algorithm RP, we have available the same choices as those discussed earlier for Algorithm C, and in addition, the order in which agents are introduced to the market can be varied – for example, singles first, or couples first, or a random choice of agent at each stage.

5 An algorithm based on satisfying blocking pairs

For a given instance $I$ of SHRC, a matching $M$ that is not stable is bound to have one or more blocking pairs, as described in Section 1. A given single applicant $a$, or couple $c$, may belong to zero or more blocking pairs. An agent that belongs to one or more blocking pairs is called a blocking agent, and for any such agent $x$, we define the best blocker for $x$ to be the blocking pair that involves the highest placed entry on $x$’s preference list. The best blocker set for $M$, denoted $\mathcal{B}_M$, is the set of best blockers. It is immediate that $M$ is stable if and only if $\mathcal{B}_M$ is empty.

The general idea of the algorithm of this section, which we refer to as Algorithm BB, is that we maintain throughout the best blocker set relative to the current matching. At each step we choose a blocking pair from this set, change the matching by satisfying this blocking pair, allowing for any required rejections and/or withdrawals, and update the best blocker set accordingly. The algorithm terminates if this set becomes empty. Several variants are possible depending on how a blocking pair is chosen at each step – for example, it may be chosen at random, or blocking pairs involving singles (or couples) could be prioritised, or priorities could be based on applicant scores, or on how often a best blocker for a particular agent has previously been chosen (which we refer to as usage).

It is interesting to note that, for the special case where there are no couples, i.e., an instance of the classical HR problem, if the matching is initially empty, the best blocker set initially contains each agent paired with the first entry on its list, and Algorithm BB reduces to an execution of the standard Gale-Shapley algorithm.

In our setting, it seems reasonable to re-use Phase 1 of Algorithm C here, since this is again applicable and will typically remove some entries from preference lists. The initial
matching is the one constructed by Phase 1, and the best blocker set is initialised to contain an element for each couple that has a non-empty list at this point, consisting of the couple paired with the first entry on its (reduced) preference list.

Although conceptually simpler than the other algorithms, Algorithm BB does present some interesting implementation challenges. When a blocking pair is satisfied, resulting rejections and withdrawals can have a substantial knock-on effect on the set of best blockers, and these have to be managed carefully to ensure a correct and efficient implementation.

Notice that Example 3 can be used again here to show that, just like Algorithm C, Algorithm RP and Algorithm BB can also fail to find a stable matching even in cases where one exists.

2

However, a variant of this approach is to base the algorithm on the complete set of blocking pairs at each stage. This variant always has the possibility of finding a stable matching when one exists – it might fortuitously choose precisely the pairs of such a matching in some order. Intuition suggests that choosing anything other than the best blocker for an agent may not be a good strategy, intuition that was borne out in practice – our implementation of a variant of Algorithm BB based on a complete set of blocking pairs did not succeed in finding a stable matching (in reasonable time) for any set of test data used in the empirical study described in Section 6.

6 An empirical study

There is a huge number of combinations of options that could be implemented for the various algorithms, so some selectivity has been essential. The following variants of the three algorithms, already referred to in previous sections, were included in the study:

Algorithm C

- C-RAN: random waiting list;
- C-STA: stack waiting list;
- C-SGL: random waiting list subject to prioritising single applicants;
- C-CPL: random waiting list subject to prioritising couples;
- C-RLP: random waiting list but with review list prioritised.

Algorithm BB

- BB-RAN: random best blocker chosen at each step;

