Stable Marriage with Ties and Unacceptable Partners

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Abstract. An instance of the classical stable marriage problem involves n men and n women, and each person ranks all n members of the opposite sex in strict order of preference. The effect of allowing ties in the preference lists has been investigated previously, and three natural definitions of stability arise. In this paper, we extend this study by allowing a preference list to involve ties and/or be incomplete. We show that, under the weakest notion of stability, the stable matchings need not be all of the same cardinality, and the decision problem related to finding a maximum cardinality stable matching is NP-complete, even if the ties occur in the preference lists of one sex only. This result has important implications for practical matching schemes such as the well-known National Resident Matching Program [9]. In the cases of the other two notions of stability, Irving [5] has described algorithms for testing whether a stable matching exists, and for constructing such a matching if one does exist, where a preference list is complete but may involve ties. We demonstrate how to extend these algorithms to the case where a preference list may be incomplete and/or involve ties.

1 Introduction

The classical *stable marriage problem* (SM) has been extensively studied in the literature. An instance of SM involves n men and n women, each of whom ranks all the members of the opposite sex in strict order of preference. Given a complete matching M of the men and women, we say that an unmatched pair (m, w) is a *blocking pair* for M if m prefers w to his partner in M, and w prefers m to her partner in M. A matching that admits no blocking pair is said to be *stable*, and *unstable* otherwise. It is known that every instance of SM admits at least one stable matching, and that such a matching may be found in $O(n^2)$ time using the Gale/Shapley algorithm [2].

A generalisation of SM occurs when one or more persons involved might find certain members of the opposite sex unacceptable. In this case, the members of the opposite sex that such a person p vetoes are missing from the preference list of p.

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We say that person p is *acceptable* to person q if p appears on the preference list of q, and *unacceptable* otherwise. We use SMI to stand for this variant of SM where preference lists may be incomplete (for simplicity we assume that the numbers of men and women are equal in a given instance of SMI). The revised notion of stability may be defined as follows: given an instance of SMI, a matching M is stable if there is no unmatched pair (x, y), each of whom is either unmatched in M and finds the other acceptable, or prefers the other to his/her partner in M. A stable matching for an instance of SMI may not be a complete matching. However, all men and women in the instance may be partitioned into two sets, one containing the persons matched in all stable matchings, and one containing the persons matched in none [3, §1.4.2]. It is a simple matter to extend the Gale/Shapley algorithm to cope with preference lists that may be incomplete.

An alternative natural extension of the original stable marriage problem arises when each person need not rank all members of the opposite sex in *strict* order. It is possible that each person involved might be indifferent among certain members of the opposite sex, so that preference lists may involve ties (in this paper we restrict attention to the case where the indifference takes the form of ties in the preference lists, but it may be verified that all results are extendable to the general case where the preference lists are arbitrary partial orders). We use SMT to stand for the variant of SM where preference lists are complete but may involve ties. Three possible definitions for stability are formulated in [5] for SMT. A matching M is *weakly stable* if there is no couple (x, y), each of whom strictly prefers the other to his/her partner in M. Also, a matching M is *strongly stable* if there is no couple (x, y) such that x strictly prefers y to his/her partner in M, and y either strictly prefers x to his/her partner in M or is indifferent between them. Finally, a matching M is *super-stable* if there is no couple (x, y), each of whom either strictly prefers the other to his/her partner in M or is indifferent between them.

By breaking the ties arbitrarily, an instance I of SMT becomes an instance I' of SM, and it is clear that a stable matching for I' is a weakly stable matching for I. Thus a weakly stable matching for I may be found in $\mathrm{O}(n^2)$ time, using the Gale/Shapley algorithm, for example. It is straightforward to construct instances of SMT which admit no strongly stable matching and/or no super-stable matching; see [5] for further details. However, Irving [5] presents $\mathrm{O}(n^4)$ and $\mathrm{O}(n^2)$ algorithms for determining whether a strongly stable matching and/or a super-stable matching exists for a given instance of SMT respectively, and if they do in either case, the algorithms will find such a matching.

