Sequent Calculus for Euler Diagrams*

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Abstract. Proof systems play a major role in the formal study of diagrammatic logical systems. Typically, the style of inference is not directly comparable to traditional sentential systems, to study the the diagrammatic aspects of inference. In this work, we present a proof system for Euler diagrams with shading in the style of sequent calculus. We prove it to be sound and complete. Furthermore we outline how this system can be extended to incorporate heterogeneous logical descriptions. Finally, we explain how small changes allow for reasoning with intuitionistic logic.

Keywords: Euler diagrams · proof systems · heterogeneous reasoning

1 Introduction

Starting from the early work on formal diagrammatic systems, the analysis of proof systems has played a major role. For example, in the seminal work of Shin [10], she developed a proof system for each system of Venn-diagrams she defined, and proved each to be sound and complete. Unsurprisingly, comparing typical rules for Euler and Venn-diagrams with sentential rules is hard. This is mainly due to two reasons. On the one hand, the former rules are inherently diagrammatic in nature and are often not directly comparable to sentential rules. For example, *introducing* a new contour into an Euler diagram is defined such that the logical information in the diagram is not affected. That is, from a logical perspective, the original diagram and the changed one are equivalent. While such changes are at least unusual for sentential transformations, diagrammatic proof systems make considerable use of equivalent transformations. On the other hand, proofs for Euler diagrams or Spider diagrams are defined as a linear progression from the assumptions to the conclusion [10,6], while sentential proofs are most of the time defined as proof-trees, where an application of a rule may split the current proof state into branches, e.g., in systems of natural deduction or sequent calculus [9]. To the best of our knowledge, the only direct comparison between diagrammatic inference systems and sentential reasoning styles was conducted by Mineshima et al. [8]. They analysed proof systems for two diagrammatic languages: Euler diagrams without shading and Venn-Diagrams, and showed, how the former relates to natural deduction, and the latter to resolution calculus.

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In this work, we present a proof system for Euler diagrams with shading in the style of sequent calculus [5]. We prove this system to be sound and complete. Furthermore, we explain how simple amendments allow us to create a system for a heterogeneous language of Euler diagrams and propositional logic.

This paper is structured as follows. In Sect. 2, we give a short definition of Euler diagrams and the semantics we use. Section 3 contains the definition of the calculus and the proofs for soundness and completeness, while we discuss extensions and relations to other systems and conclude in Sect. 4.

2 Euler Diagrams

An Euler diagram consists of a set of *contours*, dividing the space enclosed by a bounding rectangle into different, possibly shaded zones (see Fig. 1 for an example). Traditionally, each contour represents a set, and the diagram restricts the possible relations between these sets. We take a slightly different approach: contours represent propositional variables, taken from a countably infinite set Vars. A *zone* for a finite set of contours $L \subset \mathsf{Vars}$ is a tuple $(\mathsf{in}, \mathsf{out})$, where in and out are disjoint subsets of L such that $\mathsf{in} \cup \mathsf{out} = L$. The set of all zones for a given set of contours is denoted by $\mathsf{Venn}(L)$.

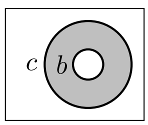


Fig. 1. Euler diagram

Definition 1 (Abstract Syntax). *A* unitary *Euler diagram is a tuple $d = (L, Z, Z^*)$, where Z and Z^* are sets of zones for L such that $Z^* \subseteq Z$, $(\emptyset, L) \in Z$, and for each $c \in L$, there is a zone $(\mathsf{in}, \mathsf{out}) \in Z$, such that $c \in \mathsf{in}$. The set Z denotes the* visible *and Z^* the* shaded zones *of d. For a unitary diagram d, we will also refer to the set of its* missing zones $\mathsf{MZ}(d) = \mathsf{Venn}(L) \setminus Z$. *The syntax of Euler diagrams is then given as $D ::= d \mid D \to D$, where d is unitary. Euler diagrams of the form $D_1 \to D_2$ are* compound.