---

2To show this, first we shall observe that, considering any of the variants of Algorithms C, RP and BB, if \((a_1, a_5)\) is matched to \((p_1, p_2)\) at any point of the algorithm then \((a_1, a_5)\) must remain matched there subsequently, so the unique stable matching (where this couple is matched to \((p_5, p_6)\)) cannot be reached. Furthermore, in each variant of Algorithms C and RP, no agent becomes matched to his/their second choice before having been rejected by his/their first choice. Suppose for a contradiction that \((a_1, a_5)\) is matched to \((p_3, p_6)\) on termination of one of these algorithms. It must be the case that \((a_1, a_5)\) has already been rejected by \((p_1, p_2)\), and at that point, say at time \(t_1\), \(p_1\) and \(p_2\) must have been occupied by \(a_2\) and \(a_4\), respectively. Let \(t_2\) denote the time when \((a_2, a_4)\) became matched to \((p_1, p_2)\) for the first time. Since \((p_1, p_2)\) is the second choice of \((a_2, a_4)\) then they must have been rejected by their first choice, \((p_2, p_6)\), at an earlier moment, say at \(t_3\). At that point \(p_2\) must have been occupied by \(a_3\). Since \(p_5\) is \(a_3\)'s second choice, he must have been rejected by \(p_1\) even earlier, say at \(t_4\), when \(p_1\) must have been occupied by \(a_1\) (and \(p_4\) by \(a_2\)), because these positions could not have been occupied by \(a_2\) and \(a_4\) (according to our assumption that they first get matched there at \(t_2 > t_4\)), a contradiction. By using a similar argument we can also show that none of the variants of Algorithm BB can reach the unique stable matching, since here no agent becomes matched to his/their second choice if he/she form a blocking pair with his/their first choice.
• BB-SCO: best blocker with highest scoring applicant chosen at each step (with score of a couple equal to the lower of the scores of its members);
• BB-USE: best blocker chosen according to usage;
• BB-USS: best blocker chosen according to usage, but with single-blockers prioritised;
• BB-SGL: random best single-blocker chosen if there is one, otherwise random best couple-blocker;
• BB-CPL: random best couple-blocker chosen if there is one, otherwise random best single-blocker.

Algorithm RP
• RP-RAN: random agent introduced at each step;
• RP-SGL: random agent introduced at each step but singles first;
• RP-CPL: random agent introduced at each step but couples first.

In this empirical study, we focused entirely on the case of a strictly ordered master list. We generated example problem instances with four different numbers of applicants – 100, 500, 1000 and 2000. For each of these sizes $n$ we varied the number of linked applicants from around 5% of the total up to $n$; this was done by using our random instance generator to create a set of 1000 base instances of size $n$ with no ties, and then randomly pairing together more and more of the applicants until all applicants were in couples. In all cases, we set the number of programmes to be one tenth of the number of applicants, the number of places, distributed randomly among the programmes, to be equal to the number of applicants, and the length of each applicant’s preference list to be (somewhat arbitrarily) six. Applicants were given random distinct scores.

Programs were allowed a fixed maximum running time on each instance – 1 second, 5 seconds, 10 seconds and 20 seconds for the instances of size 100, 500, 1000 and 2000 respectively – a failure message being output if no stable matching was found within that time. The main output from each program run was the number of instances, out of 1000, for which a stable matching was found within the allocated time. Of course, even if all program variants fail on a particular instance, this need not necessarily be because a stable matching does not exist. Our conclusions can only be in terms of how feasible it is to find a stable matching, not how likely it is that one exists. In addition for each data set, we aggregated the number of instances that were solved by at least one of the algorithm variants.

Tables 1, 2, 3 and 4 report the numbers of instances of sizes 100, 500, 1000 and 2000, respectively, that were solved, out of a total of 1000 instances in each case, using the selected variants of Algorithms C, BB and RP. Only the variants of the algorithms that were competitive on the smaller instances were run on instances of size 2000, to reduce to manageable proportions the total running time of the programs.

In each column of these tables, the most successful algorithm variant(s), in terms of the number of instances solved, is/are highlighted in bold. The final row of each table reports how many of the total of 1000 instances were solved by at least one of the algorithm variants, giving a lower bound on the number of these instances that do admit a stable matching.