In this paper, we focus on the stable marriage problem incorporating *both* extensions described above. Thus a stable marriage instance now comprises preference lists, each of which may involve ties and/or be incomplete. We use SMTI to stand for this variant of SM. The criteria for stability described above for SMI and SMT imply three conditions for stability within the context of SMTI. That is, given a matching M for an instance of SMTI, M is *weakly stable* if there is no unmatched pair (x, y), each of whom is either unmatched in M and finds the other acceptable, or strictly prefers the other to his/her partner in M. Also, M is *strongly stable* if there is no unmatched pair (x, y) such that (i) either x is unmatched in M and finds y acceptable, or x strictly prefers y to his/her partner in M, and (ii) either y is unmatched in M and finds x acceptable, or y strictly prefers x to his/her partner in M, or y is indifferent between x and his/her partner in M.[1] Finally, M is *super-stable* if there is no unmatched pair (x, y), each of whom is either unmatched in M and finds the other acceptable, or strictly prefers the other to his/her partner in M, or is indifferent between them. In each case, such a pair (x, y) that causes M to be fail the stability criterion is called a *blocking pair* with respect to the notion of stability concerned.

Given an instance of SMI, it is a consequence of [3, Theorem 1.4.2] that all stable matchings are of the same size. Similarly, given an instance of SMT, it is clear that all weakly stable matchings are complete, and by inspection of Algorithm STRONG (respectively SUPER) in [5], all strongly stable (respectively super-stable) matchings are complete, assuming that one exists. However, for a given instance of SMTI, all weakly stable matchings need not be of the same size, a fact that does not appear to have been noted explicitly in the literature previously. As a simple example, consider the following instance involving two men, m_1, m_2, and two women, w_1, w_2. Man m_1 finds only woman w_1 acceptable, and man m_2 strictly prefers woman w_1 to woman w_2. Woman w_1 is indifferent between man m_1 and man m_2, and woman w_2 finds only man m_2 acceptable. There are two weakly stable matchings for this instance, namely $\{(m_2, w_1)\}$ and $\{(m_1, w_1), (m_2, w_2)\}$.

Thus the question arises as to whether there exists an efficient algorithm to find a *maximum cardinality* weakly stable matching for a given instance of SMTI. This question has particular significance within the context of matching graduating medical students to hospitals. As is current practice with the National Resident Matching Program [9] in the U.S. and the Canadian Resident Matching Service [1], hospitals must rank a possibly large number of applicants in strict order of preference. A given hospital may be indifferent among several of its applicants, and might prefer to include ties in its preference list. With the SPA (*S*cottish *P*re-registration house officer *A*llocations) matching scheme soon to be introduced [6], hospitals may include ties in their preference lists, but these ties are broken arbitrarily so that all preference lists are strict. However, the previous example indicates that breaking the ties in different ways can affect the sizes of the subsequent stable matchings. Since the objective is always to match as many applicants as possible, we seek a strategy to break the ties so as to maximise the cardinality of the consequent stable matchings. (Note that weak stability is the stability criterion that is relevant here, since as previously mentioned, a given instance of SMTI may admit no strongly stable and/or super-stable matching.) We prove in Section 2 that the decision problem related to finding the maximum size of weakly stable matching for a given instance of SMTI is NP-complete.

However, there is more structure in the cases of strong stability and super-stability. In contrast with the case for weak stability, we show in Section 3 that, for a given instance of SMTI, the set of people may be partitioned into two sets, those who are matched in all strongly stable matchings, and those who are matched in none. A similar result holds for super-stability (noted in Section 4). Building on these results, we present $O(n^4)$ and $O(n^2)$ algorithms in Sections 3 and 4 for determining, given an instance of SMTI, whether a strongly stable matching and/or a super-stable

[1] Our definition of strong stability incorporates the assumption that a person would strictly prefer to be matched to somebody acceptable to him/her, rather than be unmatched.

matching exists respectively, and if one does in either case, the algorithms will find such a matching. These algorithms extend those of Irving [5] for SMT.

Related work. Ronn [7, 8] was possibly the first to investigate the algorithmic effect of introducing ties into the preference lists of instances of various stable matching problems. Ronn's criterion for stability was weak stability; Irving [5] was the first to define the strong stability and super-stability concepts. Spieker [10] shows that the set of super-stable matchings for a given instance of SMT forms a distributive lattice. (It is a well-known theorem that the set of all stable matchings for a given instance of SM forms a distributive lattice.) We will show in this paper that this result also carries over to SMTI – see Section 5.

Preliminaries. Henceforth, when the term *blocking pair* is used, the appropriate notion of stability will be given by the title of the section in which the term is used. Also, we assume that a person p is acceptable to a person q if and only if q is acceptable to p. The pair (m, w) is called a *weakly stable pair* if $(m, w) \in M$ for some weakly stable matching M. In this case, m is a *weakly stable partner* of w, and vice-versa. The definitions of *strongly stable/super-stable pair* and *strongly stable/super-stable partner* are analagous. We use the term *head* of a man's list to denote the set of one or more women, tied in his list, whom he strictly prefers to all other women in his list. Similarly we use the term *tail* of a woman's list to denote the set of one or more men, tied in her list, to whom she strictly prefers all other men in her list.