We allow the diagrams $\top = (\emptyset, \{(\emptyset, \emptyset)\}, \emptyset)$ and $\bot = (\emptyset, \{(\emptyset, \emptyset)\}, \{(\emptyset, \emptyset)\})$. We define negation by $\neg D \equiv D \to \bot$ and the missing connectives $D_1 \vee D_2 \equiv \neg D_1 \to D_2$, etc. A *literal* is a unitary diagram for a single contour, with exactly one shaded zone. If the zone $(\emptyset, \{c\})$ is shaded in a literal, then we call it *positive*, otherwise it is *negative* (see Fig. 2). Observe that our notion of literals slightly deviates from the original definition of Stapleton and Masthoff [11].

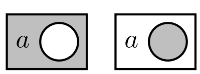


Fig. 2. Literals

Definition 2 (Semantics). *A* valuation *is a function $\nu \colon \mathsf{Vars} \to \mathbb{B}$, where $\mathbb{B} = \{\mathsf{tt}, \mathsf{ff}\}$ is the set of Boolean values. We denote the set of all valuations by Vals. Let $z = (\mathsf{in}, \mathsf{out})$ be a zone. The* semantics *of z is a subset of Vals, given by $[\![z]\!] = \{\nu \mid \forall c \in \mathsf{in} \colon \nu(c) = \mathsf{tt}$ and $\forall c \in \mathsf{out} \colon \nu(c) = \mathsf{ff}\}$. The semantics of Euler diagrams is then $[\![d]\!] = \bigcup_{z \in Z \setminus Z^*} [\![z]\!]$ and $[\![D_1 \to D_2]\!] = (\mathsf{Vals} \setminus [\![D_1]\!]) \cup [\![D_2]\!]$, where d is unitary and D_1, D_2 are arbitrary Euler diagrams. If $[\![D]\!] = \mathsf{Vals}$, then we call D* valid*, denoted by $\models D$. Otherwise, D is* falsifiable.

Note that $[\![\top]\!] = \mathsf{Vals}$ and $[\![\bot]\!] = \emptyset$, as well as $[\![D_1 \vee D_2]\!] = [\![D_1]\!] \cup [\![D_2]\!]$ and $[\![D_1 \wedge D_2]\!] = [\![D_1]\!] \cap [\![D_2]\!]$. Furthermore, the semantics of a positive literal for the contour c consists of the valuations with $\nu(c) = \mathsf{tt}$.

Definition 3 (Adjacent Zone). *Let $z = (\mathsf{in}, \mathsf{out})$ be a zone for the contours in L and $c \in L$. The* zone adjacent to z at c, *denoted by $\overline{z}^c$ is $(\mathsf{in} \cup \{c\}, \mathsf{out} \setminus \{c\})$, if $c \in \mathsf{out}$ and $(\mathsf{in} \setminus \{c\}, \mathsf{out} \cup \{c\})$ if $c \in \mathsf{in}$.*

Now we can define a way to remove contours from a unitary diagram d.

Definition 4 (Reduction). *Let $d = (L, Z, Z^*)$ be a unitary Euler diagram and $c \in L$. The* reduction *of a zone $z = (\mathsf{in}, \mathsf{out})$ is defined by $z \setminus c = (\mathsf{in} \setminus \{c\}, \mathsf{out} \setminus \{c\})$. The* reduction *of d by c is defined as $d \setminus c = (L \setminus \{c\}, Z \setminus c, Z^* \setminus c)$, where*

$$Z \setminus c = \{z \setminus c \mid z \in Z\}$$
$$Z^* \setminus c = \{z \setminus c \mid z \in Z^* \text{ and } \overline{z}^c \in Z^* \cup \mathsf{MZ}(d)\}$$

That is, we remove the contour c from all zones and only shade the reduction of a shaded zone z, if its adjacent zone at c is shaded or missing.

If each shaded or missing zone in a diagram d has a shaded or missing adjacent zone, then the conjunction of the reduction of d by each of its contours preserves the semantic information. That is, we can distribute the information contained in d among simpler diagrams.