---

3 All programs were written in Java and were run on a PC with a 2.84 GHz processor, and with 3.5 GB of RAM, running Windows XP.
<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Number of couples</th>
</tr>
</thead>
<tbody>
<tr>
<td>C-RAN</td>
<td>981 965 909 870 827 801 740 648 604 529 453</td>
</tr>
<tr>
<td>C-STA</td>
<td>978 937 831 753 676 640 605 531 508 470 407</td>
</tr>
<tr>
<td>C-SGL</td>
<td>984 952 907 862 822 801 750 685 627 545 446</td>
</tr>
<tr>
<td>C-CPL</td>
<td>974 927 821 758 712 681 646 586 554 506 431</td>
</tr>
<tr>
<td>C-RLP</td>
<td>968 920 825 708 555 424 288 193 136 84 49</td>
</tr>
<tr>
<td>BB-RAN</td>
<td>983 966 916 882 851 829 772 701 621 530 430</td>
</tr>
<tr>
<td>BB-SCO</td>
<td>968 922 819 722 604 527 444 396 248 172 107</td>
</tr>
<tr>
<td>BB-USE</td>
<td>982 962 911 872 839 816 773 705 662 591 507</td>
</tr>
<tr>
<td>BB-USU</td>
<td>968 929 863 805 751 714 686 659 647 582 507</td>
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<tr>
<td>BB-SGL</td>
<td>968 931 864 819 779 749 716 699 654 553 429</td>
</tr>
<tr>
<td>BB-CPL</td>
<td>981 952 843 867 563 496 410 344 325 329 425</td>
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<tr>
<td>RP-RAN</td>
<td>929 841 704 601 501 411 384 339 274 256 228</td>
</tr>
<tr>
<td>RP-SGL</td>
<td>975 925 796 705 613 536 477 394 336 280 211</td>
</tr>
<tr>
<td>RP-CPL</td>
<td>917 842 603 586 489 407 358 304 266 222 220</td>
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<tr>
<td>Total</td>
<td>984 967 921 888 861 848 825 793 769 728 672</td>
</tr>
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</table>

Table 1: Instances of size 100 (1 second per instance)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Number of couples</th>
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<tbody>
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<td>976 958 908 862 811 729 586 352 163 40 5</td>
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<tr>
<td>C-STA</td>
<td>965 925 809 745 660 588 481 331 191 41 10</td>
</tr>
<tr>
<td>C-SGL</td>
<td>965 957 904 861 801 752 677 504 244 61 4</td>
</tr>
<tr>
<td>C-CPL</td>
<td>964 908 804 767 709 680 426 253 122 30 5</td>
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<td>C-RLP</td>
<td>962 922 805 546 271 92 19 0 0 0 0</td>
</tr>
<tr>
<td>BB-RAN</td>
<td>976 958 911 870 800 655 412 169 54 14 0</td>
</tr>
<tr>
<td>BB-SCO</td>
<td>928 914 793 663 498 342 230 122 65 20 8</td>
</tr>
<tr>
<td>BB-USE</td>
<td>976 957 909 867 799 696 501 254 81 27 4</td>
</tr>
<tr>
<td>BB-USU</td>
<td>963 934 880 825 764 716 682 546 281 71 4</td>
</tr>
<tr>
<td>BB-SGL</td>
<td>963 934 879 828 775 720 680 529 232 44 0</td>
</tr>
<tr>
<td>BB-CPL</td>
<td>974 943 783 482 215 95 25 8 0 1 2</td>
</tr>
<tr>
<td>RP-RAN</td>
<td>888 771 579 453 320 188 119 67 35 16 4</td>
</tr>
<tr>
<td>RP-SGL</td>
<td>952 897 701 547 395 277 170 83 41 9 3</td>
</tr>
<tr>
<td>RP-CPL</td>
<td>972 778 583 424 306 181 115 63 28 11 1</td>
</tr>
<tr>
<td>Total</td>
<td>976 958 911 871 820 775 739 642 401 145 29</td>
</tr>
</tbody>
</table>

Table 2: Instances of size 500 (5 seconds per instance)

