Errata:
Keeping partners together: algorithmic results for the hospitals / residents problem with couples

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Theorem 3.8 and Corollary 3.9 in [2] are stated as follows:

**Theorem 3.8.** The problem of determining whether an HRS instance admits a stable matching is NP-complete, even if the size of each resident and the capacity of each hospital is at most 2, and the lengths of the residents’ and hospitals’ preference lists are at most 3 (these conditions holding simultaneously).

**Corollary 3.9.** The problem of determining whether an HRCC instance admits a stable matching is NP-complete, even if the joint preference list of each couple has at most 3 entries, and the capacity of each hospital is at most 2 (these conditions holding simultaneously).

However in the reduction given in the proof of Theorem 3.8 in [2], some preference lists may in fact be of length 4 (namely those of residents of the form $r_s$). A similar remark holds for Corollary 3.9 (i.e., some couples’ lists may contain as many as 4 pairs). In this note we present a revised proof of Theorem 3.8, which in turn establishes Corollary 3.9. In what follows we assume the notation and terminology used in [2].

**Proof of Theorem 3.8.** We reduce from a a restricted version of SAT. Let (2,2)-E3-SAT denote the problem of deciding, given a Boolean formula $B$ in CNF in which each clause contains exactly 3 literals and, for each variable $v_j$, each of literals $v_j$ and $\bar{v}_j$ appears exactly twice in $B$, whether $B$ is satisfiable. Berman et al. [1] showed that (2,2)-E3-SAT is NP-complete.

Hence let $B$ be an instance of (2,2)-E3-SAT. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be the set of variables and clauses respectively in $B$. Let us construct an instance of HRS in the following way.

For each variable $v_j$ there are 6 residents $r_{j1}^1, r_{j2}^2, \ldots, r_{j6}^6$, 4 residents $x_{j1}^1, x_{j2}^2, y_{j1}^1, y_{j2}^2$, 12 residents $q_{j1}^k, q_{j2}^k, q_{j3}^k$ ($1 \leq k \leq 4$), 6 hospitals $h_{j1}^1, h_{j2}^2, h_{j3}^3, h_{j4}^4, h_{j5}^T, h_{j6}^T$ and 12 hospitals
### Table 1

<table>
<thead>
<tr>
<th>Resident</th>
<th>Size Preferences</th>
<th>Hospital Capacity</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_j^1$</td>
<td>$h_j^1$</td>
<td>$h_j^1$</td>
<td>$r_j^4$, $r_j^1$, $r_j^3$</td>
</tr>
<tr>
<td>$r_j^2$</td>
<td>$h_j^2$</td>
<td>$h_j^2$</td>
<td>$r_j^3$, $r_j^2$, $r_j^4$</td>
</tr>
<tr>
<td>$r_j^3$</td>
<td>$h_j^3$</td>
<td>$h_j^3$</td>
<td>$r_j^1$, $r_j^5$</td>
</tr>
<tr>
<td>$r_j^4$</td>
<td>$h_j^4$</td>
<td>$h_j^4$</td>
<td>$r_j^2$, $r_j^6$, $r_j^7$</td>
</tr>
<tr>
<td>$r_j^5$</td>
<td>$h_j^5$</td>
<td>$h_j^5$</td>
<td>$r_j^5$, $x_j^1$, $x_j^2$</td>
</tr>
<tr>
<td>$r_j^6$</td>
<td>$h_j^6$</td>
<td>$h_j^6$</td>
<td>$r_j^6$, $y_j^1$, $y_j^2$</td>
</tr>
<tr>
<td>$x_j^1$</td>
<td>$h_j^T$</td>
<td>$z(x_j^1)$</td>
<td>$p_{j,3}^k$</td>
</tr>
<tr>
<td>$x_j^2$</td>
<td>$h_j^T$</td>
<td>$z(x_j^2)$</td>
<td>$p_{j,1}^k$, $p_{j,3}^k$</td>
</tr>
<tr>
<td>$y_j^1$</td>
<td>$h_j^F$</td>
<td>$z(y_j^1)$</td>
<td>$p_{j,2}^k$, $p_{j,3}^k$, $p_{j,4}^k$</td>
</tr>
<tr>
<td>$y_j^2$</td>
<td>$h_j^F$</td>
<td>$z(y_j^2)$</td>
<td>$p_{j,3}^k$, $p_{j,4}^k$, $p_{j,5}^k$</td>
</tr>
<tr>
<td>$q_{j,1}^k$</td>
<td>$p_{j,2}^k$</td>
<td></td>
<td>$v(q_{j,3}^k)$</td>
</tr>
<tr>
<td>$q_{j,2}^k$</td>
<td>$p_{j,3}^k$</td>
<td></td>
<td>$v(q_{j,3}^k)$</td>
</tr>
<tr>
<td>$q_{j,3}^k$</td>
<td>$p_{j,3}^k$</td>
<td></td>
<td>$v(q_{j,3}^k)$</td>
</tr>
</tbody>
</table>