Lemma 1. *Let $d = (L, Z, Z^*)$, where for each $z \in Z^* \cup \mathsf{MZ}(d)$, there is a contour $\ell \in L$ such that $\overline{z}^\ell \in Z^* \cup \mathsf{MZ}(d)$. Then $[\![d]\!] = \bigcap_{c \in L} [\![d \setminus c]\!]$*

Proof. For each $c \in L$, we have $[\![d]\!] \subseteq [\![d \setminus c]\!]$. Hence the direction from left to right is immediate. Now let $d = (L, Z, Z^*)$ and ν be such that $\nu \in [\![d \setminus c]\!]$ for all $c \in L$. Hence, for each c, there is a $z_c \in Z$, such that $\nu \in [\![z_c \setminus c]\!]$. Now we have to show that in fact there is a *single* zone $z \in Z$, such that $\nu \in [\![z \setminus c]\!]$ for all c. Observe that there are two zones $z_{\mathsf{tt}}, z_{\mathsf{ff}} \in \mathsf{Venn}(d)$ such that $\nu \in [\![z_{\mathsf{tt}} \setminus c]\!]$ and $\nu \in [\![z_{\mathsf{ff}} \setminus c]\!]$, whose only difference is that c is in the in-set of z_{tt} and in the out-set of z_{ff}. Now, assume $\nu(c) = \mathsf{tt}$, hence $\nu \in [\![z_{\mathsf{tt}}]\!]$. If $\nu \notin [\![d]\!]$, this means that $z_{\mathsf{tt}} \in Z^* \cup \mathsf{MZ}(d)$. By assumption, there is a contour ℓ such that $\overline{z_{\mathsf{tt}}}^\ell \in Z^* \cup \mathsf{MZ}(d)$. In the reduction of d by ℓ, this means that z_{tt} is either shaded or missing as well, and hence $\nu \notin [\![d \setminus \ell]\!]$, which contradicts the assumption on ν. Hence $\nu \in [\![d]\!]$. The case for $\nu(c) = \mathsf{ff}$ is similar. □

3 Sequent Calculus

Sequent calculus, as defined by Gentzen [5] is closely related to natural deduction. It is based on *sequents*, which are decomposed by rule applications.

Definition 5 (Sequent). *A* sequent *$\Gamma \Rightarrow \Delta$ consists of two multisets Γ and Δ of Euler diagrams. The multiset Γ is called the* antecedent *and Δ the* succedent*.*

If Γ (Δ) is the empty multiset, we write $\Rightarrow \Delta$ ($\Gamma \Rightarrow$, respectively). If a sequent is of the form $\Gamma, l \Rightarrow \Delta, l$, where l is a positive literal, or $\Gamma, \bot \Rightarrow \Delta$, or $\Gamma \Rightarrow \Delta, \top$ then it is called an axiom*. A sequent $D_1, \ldots, D_k \Rightarrow E_1, \ldots, E_n$ is equivalent to $(D_1 \wedge \cdots \wedge D_k) \rightarrow (E_1 \vee \cdots \vee E_n)$. The notions of validity and falsifiability carry over from the semantics of Euler diagrams.*

$$\frac{\Gamma, D, E \Rightarrow \Delta}{\Gamma, D \wedge E \Rightarrow \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma, D \Rightarrow \Delta \quad \Gamma, E \Rightarrow \Delta}{\Gamma, D \vee E \Rightarrow \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, D \quad \Gamma, E \Rightarrow \Delta}{\Gamma, D \rightarrow E \Rightarrow \Delta}\,\mathrm{L}\rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta, D \quad \Gamma \Rightarrow \Delta, E}{\Gamma \Rightarrow \Delta, D \wedge E}\,\mathrm{R}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, D, E}{\Gamma \Rightarrow \Delta, D \vee E}\,\mathrm{R}\vee \qquad \frac{\Gamma, D \Rightarrow \Delta, E}{\Gamma \Rightarrow \Delta, D \rightarrow E}\,\mathrm{R}\rightarrow$$

Fig. 3. Proof Rules for Boolean Operators

A *deduction* for a sequent $\Gamma \Rightarrow \Delta$ is a tree, where the root is labelled by $\Gamma \Rightarrow \Delta$, and the children of each node are labelled according to the rules defined below. A deduction where each leaf is labelled with an axiom is called a *proof* for $\Gamma \Rightarrow \Delta$. We denote the existence of a proof for $\Gamma \Rightarrow \Delta$ by $\vdash \Gamma \Rightarrow \Delta$. Intuitively, the prover tries to refute the sequent, i.e., she tries to find a valuation that satisfies all diagrams in the antecedent and falsifies every diagram in the succedent. If all possible ways to find such a valuation fail, i.e., each branch ends with an axiomatic sequent, then the diagram is valid. For proof search, it is beneficial to apply the rules backwards, that is from bottom to top.

