A Constraint Programming Approach to the Hospitals / Residents Problem

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Technical Report
TR-2007-236
March 2007

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Abstract

An instance I of the Hospitals / Residents problem (HR) involves a set of residents (graduating medical students) and a set of hospitals, where each hospital has a given capacity. The residents have preferences for the hospitals, as do hospitals for residents. A solution of I is a *stable matching*, which is an assignment of residents to hospitals that respects the capacity conditions and preference lists in a precise way. In this paper we present constraint encodings for HR that give rise to important structural properties. We also present a computational study using both randomly-generated and real-world instances. We provide additional motivation for our models by indicating how side constraints can be added easily in order to solve hard variants of HR.

1 Introduction

Gale and Shapley described in their seminal paper [7] the classical Hospitals / Residents problem (HR), referred to by the authors as the College Admissions problem. An instance of HR involves a set of *residents* (i.e. graduating medical students) and a set of *hospitals*. Each resident ranks in order of preference a subset of the hospitals. Each hospital has an integral *capacity*, and ranks in order of preference those residents who ranked it. We seek to match each resident to an acceptable hospital, in such a way that a hospital's capacity is never exceeded. Moreover the matching must be *stable* – a formal definition of stability follows, but informally stability ensures that no resident and hospital, not already matched together, would rather be assigned to one another than remain with their assignees. Such a resident and hospital could form a private arrangement outside the matching, undermining its integrity. Gale and Shapley [7] described a linear-time algorithm for finding a stable matching, given an instance of HR.

Many centralised matching schemes that automate the process of assigning residents to hospitals employ algorithms that solve HR and its variants [24]. For example, the National Resident Matching Program (NRMP) in the US [22] is perhaps the largest such scheme. The NRMP has been in operation since 1952 and handles the annual allocation of some 31,000 residents to hospitals. Counterparts of the NRMP elsewhere are the Canadian Resident Matching Service (CaRMS) [5] and the Scottish Foundation Allocation Scheme (SFAS) [14]. Similar matching schemes are also used in educational and vocational contexts.

*Supported by Engineering and Physical Sciences Research Council grant GR/R84597/01.

†Supported by Royal Society of Edinburgh/Scottish Executive Personal Research Fellowship and by Engineering and Physical Sciences Research Council grant EP/E011993/1.

‡Supported by an Engineering and Physical Sciences Research Council studentship.

A special case of HR occurs when each hospital has capacity 1 – this is the Stable Marriage problem with Incomplete lists (SMI). In this context, residents are referred to as *men*, whilst hospitals are referred to as *women*. A special case of SMI occurs when the numbers of men and women are equal, and each man finds all women acceptable and vice versa – this is the classical Stable Marriage problem (SM), also introduced by Gale and Shapley [7]. A specialised linear-time algorithm for SM, known as the Gale / Shapley (GS) algorithm [7], can be generalised to the SMI case [13, Section 1.4.2]. Using a process known as "cloning hospitals" (described in more detail in Section 3), a given instance I of HR may be transformed to an instance J of SMI, and the GS algorithm can be applied to J in order to give a stable matching in I. However in general this method expands the instance size, so that in practice specialised algorithms (such as those described in [13, Section 1.6]; see also Figure 2) are used to solve HR directly and achieve a better worst-case time complexity.

Over the last few decades, stable matching problems, and SM in particular, have been the focus of much attention in the literature [7, 16, 13, 26]. Several encodings of SM and its variants as a Constraint Satisfaction Problem (CSP) have been formulated [1, 9, 17, 10, 11, 12, 19, 30, 29]. Moreover, recent papers have focussed on distributed variants of SM (including the Stable Roommmates problem, a non-bipartite extension of SM) where preference lists are to be kept private [28, 27, 3, 4]. However, no encoding for HR has been considered before now.

