Non-Induced Subgraph Isomorphism

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Non-Induced Subgraph Isomorphism
The Algorithm

- Recursively build up a mapping from vertices of the pattern graph to vertices of the target graph.

- In constraint programming terms:
  - Forward-checking recursive search.
  - A variable for every pattern vertex.
  - Initially, each domain contains every target vertex.
  - After guessed assignments, infeasible values are eliminated from domains.
    - All-different constraint.
    - Adjacency constraints.
  - If we get a wipeout, we backtrack.
But wait! There’s more!

- Clever filtering at the top of search using neighbourhood degree sequences and paths, to reduce the initial values of domains.
- Pre-computed path count constraints, propagated like adjacency constraints during search.
- Bit-parallel implementation.
  - Weaker than the usual all-different propagator, but much faster.
Benchmark Instances

- 14,621 instances from Christine Solnon’s collection:
  - Randomly generated with different models.
  - Real-world graphs.
  - Computer vision problems.
  - Biochemistry problems.
  - Phase transition instances.

- At least...
  - $\geq 2,110$ satisfiable.
  - $\geq 12,322$ unsatisfiable.

- A lot of them are very easy for good algorithms.
Is It Any Good?

![Graph Showing Number of Instances Solved vs Runtime]

- **Number of Instances Solved**
- **Runtime (ms)**

Legend:
- Glasgow
- PathLAD
- VF2
Is It Any Good?

![Graph showing number of instances solved vs runtime for different datasets]

- Somewhere Exotic (not yet written)
- ESA 2018 (not yet rejected)
- LION 2016 (a)
- CP 2015
- LION 2016 (b)
- AIJ 2010

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Search Order

- Variable ordering (i.e. pattern vertices): smallest domain first, tie-breaking on highest degree.
- Value ordering (i.e. target vertices): highest degree to lowest.
Hand-Wavy Theoretical Justification

- Maximise the expected number of solutions during search?
- If $P = G(p, q)$ and $T = G(t, u)$,

$$\langle \text{Sol} \rangle = t \cdot (t - 1) \cdot \ldots \cdot (t - p + 1) \cdot u^q \cdot \binom{p}{2}$$

  injective mapping \hspace{2cm} \text{adjacency}

- Smallest domain first keeps remaining domain sizes large.
- High pattern degree makes the remaining pattern subgraph sparser, reducing $q$.
- High target degree leaves as many vertices as possible available for future use, making $u$ larger.
Sanity Check

![Graph showing the number of SAT instances solved over runtime for Degree, Random, and Anti-subgraph isomorphism problems. The x-axis represents runtime in milliseconds, ranging from $10^2$ to $10^6$, and the y-axis represents the number of SAT instances solved, ranging from 1500 to 2100. The graph includes three lines, one for Degree, one for Random, and one for Anti, each indicating the progression of solved instances as runtime increases.]

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Sanity Check

Number of Unsat Instances Solved vs Runtime (ms) for all instances

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Phase Transitions

\[ G(10, x) \rightarrow G(150, y) \]

\[ G(20, x) \rightarrow G(150, y) \]

\[ G(30, x) \rightarrow G(150, y) \]
Incidentally, Induced is Much More Complicated
However...

- Degree spread is low.
- We commit extremely heavily to the first branching choice, which is probably wrong.
Restarts

- Run search for a bit, and if we don’t find a solution, restart.
- Count number of backtracks, restart using the Luby sequence (with a magic constant multiplier).
  - 1, 1, 2, 1, 1, 2, 4, 1, 1, 2, 1, 1, 2, 4, 8, …
- Obviously, something needs to change when we restart.
  - First attempt: random value-ordering heuristic.
Restarts

Random, restarts
Degree
Random
Anti

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Restarts

Random + Restarts Search Time (ms)
Degree Search Time (ms)
Mesh sat
LV sat
Phase sat
Rand sat
Other sat
Any unsat

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Nogoods

- Whenever we restart, post new constraints eliminating parts of the search space already explored.
- Potentially exponentially many constraints.
- But they are all in the form

\[(d_1 = v_1) \land (d_2 = v_2) \land \ldots \land (d_n = v_n) \rightarrow \bot.\]

- Use two watched literals to propagate in \(O(1)\)ish time.
  - Basic idea: clauses only propagate when exactly one \((d_i = v_i)\) literal has not been set to true.
  - Watch two literals per clause that have not been set to true.
  - When unit propagating, only look at clauses with a watch corresponding to the assignment made.
  - Either find a new literal to watch, or propagate.
Nogoods

Number of Sat Instances Solved vs Runtime (ms) for different strategies:
- Random, nogoods
- Random, restarts
- Degree
- Random
- Anti

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Nogoods

Random, Nogoods Search Time (ms)

Degree Search Time (ms)

Mesh sat
LV sat
Phase sat
Rand sat
Other sat
Any unsat
Biased Value-Ordering

- Select a vertex $v'$ from the chosen domain $D_v$ with probability

$$p(v') = \frac{2^{\deg(v')}}{\sum_{w \in D_v} 2^{\deg(w)}}.$$

- Looks a lot like softmax, which uses base $e$. 
Biased Value-Ordering

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Biased Value-Ordering

The graph shows the relationship between Biased Search Time (ms) on the y-axis and Degree Search Time (ms) on the x-axis. The data points represent different categories of unsaturation: Mesh sat, LV sat, Phase sat, Rand sat, Other sat, and Any unsat. Each category is represented by a different symbol. The graph demonstrates how the search times correlate across different types of unsaturation.
Biased Value-Ordering with Restarts and Nogoods

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Biased Value-Ordering with Restarts and Nogoods

![Graph showing Biased, Nogoods Search Time vs Degree Search Time with different satisfiability states: Mesh sat, LV sat, Phase sat, Rand sat, Other sat, Any unsat.]
Ongoing Work

- Is this form of search more broadly applicable?
- Specialisations, like clique, and generalisations, like maximum common subgraph.
- Parallelism.
- Subgraphs modulo theories.
- Algorithm engineering.