Figure 1: The constructed instance of HRS

$p_{j,1}^k$, $p_{j,2}^k$, $p_{j,3}^k$ \(1 \leq k \leq 4\). For each clause $c_i$ there is one hospital $z_i$. Residents $x_j^1$ and $x_j^2$ correspond to the first and second occurrence of literal $v_j$, whilst residents $y_j^1$ and $y_j^2$ correspond to the first and second occurrence of literal $\bar{v}_j$, respectively.

The characteristics of agents and their preferences are given in Figure 1. Here, the subscripts and superscripts involving $i$, $j$ and $k$ range over the following intervals: $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq k \leq 4$. In the preference list of hospital $z_i$, the symbol $v_i^s$ means the $x$- or $y$-resident that corresponds to the literal that appears in position $s$ of clause $c_i$.

Conversely, in the preference list of $x$- or $y$-residents the symbol $z(.)$ denotes the $z$-hospital corresponding to the clause containing the corresponding literal. Also, in the preference list of $p_{j,3}^k$, the symbol $v(p_{j,3}^k)$ denotes $x_j^k$ if $1 \leq k \leq 2$ and denotes $y_j^{k-2}$ if $3 \leq k \leq 4$.

For each $j$, $1 \leq j \leq n$, let us denote

$$T_j = \{(x_j^1, h_j^T), (x_j^2, h_j^T), (r_j^6, h_j^F)\}, \quad F_j = \{(y_j^1, h_j^F), (y_j^2, h_j^F), (r_j^5, h_j^T)\}.$$  

For brevity, hospitals $h_j^T$ and $h_j^F$ will be called **decisive hospitals**.

Now, let $f$ be a satisfying truth assignment of $B$. Define a matching $M$ in $I$ as follows. For each variable $v_j \in V$, if $v_j$ is true under $f$, put the pairs $T_j$ into $M$ and if $v_j$ is false under $f$ put the pairs $F_j$ into $M$. In the former case add the pairs

$$(y_j^1, z(y_j^1)), (y_j^2, z(y_j^2)), (r_j^1, h_j^1), (r_j^2, h_j^2), (r_j^3, h_j^3), (r_j^4, h_j^4), (r_j^5, h_j^5),$$

and in the latter case add the pairs

$$(x_j^1, z(x_j^1)), (x_j^2, z(x_j^2)), (r_j^1, h_j^1), (r_j^2, h_j^2), (r_j^3, h_j^3), (r_j^4, h_j^4), (r_j^5, h_j^5).$$

Notice that as each clause $c_i \in C$ contains at most two false literals, hospital $z_i$ has enough capacity for accepting all the allocated residents. Finally, add the following pairs for each $j$ \(1 \leq j \leq n\) and $k$ \(1 \leq k \leq 4\):

$$(q_{j,1}^k, p_{j,2}^k), (q_{j,2}^k, p_{j,4}^k), (q_{j,3}^k, p_{j,3}^k).$$
It is obvious that the defined matching is feasible; it remains to prove that it is stable. We show this by considering each type of residents corresponding to variable \( v_j \) in turn. Firstly we remark that residents \( q_{j,1}^k, q_{j,2}^k, q_{j,3}^k \) each have their first choice hospital (1 \( \leq k \leq 4 \)) so cannot be involved in a blocking pair. Now suppose that \( v_j \) is true under \( f \). Then:

- residents \( x_j^1, x_j^2, r_j^1, r_j^2 \) and \( r_j^5 \) have their most-preferred hospitals, so are not blocking.
- residents \( y_j^1 \) and \( y_j^2 \) prefer hospital \( h_j^F \), but this hospital is fully occupied by \( r_j^6 \), whom it prefers.
- resident \( r_j^2 \) prefers hospital \( h_j^2 \), but this hospital is full and does not prefer \( r_j^2 \) to a set of applicants of size at least 2.
- resident \( r_j^3 \) prefers hospital \( h_j^1 \), but this hospital is fully occupied by \( r_j^1 \), whom it prefers.
- resident \( r_j^6 \) prefers hospital \( h_j^4 \), but this hospital is fully occupied by \( r_j^2 \), whom it prefers.