This paper is concerned with a Constraint Programming (CP) approach to solving HR. We firstly present in Section 3 a cloned model for HR, indicating how existing formulations of SMI as a CSP [9] can be used in order to model HR. We then present in Section 4 a constraint-based model of HR that deals directly with an HR instance without cloning, achieving improved time and space complexities. We show that the effect of Arc Consistency (AC) propagation [2] applied to this model yields the same structure as the action of established algorithms for HR [7, 13]. As a consequence, a stable matching for the given HR instance can be obtained without search (in fact we can in general obtain two complementary stable matchings following AC propagation, with optimality properties for the residents and hospitals respectively). We also demonstrate how a failure-free enumeration can be used to find all solutions for a given HR instance without search. These results therefore extend analogous results presented in [9] for SMI. In Section 5, we present a specialised n-ary constraint for HR, comparing and constrasting the time and space requirements for establishing AC with the models presented in Sections 3 and 4. Then, in Section 6, we describe the results of an empirical study which compares the various models presented in this paper in practice, on both randomly-generated and real-world data.

The models in Sections 4 and 5 are non-trivial extensions of earlier constraint models presented for SMI [9, 19, 30, 29]. In the SMI case, clearly each woman can be assigned at most one man, but to model an HR instance without cloning, the main challenges are to maintain a representation of the *set* of assignees of a given hospital h_j, and of the identity of the worst resident assigned to h_j.

The benefits of our approach are two-fold: firstly, the CSP models presented here for HR indicate that AC propagation using a CP toolkit yields the same structure as given by established linear-time algorithms for HR, from which all solutions for a given instance can be generated in a failure-free manner without search. Secondly, and more importantly, our models can be used as a basis on which additional constraints can be imposed, covering variants of HR that arise naturally in practical applications, but which cannot be accommodated easily by existing algorithms. These include variants of HR that are NP-hard, and for which no polynomial-time algorithm is currently known. Examples

Residents' preferences	M_0	M_z	Hospitals' preferences
$r_1: \; h_1 \; h_3$	–	–	$h_1: \; (2): \; \underline{r_3} \; r_7 \; \underline{r_5} \; \underline{r_2} \; r_4 \; r_6 \; r_1$
$r_2: \; \underline{h_1} \; h_5 \; \underline{h_4} \; \underline{h_3}$	h_1	h_3	$h_2: \; (3): \; r_5 \; \underline{r_6} \; r_3 \; \underline{r_4}$
$r_3: \; \underline{h_1} \; h_2 \; h_5$	h_1	h_1	$h_3: \; (1): \; \underline{r_2} \; \underline{r_5} \; r_6 \; r_1 \; r_7$
$r_4: \; h_1 \; \underline{h_2} \; h_4$	h_2	h_2	$h_4: \; (1): \; \underline{r_8} \; \underline{r_2} \; r_4 \; \underline{r_7}$
$r_5: \; \underline{h_3} \; \underline{h_1} \; h_2$	h_3	h_1	$h_5: \; (1): \; r_3 \; \underline{r_7} \; r_6 \; \underline{r_8} \; r_2$
$r_6: \; h_3 \; \underline{h_2} \; h_1 \; h_5$	h_2	h_2	
$r_7: \; h_3 \; \underline{h_4} \; \underline{h_5} \; h_1$	h_4	h_5	
$r_8: \; \underline{h_5} \; \underline{h_4}$	h_5	h_4	

Figure 1: An HR instance. The GS-list entries are underlined, and the middle two columns indicate the residents' assigned hospitals in M_0 and M_z (r_1 is unassigned in both).

of such variants, where appropriate side-constraints are suggested in three cases, are given in Section 7 to provide additional motivation for our approach.

In the next section we present notation and terminology relating to HR, which will be assumed in the remainder of this paper, and we also present some important structural and algorithmic results.