2 Weak stability

In this section we prove that the existence of an algorithm to find a maximum cardinality weakly stable matching for a given instance of SMTI is unlikely. We demonstrate NP-completeness for the following decision problem, a given instance of which involves ties in the preference lists of the women only:

> *Name:* WEAK STABILITY SMTI.
> *Instance:* n men and n women, preference list of women for each man, preference list of men for each woman, and integer $K \in \mathbb{Z}^+$. (A man's preference list may be incomplete; a woman's preference list may be incomplete and/or involve ties.)
> *Question:* Does the given instance admit a weakly stable matching M with $|M| \geq K$?

Our transformation begins from the following problem:

> *Name:* EXACT MAXIMAL MATCHING.
> *Instance:* Graph $G = (V, E)$ and integer $K \in \mathbb{Z}^+$.
> *Question:* Does G have a maximal[2] matching M with $|M| = K$?

EXACT MAXIMAL MATCHING is NP-complete, even for subdivision graphs of graphs with maximum degree three. This fact is implicit from the NP-completeness of the following decision problem for the same class of graphs:

[2]A matching M in a graph G is *maximal* if no proper superset of M is a matching in G.

Name: MINIMUM MAXIMAL MATCHING.
Instance: Graph $G = (V, E)$ and integer $K \in \mathbb{Z}^+$.
Question: Does G have a maximal matching M with $|M| \leq K$?

The NP-completeness of MINIMUM MAXIMAL MATCHING for the subdivision graphs of graphs of maximum degree three was established by Horton and Kilakos [4]. [3]

Theorem 2.1 WEAK STABILITY SMTI *is NP-complete.*

Proof: Clearly WEAK STABILITY SMTI is in NP. For, given a set M, we may easily verify that M is a matching for the given instance, and that $|M| \geq K$. Furthermore, it is straightforward to verify in polynomial time that no unmatched pair (m, w) blocks M.

To show NP-hardness, we transform from EXACT MAXIMAL MATCHING for the subdivision graphs of graphs where no vertex degree exceeds three. Let $G = (V, E)$ and $K \in \mathbb{Z}^+$ be an instance of this problem. Then G is the subdivision graph of some graph $G' = (V', E')$, so that $V = V' \cup E'$ and

$$E = \{(e, v) : e \in E' \wedge v \in V' \wedge v \text{ is incident to } e \text{ in } G'\}.$$

Also G has a bipartition (U, W), where $U = E'$ and $W = V'$. Thus every vertex in U has degree two in G, and every vertex in W has degree at most three in G. Without loss of generality we may assume that G' is connected and is not a forest, so that $|E'| \geq |V'|$, i.e. $|U| \geq |W|$. Again without loss of generality, we may assume that $|U| = |W|$ (for if $|U| = |W| + r$ for some $r > 0$, then we may add r vertices $a_1, \ldots, a_r$ to U, and $2r$ vertices $b_1, \ldots, b_r$, $c_1, \ldots, c_r$ to W, where a_i is adjacent to b_i and c_i for each i $(1 \leq i \leq r)$; clearly every vertex in the new set U has degree two in the new graph, every vertex in the new set W has degree at most three in the new graph, and G has a maximal matching of size K if and only if the transformed graph has a maximal matching of size $K + r$). Finally, without loss of generality, we may assume that $K \leq n$, where $n = |U| = |W|$.

Let $U = \{m_1, m_2, \ldots, m_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. We construct an instance I of WEAK STABILITY SMTI as follows: let $U \cup U' \cup X$ be the set of men, and let $W \cup Y \cup Z$ be the set of women, where $U' = \{m'_1, m'_2, \ldots, m'_n\}$, $X = \{x_1, x_2, \ldots, x_{n-K}\}$, $Y = \{y_1, y_2, \ldots, y_{n-K}\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$. Assume that j_i and k_i are two sequences such that $j_i < k_i$, $\{m_i, w_{j_i}\} \in E$ and $\{m_i, w_{k_i}\} \in E$ $(1 \leq i \leq n)$. For any w_j $(1 \leq j \leq n)$, let M_j contain the men m_i such that $\{m_i, w_j\} \in E$, and let M'_j contain the men m'_i such that $\{m_i, w_j\} \in E$ and $j = k_i$. Clearly $|M'_j| \leq |M_j| \leq 3$. Create preference lists for each person as follows:

$$\begin{array}{lll} m_i : & z_i \; w_{j_i} \; w_{k_i} \text{ all } y_j \text{ in any order} & (1 \leq i \leq n) \\ m'_i : & z_i \; w_{k_i} & (1 \leq i \leq n) \end{array}$$