<table>
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<th>Algorithm</th>
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<td>976 956 910 865 820 761 655 533 35 7 0</td>
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<td>C-STA</td>
<td>961 906 766 672 594 440 238 90 23 1 0</td>
</tr>
<tr>
<td>C-SGL</td>
<td>975 956 908 866 818 786 675 362 101 6 0</td>
</tr>
<tr>
<td>C-CPL</td>
<td>955 913 802 752 707 525 242 69 11 0 0</td>
</tr>
<tr>
<td>C-RLP</td>
<td>968 923 789 376 89 13 0 0 0 0 0 0</td>
</tr>
<tr>
<td>BB-RAN</td>
<td>976 956 914 871 772 424 100 11 3 0 0</td>
</tr>
<tr>
<td>BB-SCO</td>
<td>963 916 785 635 492 324 173 97 32 7 2</td>
</tr>
<tr>
<td>BB-USE</td>
<td>976 953 925 886 782 473 140 29 5 0 0</td>
</tr>
<tr>
<td>BB-SGL</td>
<td>968 934 878 824 775 635 296 45 0 0 0</td>
</tr>
<tr>
<td>BB-CPL</td>
<td>969 934 745 395 80 7 0 0 0 0 0 0</td>
</tr>
<tr>
<td>RP-RAN</td>
<td>879 770 562 376 251 167 66 27 3 1 0</td>
</tr>
<tr>
<td>RP-SGL</td>
<td>948 863 695 503 326 230 103 33 6 1 0</td>
</tr>
<tr>
<td>RP-CPL</td>
<td>879 780 559 425 256 140 64 12 2 0 0</td>
</tr>
<tr>
<td>Total</td>
<td>976 956 914 871 830 799 714 469 151 12 2</td>
</tr>
</tbody>
</table>

Table 3: Instances of size 1000 (10 seconds per instance)
Table 4: Instances of size 2000 (20 seconds per instance)

In cases where a stable matching is not found, the variants of Algorithm BB can provide potentially useful information not readily available from Algorithms C and RP, namely how near they came to finding a stable solution. It is not unreasonable to regard a matching with the smallest number of best blockers as being one that is closest to stability – since the minimum number of best blockers is the same as the minimum number of agents involved in blocking pairs.

Table 5 shows the average and maximum, taken over all instances that were not solved by any of the algorithms, of the minimum number of best blockers found by any of variants of Algorithm BB. This gives an indication of how close, in general, Algorithm BB came to finding a stable matching in cases where none of the algorithms was able to find one. The clear messages is that, except when a high proportion of the applicants are in couples, we can either find a stable matching, or come very close to one, in the vast majority of the instances that we studied.

One factor that could significantly affect the chances of a stable solution and the performance of the various algorithms is the density of the compatibility matrix used to form couples’ preference lists. The experiments reported above used a compatibility probability of 0.75, so to investigate this effect we repeated a subset of the simulations using a lower compatibility probability of 0.3. Table 6 gives the results of running the algorithms on the instances of size 100, but using that lower compatibility probability.

It is obvious that allowing more time for any particular algorithm variant might increase the number of instances solved. To obtain a feel for this, we re-ran all of the experiments on instances of size 100 but allowing 10 seconds per instance rather than 1 second, and a subset of the experiments, for the most successful algorithm variants, on instances of size 500 but allowing 50 seconds per instance rather than 5 seconds. The results are shown in Tables 7 and 8.

Finally, we conducted a much smaller-scale study of larger instances, of sizes 10000, 20000 and 30000 (the latter of the order of magnitude of NRMP instances), to give an indication as to whether the conclusions drawn from studying smaller instances would
Table 6: Instances of size 100 with 0.3 compatibility (1 second per instance)

<table>
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<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
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<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
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</thead>
<tbody>
<tr>
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<td>884</td>
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<td>645</td>
<td>574</td>
<td>521</td>
<td>431</td>
</tr>
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<td>C-STA</td>
<td>979</td>
<td>917</td>
<td>812</td>
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<td>462</td>
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<td>794</td>
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<td>695</td>
<td>660</td>
<td>589</td>
<td>532</td>
<td>431</td>
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<td>607</td>
<td>576</td>
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<td>435</td>
</tr>
<tr>
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<td>792</td>
<td>750</td>
<td>692</td>
<td>660</td>
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<td>522</td>
<td>410</td>
</tr>
<tr>
<td>BB-SSG</td>
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<td>877</td>
<td>815</td>
<td>794</td>
<td>741</td>
<td>695</td>
<td>660</td>
<td>589</td>
<td>532</td>
<td>431</td>
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<tr>
<td>RP-CPL</td>
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<td>431</td>
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</table>