2 Definitions and fundamental results

We now give a formal definition of HR. An instance I of HR comprises a set $R = \{r_1, \ldots, r_n\}$ of *residents* and a set $H = \{h_1, \ldots, h_m\}$ of *hospitals*. Each resident $r_i \in R$ has an *acceptable* set of hospitals $A_i \subseteq H$; moreover r_i ranks A_i in strict order of preference. For each $h_j \in H$, denote by $B_j \subseteq R$ those residents who find h_j acceptable; h_j ranks B_j in strict order of preference. Finally, each hospital $h_j \in H$ has an associated *capacity*, denoted by $c_j \in \mathbb{Z}^+$, indicating the number of *posts* that h_j has. For each $r_i \in R$, let l_i^r denote the length of r_i's preference list, and for each $h_j \in H$, let l_j^h denote the length of h_j's preference list; we assume that $c_j \leq l_j^h$. Let L denote the total length of the residents' preference lists in I. Given $r_i \in R$ and $h_j \in A_i$, define $rank(r_i, h_j)$ to be the position of h_j in r_i's preference list; $rank(h_j, r_i)$ is defined similarly. An example HR instance is shown in Figure 1 (the hospital capacities are indicated in brackets).

An *assignment* M is a subset of $R \times H$ such that $(r_i, h_j) \in M$ implies that $h_j \in A_i$ (i.e. r_i finds h_j acceptable). If $(r_i, h_j) \in M$, we say that r_i is *assigned to* h_j, and h_j is *assigned* r_i. For any $q \in R \cup H$, we denote by $M(q)$ the set of assignees of q in M. If $r_i \in R$ and $M(r_i) = \emptyset$, we say that r_i is *unassigned*, otherwise r_i is *assigned*. Similarly, any hospital $h_j \in H$ is *under-subscribed*, *full* or *over-subscribed* according as $|M(h_j)|$ is less than, equal to, or greater than c_j, respectively.

A *matching* M is an assignment such that $|M(r_i)| \leq 1$ for each $r_i \in R$ and $|M(h_j)| \leq c_j$ for each $h_j \in H$ (i.e. each resident is assigned to at most one hospital, and no hospital is over-subscribed). For convenience, given a resident $r_i \in R$ such that $M(r_i) \neq \emptyset$, where there is no ambiguity the notation $M(r_i)$ is also used to refer to the single member of $M(r_i)$

A *blocking pair* relative to a matching M is a (resident,hospital) pair $(r_i, h_j) \in (R \times H) \backslash M$ such that (i) $h_j \in A_i$, (ii) either r_i is unassigned in M or prefers h_j to $M(r_i)$, and (iii) either h_j is under-subscribed or prefers r_i to at least one member of $M(h_j)$. A matching is *stable* if it admits no blocking pair.

Gale and Shapley [7] described an algorithm for finding a stable matching in a given HR instance I, which is known as the *resident-oriented* Gale/Shapley (RGS) algorithm [13, Section 1.6.3]. This algorithm finds the *resident-optimal* stable matching M_0 in I, in

$M = \emptyset$;
while (some $r_i \in R$ is unassigned
  and r_i has a non-empty list)
  h_j = first hospital on r_i's list;
  /* *r_i applies to h_j* */
  $M = M \cup \{(r_i, h_j)\}$;
  if (h_j is over-subscribed)
    r_k = worst resident assigned to h_j;
    $M = M \backslash \{(r_k, h_j)\}$;
  if (h_j is full)
    r_k = worst resident assigned to h_j;
    for (each successor r_z of r_k on h_j's list)
      delete the pair (r_z, h_j);

$M = \emptyset$;
while (some $h_j \in H$ is under-subscribed
  and some $r_i \in B_j$ is not assigned to h_j)
  r_i = first such resident on h_j's list;
  /* *h_j offers a post to r_i* */
  if (r_i is assigned)
    $h_k = M(r_i)$;
    $M = M \backslash \{(r_i, h_k)\}$;
  $M = M \cup \{(r_i, h_j)\}$;
  for (each successor h_z of h_j on r_i's list)
    delete the pair (r_i, h_z);

Figure 2: RGS algorithm for HR; HGS algorithm for HR.