[3] In fact Horton and Kilakos proved that MINIMUM EDGE DOMINATING SET is NP-complete for this class of graphs. The MINIMUM EDGE DOMINATING SET problem is to determine, given a graph $G = (V, E)$ and an integer $K \in \mathbb{Z}^+$, whether G contains an *edge dominating set* of size at most K. A set of edges S is an edge dominating set in G if every edge in $E \backslash S$ is adjacent to some edge in S. It is known that MINIMUM MAXIMAL MATCHING and MINIMUM EDGE DOMINATING SET are polynomially equivalent; indeed the size of a minimum maximal matching of a given graph G is equal to the size of a minimum edge dominating set of G [11].

x_i :	all w_j in any order	$(1 \leq i \leq n-K)$
w_j :	(members of M_j and M'_j) $(x_1 \ldots x_{n-K})$	$(1 \leq j \leq n)$
y_j :	$(m_1 \ldots m_n)$	$(1 \leq j \leq n-K)$
z_j :	$(m_j\ m'_j)$	$(1 \leq j \leq n)$

In a preference list, persons within parentheses are tied. To complete the construction of the instance, we set the target value to be $K' = 3n - K$. Clearly the maximum size of weakly stable matching for this instance is K'. We claim that G has a maximal matching of size exactly K if and only if the stable marriage instance admits a weakly stable matching of size K'.

For, suppose that G has a maximal matching M, where $|M| = K$. We construct a matching M' in I as follows. For each edge $\{m_i, w_j\}$ in M, if $j = j_i$, then we add (m_i, w_{j_i}) and (m'_i, z_i) to M'. If $j = k_i$, then we add (m'_i, w_{k_i}) and (m_i, z_i) to M'. There remain $2(n-K)$ men of the form m_{p_i}, m'_{p_i} $(1 \leq i \leq n-K)$ who are as yet unmatched. Add (m_{p_i}, y_i) and (m'_{p_i}, z_{p_i}) to M' $(1 \leq i \leq n-K)$. Similarly there remain $n-K$ women of the form w_{q_i} $(1 \leq i \leq n-K)$ who are as yet unmatched. Add (x_i, w_{q_i}) to M' $(1 \leq i \leq n-K)$. Clearly M' is a matching of size $2K + 2(n-K) + (n-K) = K'$. It remains to show that M' is weakly stable.

It is straightforward to verify that no man of the form x_i, and no woman of the form y_j or z_j, can be involved in a blocking pair of M'.

No unmatched pair (m_i, w_j) blocks M'. For if this occurs, then $(m_i, y_k) \in M'$ for some y_k. Thus no edge of M is incident to m_i. Hence by maximality of M, $(m^*_l, w_j) \in M'$ for some m^*_l, where $m^*_l \in \{m_l, m'_l\}$. But w_j is indifferent between m_i and m^*_l. Hence (m_i, w_j) does not block M'.

Additionally, no unmatched pair (m'_i, w_j) blocks M', for either $(m'_i, z_i) \in M'$ or $(m'_i, w_{k_i}) \in M'$ holds. Thus M' is weakly stable.

Conversely suppose that M' is a weakly stable matching for I, where $|M'| = K'$. Then everybody has a partner in M'. For each i $(1 \leq i \leq n)$, at most one of m_i and m'_i is matched to a woman of the form w_j in M', for otherwise z_i is unmatched, a contradiction. Thus

$$M = \{\{m_i, w_j\} \in E : 1 \leq i, j \leq n \wedge ((m_i, w_j) \in M' \vee (m'_i, w_j) \in M')\}$$

is a matching in G. There are exactly $n-K$ men m_{r_i} $(1 \leq i \leq n-K)$ who have a partner among the y_k in M'. Since man m'_{r_i} must have z_{r_i} as his partner in M' $(1 \leq i \leq n-K)$, then $|M| = K$.

To complete the proof, it remains to show that M is maximal. For, suppose not. Then there is some edge $\{m_i, w_j\}$ in G such that no edge of M is incident on either m_i or w_j. Thus $(m_i, y_k) \in M'$ for some y_k, and $(w_j, x_l) \in M'$ for some x_l. But then (m_i, w_j) blocks M', for m_i strictly prefers w_j to y_k, and w_j strictly prefers m_i to x_l. This contradiction to the weak stability of M' implies that M is indeed maximal. ■

A simpler transformation exists, also starting from EXACT MAXIMAL MATCHING, if we allow ties to occur in the preference lists of both sexes (we return to this issue in Section 5).