Lemma 2. *A sequent containing only positive literals is valid iff it is an axiom.*

Proof. The right to left direction is immediate. Now let $d_1, \ldots, d_k \Rightarrow e_1, \ldots, e_n$ be valid, where each d_i and e_j is a positive literal, and assume it is not an axiom. Hence, for no i and j, we have that $d_i = e_j$. Then the valuation ν with $\nu(d_i) = \mathsf{tt}$ and $\nu(e_j) = \mathsf{ff}$ falsifies the sequent, which contradicts our assumption. □

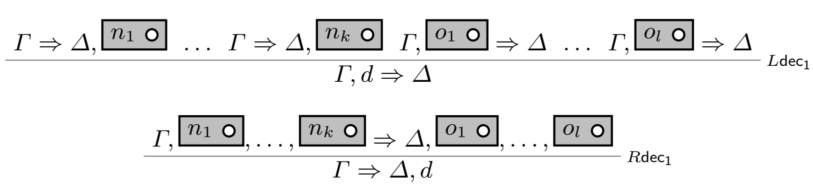
The rules to treat compound diagrams, as shown in Fig. 3, are directly taken from sequent calculus for propositional logic and are sound [9].

Lemma 3 (Soundness). *The rules for Boolean operators are sound.*

$$\frac{\Gamma, d_1, d_2 \Rightarrow \Delta}{\Gamma, d \Rightarrow \Delta}\,L\mathsf{s} \qquad \frac{\Gamma, d^{z} \Rightarrow \Delta}{\Gamma, d \Rightarrow \Delta}\,L\mathsf{MZ} \qquad \frac{\Gamma, d \setminus c_1, \ldots, d \setminus c_k \Rightarrow \Delta}{\Gamma, d \Rightarrow \Delta}\,L\mathsf{r}$$

$$\frac{\Gamma \Rightarrow \Delta, d_1 \quad \Gamma \Rightarrow \Delta, d_2}{\Gamma \Rightarrow \Delta, d}\,R\mathsf{s} \qquad \frac{\Gamma \Rightarrow \Delta, d^{z}}{\Gamma \Rightarrow \Delta, d}\,R\mathsf{MZ} \qquad \frac{\Gamma \Rightarrow \Delta, d \setminus c_1 \;\ldots\; \Gamma \Rightarrow \Delta, d \setminus c_k}{\Gamma \Rightarrow \Delta, d}\,R\mathsf{r}$$

(a) (b) (c)



(d)

Fig. 4. Proof Rules to Decompose Unitary Diagrams

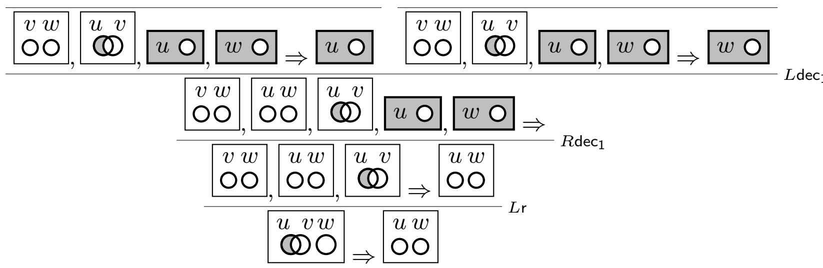


Fig. 5. Example of a Deduction

Let $d = (L, Z, Z^*)$ with $|Z^*| > 1$, and let $d_i = (L, Z, Z_i^*)$, for $i \in \{1, 2\}$, such that $Z^* = Z_1^* \cup Z_2^*$. Then the rules $L\mathsf{s}$ and $R\mathsf{s}$ shown in Fig. 4a *separate* d. These rules are closely related to the *Combine* equivalence rule for Spider diagrams [6].