which each assigned resident is assigned to the best hospital that he could obtain in any stable matching. On the other hand, the *hospital-oriented* (HGS) algorithm [13, Section 1.6.2] is a second algorithm for HR that finds the *hospital-optimal* stable matching M_z in I, in which each hospital is assigned the best set of residents that it could obtain in any stable matching. Figure 1 includes columns that give M_0 and M_z for the example HR instance shown. In general, the optimality property of each of M_0 and M_z is achieved at the expense of the hospitals and residents respectively (the "pessimality" of each of these matchings for the relevant parties is discussed in Sections 1.6.2 and 1.6.5 of [13]). The RGS and HGS algorithms for HR are shown in Figure 2 (the term "delete the pair (r_i, h_j)" refers to the operations of deleting r_i from h_j's preference list and vice versa). Using a suitable choice of data structures (extending those described in [13, Section 1.2.3]), both the RGS and the HGS algorithms can be implemented to run in $O(L)$ time and $O(nm)$ space.

The deletions made by each of the RGS and HGS algorithms have the effect of reducing the original set of preference lists in I. The reduced lists returned by the RGS (respectively HGS) algorithm are known as the *RGS-lists* (respectively *HGS-lists*). The intersection of the RGS-lists and the HGS-lists yields the *GS-lists*. (E.g. the GS-lists for the HR instance shown in Figure 1 are represented as underlined preference list entries.) The GS-lists in I have several useful properties, which are summarised below (these properties follow as a consequence of Lemmas 1.6.2 and 1.6.4, and Theorems 1.6.1 and 1.6.2 of [13]):

Theorem 1. *For a given instance of HR,*
(i) all stable matchings are contained in the GS-lists;
(ii) in M_0, each resident with a non-empty GS-list is assigned to the first hospital on his GS-list, whilst each resident with an empty GS-list is unassigned;
(iii) in M_z, each hospital h_j is assigned the first m_j members of its GS-list, where $m_j = \min\{c_j, g_j^h\}$ and g_j^h is the length of h_j's GS-list.

Given any $q \in R \cup H$, we denote by $GS(q)$ the set of hospitals or residents (as appropriate) that belong to q's GS-list in I.

Additional important results, attributed to Gale and Sotomayor [8] and Roth [25], concern residents who are unassigned, and hospitals that are under-subscribed, in stable matchings in I. These results are collectively known as the *Rural Hospitals Theorem* [13, Section 1.6.4], and may be stated as follows:

Theorem 2. *For a given instance of HR,*
(i) each hospital is assigned the same number of residents in all stable matchings;

(ii) exactly the same residents are unassigned in all stable matchings;
(iii) any hospital that is under-subscribed in one stable matching is assigned precisely the same set of residents in all stable matchings.

3 A cloned model

In this section we indicate how an instance of HR may be reduced to an instance of SMI by "cloning" hospitals. This technique is described in [13, p.38]; see also [26, pp.131-132]. For completeness, we briefly restate the construction here. Let I be an instance of HR. We form an instance J of SMI by replacing each hospital $h_j \in H$ by c_j women in J, denoted by h_j^k $(1 \leq k \leq c_j)$. The preference list of h_j^k in J is identical to that of h_j in I. Each resident r_i in I corresponds to a man r_i in J, and each hospital h_j in r_i's list in I is replaced by $h_j^1\ h_j^2\ \ldots\ h_j^{c_j}$, in that order, in J. It may then be shown that the stable matchings in I are in one-one correspondence with the stable matchings in J.

In order to obtain the GS-lists of I, we can model J using the "conflict matrices" encoding of SMI as presented in [9]. In general AC may be established in $O(ed^r)$ time, where e is the number of constraints, d is the domain size, and r is the arity of each constraint [2]. Due to the cloning technique, the number of women in J is $\sum_{j=1}^{m} c_j = O(cm)$, where $c = \max\{c_j : h_j \in H\}$. Given the construction of the encoding in J [9], it follows that $e = O(nmc)$, $d = O(n + m)$ and $r = 2$, so that the time and space complexities for finding the GS-lists in I using the cloned model are $O((n + m)^4 c)$ and $O((nmc)^2)$ respectively.