3 Strong stability

We begin this section by demonstrating that, for a given instance of SMTI, the set of people may be partitioned into two sets, those matched in all strongly stable matchings, and those matched in none.

Lemma 3.1 *For a given instance of SMTI, let M and M' be two strongly stable matchings. Then for any person p in the instance, p is matched in M if and only if p is matched in M'.*

Proof: We suppose that p is some man m; the argument for the case that p is a woman is similar. Suppose that m is matched in M, to w say, and m is unmatched in M'. Then w has a partner $m' \neq m$ in M', such that w strictly prefers m' to m, for otherwise (m, w) blocks M'. Similarly, m' has a partner $w' \neq w$ in M, such that m' strictly prefers w' to w, for otherwise (m', w) blocks M.

We claim that there is a sequence $\langle m_j \rangle_{j \geq 0}$ of men, and a sequence $\langle w_j \rangle_{j \geq 0}$ of women, such that, for each $i \geq 1$,

1. $m_0, \ldots, m_i, w_0, \ldots, w_i$ are distinct people.
2. $(m_i, w_{i-1}) \in M'$ and $(m_i, w_i) \in M$.
3. m_i strictly prefers w_i to w_{i-1}.

We prove the claim by induction on i. The base case $i = 1$ clearly holds with $m_0 = m$, $m_1 = m'$ and $w_0 = w$, $w_1 = w'$. Suppose that some $r \geq 1$ is given, and assume that the claim is true for $i = r$. We show that the claim holds for $i = r + 1$. Woman w_r has a partner, m_{r+1} say, in M', such that w_r strictly prefers m_{r+1} to m_r, for otherwise (m_r, w_r) blocks M'. Clearly $m_{r+1} \neq m_j$ for any j $(1 \leq j \leq r)$, and also $m_{r+1} \neq m_0$ as $m_0 = m$ is unmatched in M'. Additionally m_{r+1} has a partner, w_{r+1} say, in M, such that m_{r+1} strictly prefers w_{r+1} to w_r, for otherwise (m_{r+1}, w_r) blocks M. Clearly $w_{r+1} \neq w_j$ for any j $(0 \leq j \leq r)$. This completes the induction step.

As the sequence of distinct men and women is infinite, we reach an immediate contradiction. Hence m is matched in M'. ∎

For a given instance of SMTI, Algorithm STRONG2 shown in Figure 1 determines whether a strongly stable matching exists, and if so will find such a matching. This algorithm is an extension of Algorithm STRONG in [5]. We require to define some terminology used in the description of Algorithm STRONG2. By *delete the pair* (m, w), we mean that m should be deleted from the preference list of w, and w should be deleted from the preference list of m. Given a bipartite graph $G = (V, E)$ with bipartition $V = X \cup Y$ and a subset Z of X, the *neighbourhood* of Z, $N_G(Z)$, is the set of vertices in Y adjacent to at least one vertex in Z. The *deficiency* of Z, $\delta(Z)$, is defined by $\delta(Z) = |Z| - N_G(Z)$. The deficiency of G, $\delta(G)$, is the maximum deficiency over all subsets of X. It is a classical result that the maximum size of matching in G is equal to $|X| - \delta(G)$. A subset Z of X such that $\delta(Z) = \delta(G)$, and such that no $Z' \subset Z$ satisfies $\delta(Z') = \delta(G)$, is called a *critical* subset of X. Clearly Z may be empty; also it may be shown that there is a unique critical subset of X.

```
assign each person to be free;
for each woman w do
    proposed(w) := false;
repeat
   while some man m is free and has a nonempty list do
      for each woman w at the head of m's list do
      begin
         m proposes, and becomes engaged, to w;
         proposed(w) := true;
         for each strict successor m' of m on w's list do
         begin
            if m' is engaged to w then
               break the engagement;
            delete the pair (m', w)
         end
      end;
   let G be the current engagement graph;
   Z := critical set of men in G;
   U := N_G(Z);
   for each woman w ∈ U do
   begin
      break all engagements involving w;
      for each man m at the tail of w's list do
         delete the pair (m, w)
   end
until U = ∅;
let M be a maximum matching in G;
if some woman w is unmatched in M and proposed(w) then
    no strongly stable matching exists
else
    M is a strongly stable matching in the original instance;
```

Figure 1: Algorithm STRONG2.

In order to establish the correctness of Algorithm STRONG2, a number of lemmas follow. Henceforth, a person p's preference list at the termination of Algorithm STRONG2 will be referred to as the *reduced list* of p.