For $d = (L, Z, Z^*)$ with $|\mathsf{MZ}(d)| > 0$ and $z \in \mathsf{MZ}(d)$, let $d^z = (L, Z \cup \{z\}, Z^* \cup \{z\})$. The rules $L\mathsf{MZ}$ and $R\mathsf{MZ}$ in Fig. 4b *introduce the missing zone* z.

Now let $d = (L, Z, Z^*)$, where for each $z \in Z^* \cup \mathsf{MZ}(d)$ there is a contour $\ell \in L$, such that $\overline{z}^\ell \in Z^* \cup \mathsf{MZ}(d)$. Let $L = \{c_1, \ldots, c_k\}$. Then we can *reduce* d according to the rules $L\mathsf{r}$ and $R\mathsf{r}$ shown in Fig. 4c.

Finally, let $d = (L, Z, Z^*)$, where d is not a positive literal or $\bot$, and either $|Z^*| = 1$ and $|\mathsf{MZ}(d)| = 0$ or $|\mathsf{MZ}(d)| = 1$ and $|Z^*| = 0$. Let $z = (\mathsf{in}, \mathsf{out})$ be the corresponding shaded or missing zone, where $\mathsf{in} = \{n_1, \ldots, n_k\}$ and $\mathsf{out} = \{o_1, \ldots, o_l\}$. Then the rules $L\mathsf{dec}_1$ and $R\mathsf{dec}_1$ (see Fig. 4d) decompose d into positive literals.

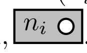
An example of a proof is shown in Fig. 5. In the applications of $L\mathsf{dec}_1$ and $R\mathsf{dec}_1$, the diagram denoting the disjointness of u and w is decomposed on the left side (right side, resp.) of the sequent. The application of $L\mathsf{r}$ is possible, since for each shaded or missing zone z, there is a contour c such that $\overline{z}^c$ is also shaded or missing. E.g., consider $z = (\{u\}, \{v, w\})$. Then $\overline{z}^w = (\{u, w\}, \{v\})$ is missing. Hence, in the reduction of the diagram by w, the zone $(\{u\}, \{v\})$ is also shaded. That is, we can decompose a complex diagram into simpler diagrams, whose conjunction comprises the same information as the original.

Lemma 4. *The conclusion of each rule in Fig. 4 is falsifiable, if and only if, at least one of its premises is falsifiable.*

Proof. First we consider $L\mathsf{s}$. Let $d = (L, Z, Z^*)$, where $|Z^*| > 1$, and d_1, d_2 be as required for an application of $L\mathsf{s}$. Furthermore, let ν be a valuation that falsifies $\Gamma, d \Rightarrow \Delta$, i.e., ν satisfies Γ and d, and falsifies Δ. Since $Z^* = Z_1^* \cup Z_2^*$, we have $[\![d]\!] = [\![d_1]\!] \cap [\![d_2]\!]$. Hence ν falsifies $\Gamma, d \Rightarrow \Delta$ if and only if ν falsifies $\Gamma, d_1, d_2 \Rightarrow \Delta$. For $R\mathsf{s}$, let ν falsify $\Gamma \Rightarrow \Delta, d$. That is, ν falsifies d. Since $[\![d]\!] = [\![d_1]\!] \cap [\![d_2]\!]$, this is equivalent to ν falsifying at least one of d_1 and d_2.

The rules $L\mathsf{r}$ and $R\mathsf{r}$ can be proven sound similarly, due to Lemma 1.