4 A direct CSP-based model

We now present a direct CSP encoding of an HR instance that avoids cloning. Let I be an instance of HR. For $r_i \in R$ and $h_j \in H$, we use the terminology r_i *applies (or is assigned) to* h_j*'s* k^{th} *post* $(1 \leq k \leq c_j)$ in the case that h_j prefers exactly $k-1$ members of $M(h_j)$ to r_i. Also given a matching M, we denote the resident who is assigned to h_j's k^{th} post in M by $M_k(h_j)$ $(1 \leq k \leq |M(h_j)|)$.

We construct a CSP instance J with variables $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_{j,k} : 1 \leq j \leq m \wedge 0 \leq k \leq c_j\}$, whose domains are initially defined as follows:

$$\begin{array}{ll}
dom(x_i) = \{1, 2, \ldots, l_i^r\} \cup \{m+1\} & (1 \leq i \leq n) \\
dom(y_{j,0}) = \{0\} & (1 \leq j \leq m) \\
dom(y_{j,k}) = \{k, k+1, \ldots, l_j^h\} \cup \{n+k\} & (1 \leq j \leq m \wedge 1 \leq k \leq c_j).
\end{array}$$

For the x_i variables $(1 \leq i \leq n)$, the value $m+1$ corresponds to the case that r_i's GS-list is empty, whilst the remaining values correspond to the ranks of preference list entries that belong to the GS-lists. A similar meaning applies to the $y_{j,k}$ variables $(1 \leq j \leq m$, $1 \leq k \leq c_j)$, except that the value $n+k$ corresponds to the case that h_j's GS-list contains fewer than k entries.

More specifically, if $\min(dom(x_i)) \geq p$ $(1 \leq p \leq l_i^r)$, then during the RGS algorithm, r_i applies to his p^{th}-choice hospital or worse, so that in M_0, either r_i is assigned to such a hospital or is unassigned. Similarly if $\max(dom(x_i)) \leq p$, then during the HGS algorithm, r_i was offered a post by his p^{th}-choice hospital or better, so that r_i is assigned to such a hospital in M_z.

From the hospitals' point of view, if $\min(dom(y_{j,k})) \geq q$ $(1 \leq q \leq l_j^h)$, then during the HGS algorithm, h_j offers its k^{th} post to its q^{th}-choice resident or worse, so that in M_z, either h_j's k^{th} post is filled by such a resident, or is unfilled. Similarly if $\max(dom(y_{j,k})) \leq q$, then

1.	$y_{j,k} < y_{j,k+1}$	$(1 \leq j \leq m, 1 \leq k \leq c_j - 1)$
2.	$y_{j,k} \geq q \Rightarrow x_i \leq p$	$(1 \leq j \leq m, 1 \leq k \leq c_j, 1 \leq q \leq l_j^h)$
3.	$x_i \neq p \Rightarrow y_{j,k} \neq q$	$(1 \leq i \leq n, 1 \leq p \leq l_i^r, 1 \leq k \leq c_j)$
4.	$(x_i \geq p \wedge y_{j,k-1} < q) \Rightarrow y_{j,k} \leq q$	$(1 \leq i \leq n, 1 \leq p \leq l_i^r, 1 \leq k \leq c_j)$
5.	$y_{j,c_j} < q \Rightarrow x_i \neq p$	$(1 \leq j \leq m, c_j \leq q \leq l_j^h)$

Figure 3: Constraints for the CSP model of an HR instance.

during the RGS algorithm, some resident r_i applied to h_j's k^{th} post, where $rank(h_j, r_i) \leq q$, so that h_j's k^{th} post is filled by r_i or better in M_0.

The constraints in J are given in Figure 3 (in the context of Constraints 2-5, p denotes the rank of h_j in r_i's list and q denotes the rank of r_i in h_j's list). An interpretation of the constraints is now given. Constraint 1 ensures that h_j's filled posts are occupied by residents in preference order, and that if post $k-1$ is unfilled then so is post k. Constraint 2 states that if h_j's k^{th} post is filled by a resident no better than r_i or is unfilled, then r_i must be assigned to a hospital no worse than h_j. Constraints 3 and 5 reflect the consistency of deletions carried out by the HGS and RGS algorithms respectively (i.e. if h_j is deleted from r_i's list, then r_i is deleted from h_j's list, and vice versa). Finally Constraint 4 states that if r_i is assigned to a hospital no better than h_j or is unassigned, and h_j's first $k-1$ posts are filled by residents better than r_i, then h_j's k^{th} post must be filled by a resident at least as good as r_i.