Lemma 3.2 *If the pair (m, w) is deleted during an execution of Algorithm STRONG2, then that pair cannot block any matching output by Algorithm STRONG2, comprising pairs that are never deleted.*

Proof: Let M be a matching output by Algorithm STRONG2, comprising pairs that are never deleted, and suppose that (m, w) is deleted during execution of the algorithm. If w is matched in M, then w strictly prefers her partner in M to m, since m is a strict successor of any undeleted entries in the reduced list of w. Hence (m, w) does not block M in this case. Now suppose that w is unmatched in M. It is clear that, in order for the pair (m, w) to be deleted by the algorithm, w must have received a proposal from some man during the execution of the algorithm. Since

w is unmatched in M, the algorithm would have reported that no strongly stable matching exists, rather than outputting M, a contradiction. ∎

Lemma 3.3 *A matching output by Algorithm STRONG2 is strongly stable.*

Proof: Suppose that some execution of Algorithm STRONG2 outputs a matching M, and suppose that M is blocked by some pair (m, w). Then m and w are acceptable to each other, so that each is on the original preference list of the other. By Lemma 3.2, the pair (m, w) has not been deleted. Hence each is on the reduced list of the other.

At the termination of the main loop of Algorithm STRONG2, $U = \emptyset$. Let G be the engagement graph at this point. Every man x who is engaged in G to some woman is matched in M. For otherwise $x \in Z$ by Lemma 4.6 of [5], and thus $U \neq \emptyset$, a contradiction. Similarly every woman y who is engaged in G to some man is matched in M. For otherwise the algorithm reports that no strongly stable matching exists, since y has received a proposal, a contradiction.

As the reduced list of m is nonempty, m is engaged to one or more women in G. Hence by the previous paragraph, m has a partner, w' say, in M. Now $w \neq w'$, as (m, w) blocks M. If m strictly prefers w to w', then the pair (m, w) has been deleted, since w' is at the head of the reduced list of m, a contradiction. Thus m is indifferent between w and w', so that m proposed to w during the execution of the algorithm. Thus w is engaged to m in G, for otherwise the pair (m, w) would have been deleted, a contradiction. Again by the previous paragraph, w has a partner, m' say, in M. But (m, w) blocks M, so that w strictly prefers m to m'. Hence the pair (m', w) would have been deleted as a result of m proposing to w, a contradiction. ∎

Lemma 3.4 *No strongly stable pair is ever deleted during an execution of Algorithm STRONG2.*

Proof: The proof of this lemma is almost identical to that of Lemma 4.4 in [5]; only minor modifications of the latter proof are required in order to cope with the case that a preference list may be incomplete. We omit the details. ∎

Lemma 3.5 *If, during the execution of Algorithm STRONG2, some woman w receives a proposal and is unmatched in the maximum matching M, then no strongly stable matching exists for the given instance.*

Proof: Let m be a man who proposes to w during execution of the algorithm and let G be the engagement relation at the termination of the algorithm. Suppose, for a contradiction, that there is a strongly stable matching M' for the given instance. Suppose firstly that w is unmatched in M'. If m is unmatched in M' then (m, w) blocks M', since m and w are mutually acceptable. Thus m has a partner, x say, in M'. If m strictly prefers x to w, then in order for m to propose to w, the strongly stable pair (m, x) would have been deleted by the algorithm, a contradiction to Lemma 3.4. Thus either m is indifferent between w and x, or m strictly prefers w to x. In either case (m, w) blocks M'.

Thus w has a partner, m' say, in M' (possibly $m = m'$). Now m' is engaged to some woman in G. For if not, then the reduced list of m' is empty, so that m' has no stable partners by Lemma 3.4, a contradiction. Since M matches every man who is engaged to at least one woman in G (shown in the second paragraph of the proof of Lemma 3.3), then m' has a partner, w' say, in M. As $(m', w) \notin M$, then $w \neq w'$. If m' strictly prefers w to w', then the strongly stable pair (m', w) would have been deleted in order for m' to propose to w', a contradiction to Lemma 3.4. Thus m' is either indifferent between w' and w, or m' strictly prefers w' to w.

We claim that there is a sequence $\langle m_j \rangle_{j \geq 1}$ of distinct men, and a sequence $\langle w_j \rangle_{j \geq 0}$ of distinct women, such that, for each $i \geq 1$,

1. $m_1, \ldots, m_i, w_0, \ldots, w_i$ are distinct people.
2. $(m_i, w_{i-1}) \in M'$ and $(m_i, w_i) \in M$.
3. m_i is either indifferent between w_i and w_{i-1}, or m_i strictly prefers w_i to w_{i-1}.