Table 7: Instances of size 100 (10 seconds per instance)

<table>
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<tr>
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<th>10</th>
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<td>737</td>
<td>686</td>
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<td>432</td>
</tr>
<tr>
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<td>906</td>
<td>861</td>
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<td>782</td>
<td>737</td>
<td>686</td>
<td>632</td>
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<td>502</td>
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<td>674</td>
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</table>

Table 8: Instances of size 500 (50 seconds per instance)

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<td>654</td>
<td>553</td>
<td>453</td>
<td>353</td>
<td>253</td>
</tr>
</tbody>
</table>

Table 8: Instances of size 500 (50 seconds per instance)
continue to hold in such cases. Table 9 summarises the numbers of instances solved for the cases of size 30000, where just 10 instances in each set were involved. Note that here, each algorithm was allowed 30 seconds execution time on each instance, but, with only a handful of exceptions, the solved instances were, in practice, solved in under 1 second. The pattern of solved instances was similar for problem sizes 10000 and 20000.

<table>
<thead>
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<th>3000</th>
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<th>7500</th>
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<td>4</td>
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<td>0</td>
</tr>
<tr>
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<td>9</td>
<td>6</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>Total</td>
<td>10</td>
<td>9</td>
<td>10</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>5</td>
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<td>0</td>
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</tr>
</tbody>
</table>

Table 9: Instances of size 30000 solved by various algorithms (30 seconds per instance)

This smaller scale study of instances of larger size confirms the general trends discernable for smaller instances. When the proportion of couples is small, the likelihood of finding a stable solution remains high. However, as the ratio of couples to single applicants increases the decrease in the proportion of solved instances becomes steeper, reinforcing the belief that the absolute number of couples, and not just the proportion, is significant in this respect. For instances of size 10000 or greater, none of the algorithms was able to solve a single instance in which more than 60% of the applicants were in couples. The other notable feature for larger instances is the prominence of Algorithm C-SGL; this variant dominated all of the others in the sense that every single instance of size 10000 or more that was solved by at least one of the algorithm variants was solved by this particular one.

7 Discussion and conclusions

In this section we make a number of observations, on the basis of the empirical study, regarding the effectiveness of the various algorithms and the question of the existence of stable matchings.

Algorithm C

Variants C-RAN (random waiting list) and C-SGL (random waiting list subject to prioritising single applicants) are the most successful, the former generally being superior when the number of linked applicants is up to about 40% of the total, and the latter when this ratio is exceeded. However C-SGL appears to become pre-eminent in the case of larger instances. Use of a stack waiting list is generally less successful (except for a curious unexplained outlier in the 500 data set when all applicants are in couples) – this is a significant observation in view of the tendency for implementations of Gale-Shapley like algorithms to organise applicants in a stack (as in the algorithm described by Klaus et al [5]). Prioritising the review list appears to be a bad strategy.

Algorithm BB

The relative success rate of the variants of Algorithm BB is very much dependent on the instance size and number of couples, but the effectiveness of this conceptually simple
approach, at least when the proportion of couples is relatively small, came as something of a surprise. It is curious how BB-USE is better than BB-USS on data sets of size 100, but BB-USS is markedly better on the larger data sets. Also strange is the fact that BB-CPL consistently outperforms BB-SGL when the number of couples is very small, but otherwise BB-SGL is greatly superior. BB-RAN is never beaten when the proportion of linked applicants is small (up to 30% of applicants). On larger data sets with many couples BB-USS and BB-SGL fare much better than the other variants, so it seems to be important to prioritise single applicants. Prioritising by score seems to be a poor strategy (perhaps because it encourages cyclic behaviour) – but again there are curious outliers in the 500 and 1000 data sets when all applicants are in couples.

**Algorithm RP**

It appears that our version of the Roth-Peranson algorithm is generally not competitive with the other algorithms. This may not be entirely surprising, since in this approach stability has to be achieved over and over again as each successive agent is admitted. RP-SGL (single applicants admitted first) is consistently the best of the three variants, confirming the findings reported in the original paper of Roth and Peranson [12].