It turns out that establishing AC in J yields a set of domains that correspond to the GS-lists in I. We prove this using three lemmas. The first two lemmas show that the arc consistent domains correspond to subsets of the HGS-lists and the RGS-lists respectively. The third lemma shows that the GS-lists correspond to arc consistent domains.

Lemma 3. *(i) For a given j $(1 \leq j \leq m)$, let q be an integer $(q \leq n)$ such that $q \in dom(y_{j,k})$ for some k $(1 \leq k \leq c_j)$ after AC propagation. Then the resident r_i at position q on hospital h_j's preference list belongs to the HGS-list of h_j.*
(ii) For a given i $(1 \leq i \leq n)$, let p be an integer $(p \leq m)$ such that $p \in dom(x_i)$ after AC propagation. Then hospital h_j at position p on resident r_i's preference lists belongs to the HGS-list of r_i.

Proof. The HGS-lists are constructed as a result of the deletions made by the HGS algorithm. We show that the corresponding deletions are made to the variables' domains during AC propagation.

The following proof uses induction on the number of iterations of the main loop during an execution E of the HGS algorithm to show that, if iteration z consists of some hospital h_j offering some resident r_i its k^{th} post, then $x_i \leq p$, proving (ii) above, $y_{j,k} \geq q$, and $y_{v,b} \neq t$ $(1 \leq b \leq c_v)$, proving (i) above, for each hospital h_v such that $rank(r_i, h_v) > p$, where $t = rank(h_v, r_i)$, $p = rank(r_i, h_j)$ and $q = rank(h_j, r_i)$.

First consider the case where $z = 1$. On the first iteration of the main loop, hospital h_j offers resident r_i its first post, where $q = rank(h_j, r_i) = 1$ and $p = rank(r_i, h_j)$. By the domain initialisations, $y_{j,k} \geq 1$ $(1 \leq k \leq c_j)$, therefore propagation of Constraint 2 yields $x_i \leq p$. Finally, consider each hospital h_v where $rank(r_i, h_v) > p$. By propagation of Constraint 3 we obtain $y_{v,b} \neq t$ $(1 \leq b \leq c_v)$, where $t = rank(h_v, r_i)$, giving the required result.

Now suppose that $z = d > 1$, and that the result holds for $z < d$. We consider the two cases where (i) $k = 1$ and (ii) $k > 1$.

Case (i). Suppose that $k = 1$. Then h_j offers its first post to the resident r_i such that $rank(h_j, r_i) = q$. If $q = 1$, the proof is similar to that of the base case. Hence suppose that $q > 1$. Let r_{u_2} be any resident such that $rank(h_j, r_{u_2}) = t_2 < q$. Then r_{u_2} has been deleted from h_j's list. Let $s_2 = rank(r_{u_2}, h_j)$. Then r_{u_2} must have received an offer from some hospital h_v whom he prefers to h_j, where $rank(r_{u_2}, h_v) = s_3 < s_2$. Let $t_3 = rank(h_v, r_{u_2})$. Then h_v offered its a^{th} post, for some a $(1 \leq a \leq c_v)$, to r_{u_2} before the d^{th} iteration. By the induction hypothesis, it follows that $y_{v,a} \geq t_3$, $x_{u_2} \leq s_3$ and $y_{j,k} \neq t_2$ $(1 \leq k \leq c_j)$. However r_{u_2} was arbitrary, and hence $y_{j,k} \neq t_2$ for all t_2 such that $1 \leq t_2 \leq q - 1$. Hence $y_{j,k} \geq q$. The rest of the proof is similar to that of the base case.