We prove the claim by induction on i. The base case $i = 1$ clearly holds with $m_1 = m'$ and $w_0 = w$, $w_1 = w'$. Suppose that some $r \geq 1$ is given, and assume that the claim is true for $i = r$. We show that the claim holds for $i = r+1$. Suppose that w_r is unmatched in M'. Now m_r is acceptable to w_r, and either m_r strictly prefers w_r to w_{r-1}, or m_r is indifferent between w_r and w_{r-1}. Hence (m_r, w_r) blocks M', a contradiction. Thus w_r has a partner, m_{r+1} say, in M'. Clearly $m_{r+1} \neq m_j$ for any j $(1 \leq j \leq r)$. Now m_{r+1} is engaged in G, for if not, then the reduced list of m_{r+1} is empty, so that m_{r+1} has no stable partners by Lemma 3.4, a contradiction. As above, M matches every man who is engaged to at least one woman in G, so that m_{r+1} has a partner, w_{r+1} say, in M. Clearly $w_{r+1} \neq w_j$ for any j $(1 \leq j \leq r)$, and $w_{r+1} \neq w_0$ as $w_0 = w$ is unmatched in M. If m_{r+1} strictly prefers w_r to w_{r+1}, then the strongly stable pair (m_{r+1}, w_r) would have been deleted in order for m_{r+1} to propose to w_{r+1}, a contradiction to Lemma 3.4. Thus m_{r+1} is either indifferent between w_{r+1} and w_r, or m_{r+1} strictly prefers w_{r+1} to w_r. This completes the induction step.

As the sequence of distinct men and women is infinite, we reach an immediate contradiction. Thus no strongly stable matching exists for the given instance. ■

Theorem 3.6 *For a given instance of SMTI, Algorithm STRONG2 determines whether or not a strongly stable matching exists. If such a matching does exist, all possible executions of the algorithm find one in which every man has as good a partner, and every woman as bad a partner, as in any strongly stable matching.*

Proof: Let I be the given instance of SMTI. Clearly the main loop of Algorithm STRONG2 terminates. For, if $U \neq \emptyset$, then at least one pair (m, w) is deleted, where $w \in U$ and m is a member of the critical set of men. Thus, each man's list is bound to become empty eventually; in this case, $U = \emptyset$.

Let M be a maximum matching in the final engagement relation G. If some woman who is unmatched in M received a proposal during execution of the algorithm, then I has no strongly stable matching by Lemma 3.5. If there is no such woman, then the algorithm outputs the matching M. By Lemma 3.3, M is a strongly stable matching.

Finally, if M' is any strongly stable matching for I, and M is the matching generated by the algorithm, then every man M has as good a partner in M as in M', and every woman has as bad a partner in M as in M'. This fact is a consequence of Lemmas 3.1 and 3.4. ■

The $O(n^4)$ time bound computed by Irving for Algorithm STRONG in [5] also applies to Algorithm STRONG2.

```
assign each person to be free;
for each woman w do
    proposed(w) := false;
repeat
    while some man m is free and has a nonempty list do
        for each woman w at the head of m's list do
        begin
            m proposes, and becomes engaged, to w;
            proposed(w) := true;
            for each strict successor m' of m on w's list do
            begin
                if m' is engaged to w then
                    break the engagement;
                delete the pair (m', w)
            end
        end;
    for each woman w who is multiply engaged do
    begin
        break all engagements involving w;
        for each man m at the tail of w's list do
            delete the pair (m, w);
    end;
until each man is either engaged or has an empty list;
let M be a maximum matching in the engagement relation;
if some woman w is unmatched in M and proposed(w) then
    no super-stable matching exists
else
    M is a super-stable matching in the original instance;
```

Figure 2: Algorithm SUPER2.

4 Super-stability

We begin this section by demonstrating that the analagous result to Lemma 3.1 holds in the super-stability case. That is, for a given instance of SMTI, the set of people may be partitioned into two sets, those matched in all super-stable matchings, and those matched in none.

Lemma 4.1 *For a given instance of SMTI, let M and M' be two super-stable matchings. Then for any person p in the instance, p is matched in M if and only if p is matched in M'.*

Proof: The proof is identical to that of Lemma 3.1. ∎

For a given instance of SMTI, Algorithm SUPER2 shown in Figure 2 determines whether a super-stable matching exists, and if so will find such a matching. As before, this algorithm is an extension of Algorithm SUPER in [5]. The terms *delete the pair* (m, w) and *reduced list*, defined in Section 3, are defined analagously here.

We establish the correctness of Algorithm SUPER2, following a similar approach to that of Section 3.