For $L\mathsf{dec}_1$ and $R\mathsf{dec}_1$, let $z = (\mathsf{in}, \mathsf{out})$ be the single shaded zone in d (the case for z being missing is similar). Now consider $L\mathsf{dec}_1$. Let ν be a valuation that falsifies $\Gamma, d \Rightarrow \Delta$. Hence, $\nu \in [\![d]\!]$. That is, either $\nu(n) = \mathsf{ff}$ for at least one

$n \in \mathsf{in}$, or $\nu(o) = \mathsf{tt}$ for at least one $o \in \mathsf{out}$. Assume that $\nu(n_i) = \mathsf{ff}$ (the other case is similar). This is equivalent to ν falsifying $\Gamma \Rightarrow \Delta,$ . Consider $R\mathsf{dec}_1$. If ν falsifies $\Gamma \Rightarrow \Delta, d$, then $\nu \notin [\![d]\!]$. Since $[\![d]\!] = \mathsf{Vals} \setminus [\![z]\!]$, we have $\nu \in [\![z]\!]$. That is, $\nu(n) = \mathsf{tt}$ and $\nu(o) = \mathsf{ff}$ for all $n \in \mathsf{in}$ and $o \in \mathsf{out}$. Hence ν falsifying the premiss of $R\mathsf{dec}_1$ is equivalent to ν falsifying $\Gamma \Rightarrow \Delta, d$.

$L\mathsf{MZ}$ and $R\mathsf{MZ}$ are sound, since missing and shaded zones are equivalent. □

From Lemma 3 and 4, we immediately get the necessary soundness theorem.

Theorem 1 (Soundness). $\vdash \Gamma \Rightarrow \Delta$ *implies* $\models \Gamma \Rightarrow \Delta$.

Proof. By induction on the length of proofs, using Lemma 3 and 4. □

To prove completeness, we need to show that each diagram can be decomposed into positive literals. That is, each deduction can be maximised until only positive literals remain. Note that the example in Fig. 5 is not maximal.

Lemma 5. *Every deduction for a sequent $\Gamma \Rightarrow \Delta$ can be extended to a maximal deduction, where all diagrams in each leaf are either positive literals, $\bot$ or $\top$.*

Proof. Assume we have a deduction for $\Gamma \Rightarrow \Delta$, where one of the leaves contains a diagram D, which is not a literal. If D is compound, we use the rules for Boolean operators to decompose D, until we reach a sequent where D is reduced to a set of unitary diagrams (possibly on both the left and the right side of the sequent). Now, let d be such a unitary diagram. If d contains only one shaded or missing zone, then depending on the side on which d appears, we can apply $L\mathsf{dec}_1$ or $R\mathsf{dec}_1$ to decompose d to literals. Otherwise, we have to distinguish two cases. If d contains more than one missing zone, we can apply $L\mathsf{MZ}$ or $R\mathsf{MZ}$ to change them to shaded zones. If d contains more than one shaded zone, we can repeatedly apply $L\mathsf{s}$ or $R\mathsf{s}$ to separete d to diagrams which only contain a single shaded zone. Finally, if d does not contain any shaded or missing zones, we can reduce it by using $L\mathsf{r}$ and $R\mathsf{r}$. We can repeat these steps for every diagram in the leaves for the derivation of $\Gamma \Rightarrow \Delta$. Since each step reduces the number of operators or of missing or shaded zones, this yields a maximal derivation. □

Theorem 2 (Completeness). $\models \Gamma \Rightarrow \Delta$ *implies* $\vdash \Gamma \Rightarrow \Delta$.

Proof. Assume $\models \Gamma \Rightarrow \Delta$. By Lemma 5, we can create a maximal derivation for $\Gamma \Rightarrow \Delta$. Since $\Gamma \Rightarrow \Delta$ is valid, the premises constructed in each step are valid as well, due to Lemmas 3 and 4. Hence the leaves of the deduction tree are valid, and the only valid leaves are axioms by Lemma 2. Accordingly, $\vdash \Gamma \Rightarrow \Delta$. □

4 Discussion

In this section, we compare our calculus with existing proof systems for Euler diagrams and discuss its properties, implications and possible extensions.

Burton et al. analysed strategies for completeness proofs of diagrammatic languages [3]. They emphasise that usually the strategy how to prove an Euler diagram E from the assumptions $D_1, \ldots, D_n$ is to first create a maximal

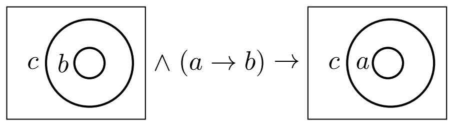


Fig. 6. Heterogeneous Euler Diagrams

diagram D_{max} incorporating the information from all of the D_i. Then, all information that is not part of E will be removed from D_{max}. We do not apply this strategy. Instead, the rules decompose the diagrams, with the only exception being the rules to introduce missing zones. This is due to the similarity of our calculus to typical sentential calculi.