Case (ii). Now suppose that $k > 1$. Let r_{u_1} be the last resident to which h_j offered its $(k-1)^{th}$ post. This occurred during the g^{th} iteration for some g $(g < d)$. Suppose that $rank(h_j, r_{u_1}) = t_1 < q$. Then by the induction hypothesis we have $y_{j,k-1} \geq t_1$, therefore propagation of Constraint 1 yields:

$$y_{j,k} \geq t_1 + 1. \tag{1}$$

If $q = t_1 + 1$, then the rest of the proof is similar to that of the base case. Hence suppose that $q > t_1 + 1$. Let r_{u_2} be any resident such that $rank(h_j, r_{u_2}) = t_2$ $(t_1 + 1 \leq t_2 \leq q - 1)$. Then r_{u_2} has been deleted from h_j's list. Now suppose $rank(r_{u_2}, h_j) = s_2$. Then r_{u_2} must have received an offer from some hospital h_v whom he prefers to h_j, where $rank(r_{u_2}, h_v) = s_3 < s_2$. Let $t_3 = rank(h_v, r_{u_2})$. Then h_v offered its a^{th} post, for some a $(1 \leq a \leq c_v)$, to r_{u_2} before the d^{th} iteration. By the induction hypothesis, it follows that $y_{v,a} \geq t_3$, $x_{u_2} \leq s_3$ and $y_{j,k} \neq t_2$ $(1 \leq k \leq c_j)$. However, r_{u_2} was arbitrary, so that:

$$y_{j,k} \neq t_2 \text{ for } t_1 + 1 \leq t_2 \leq q - 1. \tag{2}$$

Thus from Inequalities 1 and 2, we have $y_{j,k} \geq q$. The rest of the proof is similar to that of the base case. □

Lemma 4. *(i) For a given i $(1 \leq i \leq n)$, let p be an integer $(p \leq m)$ such that $p \in dom(x_i)$ after AC propagation. Then hospital the h_j at position p on resident r_i's preference lists belongs to the RGS-list of r_i.*
(ii) For a given j $(1 \leq j \leq m)$, let q be an integer $(q \leq m)$ such that $q \in dom(y_{j,k})$ for some k $(1 \leq k \leq c_j)$ after AC propagation. Then the resident r_i at position q on h_j's preference list belongs to the RGS-list of r_i.

Proof. The RGS-lists are constructed as a result of the deletions made by the RGS algorithm. We show that the corresponding deletions are made to the variables' domains during AC propagation.

The following proof uses induction on the number of iterations of the main loop during an execution E of the RGS algorithm to show that, if the z^{th} iteration of the main loop involves some resident r_i applying to some hospital h_j, and at the termination of this same iteration, residents $r_{i_1}, \ldots, r_{i_{d_j}}$ are assigned to h_j, where $d_j \leq c_j$, then $y_{j,k} \leq q_k$ $(1 \leq k \leq d_j)$, where $q_k = rank(h_j, r_i)$ and $0 < q_1 < q_2 < \ldots < q_{d_j}$, and also $x_{i_k} \geq p_{i_k}$, where $p_{i_k} = rank(r_{i_k}, h_j)$ $(1 \leq k \leq d_j)$. We use this result (in the case that $d_j = c_j$) to show that (ii) above is satisfied, and then propagation of Constraint 5 to show that (i) is also satisfied.

First consider the base case where $z = 1$. Then during the first iteration of the main loop, some resident r_i applies for the first post at hospital h_j, where $p = rank(r_i, h_j) = 1$, and $q = rank(h_j, r_i)$. Thus $x_i \geq p$ (by construction of the x_i variables' domains), and $y_{j,k-1} < q$, since $k = 1$ and $y_{j,0} = 0$ by definition. Therefore propagation of Constraint 4 yields $y_{j,k} \leq q$ as required.

Now suppose that $z = d > 1$, and that the result holds for $z < d$. Then during the d^{th} iteration, some resident r_i applies to some hospital h_j, and we let d_j denote the number of residents assigned to h_j just before r_i applies, where $d_j \geq 0$. We consider the cases where (i) $p = 1$ and (ii) $p > 1$.