Lemma 4.2 *If the pair (m, w) is deleted during an execution of Algorithm SUPER2, then that pair cannot block any matching output by Algorithm SUPER2, comprising pairs that are never deleted.*

Proof: The proof is identical to that of Lemma 3.2. ∎

Lemma 4.3 *A matching output by Algorithm SUPER2 is super-stable.*

Proof: Suppose that some execution of Algorithm SUPER2 outputs a matching M, and suppose that M is blocked by some pair (m, w). Then m and w are acceptable to each other, so that each is on the original preference list of the other. By Lemma 4.2, the pair (m, w) has not been deleted. Hence each is on the reduced list of the other.

Let G be the engagement relation at the termination of the algorithm. Clearly each man x who is engaged in G to some woman is matched in M, since each woman has degree at most one in G. Similarly each woman y who is engaged in G to some man is matched in M. For otherwise the algorithm reports that no super-stable matching exists, since y has received a proposal, a contradiction.

As the reduced list of m is nonempty, m is engaged to one or more women in G. Hence by the previous paragraph, m has a partner, w' say, in M. Now $w \neq w'$, as (m, w) blocks M. If m strictly prefers w to w', then the pair (m, w) has been deleted, since w' is at the head of the reduced list of m, a contradiction. Thus m is indifferent between w and w', so that m proposed to w during the execution of the algorithm. Hence w is engaged to m in G, for otherwise the pair (m, w) would have been deleted, a contradiction. By the previous paragraph, w has a partner in M; since w is engaged to at most one man, then this partner is m. Thus $(m, w) \in M$, a contradiction. ∎

Lemma 4.4 *No super-stable pair is ever deleted during an execution of Algorithm SUPER2.*

Proof: The proof of this lemma is almost identical to that of Lemma 3.3 in [5]; only minor modifications of the latter proof are required in order to cope with the case that a preference list may be incomplete. We omit the details. ∎

Lemma 4.5 *If, during the execution of Algorithm SUPER2, some woman receives a proposal and is unmatched in the maximum matching M, then no super-stable matching exists for the given instance.*

Proof: The proof is identical to that of Lemma 3.5. ∎

Theorem 4.6 *For a given instance of SMTI, Algorithm SUPER2 determines whether or not a super-stable matching exists. If such a matching does exist, all possible executions of the algorithm find one in which every man has as good a partner, and every woman as bad a partner, as in any strongly stable matching.*

Proof: Let I be the given instance of the SMTI. Clearly the main loop of the algorithm terminates. For, if some man m is free and has a nonempty list at the end of some iteration of the main loop, then m proposes to some woman w at the head of his list during the next iteration. Either m is engaged at the end of the this iteration, or the pair (m, w) is deleted. Thus we are guaranteed that eventually the termination condition will be satisfied.

Let M be a maximum matching in the final engagement relation. If some woman who is unmatched in M received a proposal during execution of the algorithm, then I has no super-stable matching by Lemma 4.5. If there is no such woman, then Algorithm SUPER2 outputs the matching M. By Lemma 4.3, M is a super-stable matching.

Finally, if M' is any strongly stable matching for I, and M is the matching generated by the algorithm, then every man M has as good a partner in M as in M', and every woman has as bad a partner in M as in M'. This fact is a consequence of Lemmas 4.1 and 4.4. ∎

The $O(n^2)$ time bound computed by Irving for Algorithm SUPER in [5] also applies to Algorithm SUPER2.

5 Conclusion and open problems

As mentioned in Section 2, an instance of the NP-complete problem WEAK STABILITY SMTI contains ties only on the women's side. Thus, with women representing hospitals, a natural restriction of the problem arises if, in an applicant-hospital matching scheme, we ask each hospital to rank some applicants in strict order, but then permit indifference among the remaining applicants, so that each tie will occur at the tail of some hospital's list. Our transformation as it stands does not prove NP-completeness for this restriction of WEAK STABILITY SMTI, however we conjecture that this restricted version of the problem is NP-complete nevertheless.

As previously mentioned, the set of all super-stable matchings for a given instance of SMT forms a distributive lattice. A consequence of Lemma 4.1 is that this result also holds for SMTI. However, no such result holds for weakly stable matchings: Roth [9] constructs an instance of SMTI, comprising three men and three women, which admits no man-optimal or woman-optimal weakly stable matching. Regarding strongly stable matchings, Spieker's results in [10] may be easily adapted to show that the set of all such matchings for a given instance of SMTI forms a distributive lattice, if ties occur in the preference lists of one sex only. It remains open to characterise the structure in the general case.

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