The proof system presented in this paper is related to both systems presented by Mineshima et al. [8]. It is oriented towards refutations, like the resolution calculus for Venn-diagrams, but also contains rules for the connectives and diagrammatic elements, like natural deduction for Euler diagrams. However, our language comprises both Venn-diagrams and Euler diagrams without shading.

We can extend our calculus to facilitate heterogeneous sequents in a rather simple way. We can allow compound diagrams to be mixed with propositional formulas, as for example shown in Fig. 6. The rules for Boolean operators can then be directly applied to propositional formulas. The only extension we need to incorporate into the calculus are heterogeneous axioms.

$$\Gamma, a \Rightarrow \Delta, \boxed{a \circ} \qquad\qquad \Gamma, \boxed{a \circ} \Rightarrow \Delta, a$$

This system then allows us to reason about heterogeneous Euler diagrams. However, it is hardly a heterogeneous *reasoning* system in the sense of Barwise and Etchemendy [2], since it does not include rules to transfer information from one representation into the other.

Furthermore, it is simple to amend our calculus to represent intuitionistic logic instead of classical propositional logic. To that end, we restrict the succedent of sequents to be a single formula, and change the Boolean rules accordingly[1]. For most of the diagrammatic rules, this change is sufficient as well, the only exceptions are $L\mathsf{dec}_1$ and $R\mathsf{dec}_1$. However, we can change these rules as follows.

$$\frac{\Gamma, d \Rightarrow \boxed{n_1 \circ} \;\; \dots \;\; \Gamma, d \Rightarrow \boxed{n_k \circ} \quad \Gamma, \boxed{o_1 \circ} \Rightarrow D \;\; \dots \;\; \Gamma, \boxed{o_l \circ} \Rightarrow D}{\Gamma, d \Rightarrow D}\, L\mathsf{dec}_1{}^{I}$$

$$\frac{\Gamma, \boxed{n_1 \circ}, \dots, \boxed{n_k \circ} \Rightarrow \boxed{o_i \circ}}{\Gamma \Rightarrow d}\, R\mathsf{dec}_1{}^{I}$$

That is, in $L\mathsf{dec}_1{}^{I}$, we keep the diagram in the antecedent for the branches with the new literals in the succedent, while we omit it in the branches, where we add literals to the antecedent. For $R\mathsf{dec}_1{}^{I}$, we choose a single occurrence of a literal to keep in the succedent. These changes are similar to the changes for the Boolean operators. Observe that the semantics presented in Sect. 2 is no longer suited for this proof system. We would have to define a semantics based on intuitionistic models, for example Heyting algebras. However, how such a semantics should

[1] Compare with the textbook by Negri et al. [9]

look like is not obvious. It would be interesting to study the connection of this proof system to traditional proof systems for Euler diagrams, since the graphical notations for intuitionistic logic are sparse. Notable exceptions are the work of de Freitas and Viana [4], defining a graphical calculus for relational reasoning, and Alves et al. [1], in which they present a visualisation of intuitionistic proofs. In a similar way, we could try to change the system to reflect substructural logics, i.e., logics for which the structural rules of weakening, contraction and/or permutation do not hold[2]. However, in these logics, new operators arise and would have to be reflected in the diagrams as well. Such a radical change is not part of this paper, and left as future work. Of course, classical diagrammatic systems are possible ways to extend our calculus as well. A natural next step is an extension to treat Spider diagrams or Constraint diagrams.

A sequent calculus style proof system is suited for automatic proof search. Hence, an implementation into the theorem prover Speedith [12] is obvious future work, since it already supports backward reasoning and several proof branches. Furthermore, extending the tactics within Speedith [7] to our calculus would allow us to delay the application of rules creating new branches in the proof.

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[2] More precisely, these logics are typically *defined* by the lack of these rules.