**Overall comparison**

Variants of Algorithm BB – BB-RAN and BB-USE in particular – are the clear winners on data sets of size 100. On larger data sets, variants of Algorithm C – C-RAN and C-SGL in particular – are competitive, and the latter variant clearly outperforms any variant of Algorithm BB as the proportion of couples grows, and indeed is unambiguously the most effective variant for larger instances. The general conclusion is that a number of algorithm variants should be available to maximise the chances of finding a stable solution for any particular instance.

**General**

When the proportion of couples is low, the best algorithms solve all, or almost all, of the instances that can be solved by any of the algorithms. By contrast, for higher proportions of couples, the total number of instances solved, aggregated over all of the algorithm variants, can be substantially greater than the number solved by any one algorithm. An increase in both the number of couples and the proportion of couples makes it harder to find a stable matching, but we don’t have the evidence to judge to what extent this is because a stable matching is less likely to exist.

**Additional observations**

Running the various algorithms on numerous data sets revealed a number of additional interesting facts. For example, just as the number of solved instances decreased with an increasing couples to singles ratio, so the average number of proposal steps for the solved instances increased. For small number of couples, and even for the less effective algorithms, it appeared that most instances that were solved at all were solved very quickly. However, for larger numbers of couples many of the solved instances required a large number of proposals.

When the compatibility probability is reduced, so that couples’ preference lists are typically shorter, the numbers of instances for which a stable solution was found were generally somewhat reduced. The overall pattern of results was somewhat similar to the earlier case. However, the successful variants of Algorithm C seemed to be less affected than the successful variants of Algorithm BB, and the variants of Algorithm RP seemed to be most severely affected. Somewhat bizarrely, and inexplicably, the least effective of all of the variants, Algorithms C-RLP and BB-SCO, improve when the compatibility probability is lower. The use of a relatively high compatibility probability in our main experiments can be justified by the fact that, in practice, linked applicants tend to submit
highly correlated preference lists, typically focusing on one geographical region in which a
high proportion of pairs of programmes are compatible.

When the algorithms are allowed more time to find stable matchings, there is very
little change in the results pattern for low or moderate numbers of couples. Indeed, in
the trials that we conducted, few if any additional instances of this kind were solved when
the time available was increased by a factor of 10. In one or two cases, the shorter runs
actually solved more such instances (presumably because in just one or two instances a
‘lucky’ sequence of random choices was made). For larger numbers of couples, there were
some significant increases in the numbers of instances solved. With one notable exception,
this trend of improvement was similar for all variants, though slightly more accentuated
for variants of Algorithm C when the number of couples was very high. The exception
was Algorithm BB-SCO, where the results, in all cases, were completely unchanged when
extra time was allowed, revealing the fact that failure to find a stable matching in this case
is the result of cyclic behaviour that appears to be inevitable in any particular instance
because of the fixed order in which best blockers are chosen.

More general contexts
Although the results and observations presented above are based on SHRC, the particular
version of the HRC problem that is relevant in our application domain, we believe that they
are indicative of the likely behaviour of the various algorithms were they to be tailored for
more general settings, where both programmes and couples have the freedom to form their
own preference lists. We note that our definition of stability is appropriate for this more
general context provided that the concept of superiority is replaced by that of position
within individual programme preference lists. Each of algorithms C, BB and RP can easily
be amended to handle the more general problem, except that, in the case of Algorithms
C and BB, Phase 1 is no longer applicable and Phase 2 therefore starts with an empty
matching.

Our empirical study was based exclusively on instances with a strictly ordered master
list. If, in a practical setting such as SFAS, the master list has ties, then these can, of
course, be broken arbitrarily to produce a strictly ordered list. However, the prospects of
finding a stable solution are even greater in this case, since different instances of SHRC
can be created by breaking ties in different ways, and failure to find a solution for one
such instance need not imply failure for another.

References


[5] B. Klaus, F. Klijn, and J. Massó. Some things couples always wanted to know about
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