The case of a false variable can be proved similarly.

For the converse implication let us first prove two claims.

**Claim 1.** Each stable matching \( M \) contains for each \( j \) either all the pairs in \( T_j \) or all the pairs in \( F_j \).

**Proof.** Let \( M \) be a stable matching. Fix \( j \in \{1, 2, \ldots, n\} \). Notice first that both hospitals \( h_j^1 \) and \( h_j^F \) must be full, otherwise either \( h_j^1 \) will form a blocking pair with at least one of \( x_j^1 \) and \( x_j^2 \), or \( h_j^F \) will form a blocking pair with at least one of \( y_j^1 \) and \( y_j^2 \). Further, let us distinguish the following cases.

- \( \{(r_j^3, h_j^3), (r_j^6, h_j^F)\} \subseteq M \). Then, as there are no blocking pairs, \( \{(r_j^1, h_j^3), (r_j^2, h_j^F)\} \subseteq M \), which further implies \( \{(r_j^3, h_j^3), (r_j^1, h_j^F)\} \subseteq M \). This, however means that \( (r_j^3, h_j^3) \) and \( (r_j^1, h_j^F) \) are blocking pairs for \( M \), a contradiction.

- \( \{(x_j^1, h_j^3), (x_j^2, h_j^T), (y_j^1, h_j^F), (y_j^2, h_j^F)\} \subseteq M \). Now, to avoid blocking pairs, \( \{(r_j^5, h_j^3), (r_j^6, h_j^F), (r_j^3, h_j^3), (r_j^1, h_j^F)\} \subseteq M \). Then there are blocking pairs \( (r_j^3, h_j^3) \) and \( (r_j^1, h_j^F) \), again a contradiction.

**Claim 2.** In each stable matching \( M \) every resident in the set \( \{x_j^1, x_j^2, y_j^1, y_j^2 : 1 \leq j \leq n\} \) is matched to her first- or second-choice hospital.

**Proof.** For some \( j \) (1 \( \leq j \leq n \)), consider resident \( x_j^1 \) (the argument for \( x_j^2, y_j^1, y_j^2 \) is similar). Suppose firstly that \( x_j^1 \) is unmatched in \( M \). Then \( (x_j^1, p_{j,3}^3) \) blocks \( M \), a contradiction.

Now suppose that \( (x_j^1, p_{j,3}^1) \in M \). If \( (q_{j,1}^1, p_{j,1}^1) \in M \) then \( (q_{j,1}^1, p_{j,1}^2) \in M \), for otherwise \( (q_{j,1}^1, p_{j,1}^1) \) blocks \( M \). But then \( (q_{j,2}^1, p_{j,2}^1) \) blocks \( M \), a contradiction. Thus \( q_{j,3}^1 \) is unmatched in \( M \).

Now suppose that \( (q_{j,3}^1, p_{j,3}^1) \in M \). For otherwise \( (q_{j,2}^1, p_{j,1}^1) \) blocks \( M \). Also \( (q_{j,1}^1, p_{j,1}^1) \) blocks \( M \). Hence \( (q_{j,3}^1, p_{j,3}^1) \) blocks \( M \), a contradiction.

Conversely, suppose that \( M \) is a stable matching in \( I \). We form a truth assignment \( f \) in \( B \) as follows. Let \( j \) (1 \( \leq j \leq n \)) be given. If \( T_j \subseteq M \), set \( f(v_j) = T \), otherwise set \( f(v_j) = F \). Now let \( v_j \in V \) and suppose that \( f(v_j) = T \). Then by Claim 2, each of \( y_{j,1} \) and \( y_{j,2} \) is matched to her second choice hospital. Now suppose that \( f(v_j) = F \). Then by Claims 1 and 2, each of \( x_{j,1} \) and \( x_{j,2} \) is matched to her second choice hospital. Now let \( c_i \in C \) and suppose that all literals in \( c_i \) are false. By the preceding remarks about \( x_{j,1}, x_{j,2}, y_{j,1} \) and \( y_{j,2} \) we deduce that \( z_i \) is over-subscribed, a contradiction. Thus \( f \) is a satisfying truth assignment.
Corollary 3.9 then follows immediately by Theorem 3.8 and by Lemma 2.1 in [2].

References