Case (i). Suppose that $p = 1$, and therefore $x_i \geq p$ by initialisation of the variables' domains. We firstly note that if $d_j = 0$, the proof is similar to that of the base case. Now suppose that $d_j \geq 1$. Then there exists an iteration $g < d$ of the main loop, where some resident applies to h_j, such that iteration g' of the main loop, for $g < g' < d$, does not involve a resident applying to h_j. Then at the end of the g^{th} iteration, residents $r_{i_1}, \ldots, r_{i_{d_j}}$ are assigned to h_j, and by the induction hypotheses, $y_{j,k} \leq q_k$ $(1 \leq k \leq d_j)$, where $q_k = rank(h_j, r_{i_k})$ and $0 < q_1 < q_2 < \ldots < q_{d_j}$. Now consider the two subcases where (a) $d_j < c_j$ and (b) $d_j = c_j$.

> *Subcase (a).* Suppose that $d_j < c_j$. If $q > q_{d_j}$, then at the d^{th} iteration, r_i is assigned to h_j's $(d_j + 1)^{th}$ post. From above we have that $y_{j,d_j} \leq q_{d_j} < q$ and since $x_i \geq p$, propagation of Constraint 4 yields $y_{j,d_j+1} \leq q$, as required. Now suppose that $q < q_{d_j}$. Then there exists b $(1 \leq b \leq d_j)$ such that $q_{b-1} < q < q_b$ (for convenience we define $q_0 = 0$). Therefore at the d^{th} iteration, r_i is assigned to h_j's b^{th} post. Thus from above $y_{j,b-1} \leq q_{b-1} < q$, and since $x_i \geq p$, propagation of Constraint 4 yields $y_{j,b} \leq q$. Furthermore, $y_{j,b} \leq q < q_b$, and by the induction hypothesis $x_{i_b} \geq p_{i_b}$, where $p_{i_b} = rank(r_{i_b}, h_j)$. Again propagation of Constraint 4 yields $y_{j,b+1} \leq q_b$. Continuing in this manner we obtain $y_{j,k} \leq q_{k-1}$ for all k $(b + 1 \leq k \leq d_j + 1)$, as required.
>
> *Subcase (b).* Now suppose that $d_j = c_j$. Then when r_i applies to h_j at the d^{th} iteration, h_j becomes oversubscribed. Hence during the g^{th} iteration of the main loop, h_j must have become full. When this happens as part of the RGS algorithm, the worst assigned resident is identified, and all its successors on h_j's list are deleted. It follows that $q < q_{c_j}$. Also, during the d^{th} iteration, resident $r_{i_{d_j}}$ is rejected from h_j. The remainder of the proof is similar to that used in Subcase (a) when $q < q_{d_j}$.

Case (ii). Now suppose that $p > 1$. Let h_v be a hospital such that $rank(r_i, h_v) = s_1 < p$. Let $t_1 = rank(h_v, r_i)$. Then h_v has been deleted from r_i's list during the execution of the RGS algorithm. This can only happen if h_v became full at the g^{th} iteration (for some $g < d$) of the RGS algorithm. At this point the worst resident r_u assigned to h_v is identified, where $rank(h_v, r_u) = t_2 < t_1$. Since h_v is full, r_u is assigned to h_v's c_v^{th} post at the end of g^{th} iteration, so by the induction hypothesis $y_{v,c_v} \leq t_2 < t_1$. Thus propagation of Constraint 5 yields $x_i \neq s_1$. But h_v was arbitrary and hence $x_i \neq s_1$ for all s_1 such that $1 \leq s_1 \leq p - 1$, so $x_i \geq p$. The rest of the proof is similar to that used in Case (i). $\square$

To demonstrate that the GS-lists give rise to arc consistent domains, we define some additional notation. For each j $(1 \leq j \leq m)$, define $S_j = \{rank(h_j, r_i) : r_i \in GS(h_j)\}$. Let d_j denote the number of residents assigned to hospital h_j in M_0 (or indeed in any stable matching in I, by Theorem 2(i)). For each k $(1 \leq k \leq d_j)$, let $q_{j,k} = rank(h_j, M_{z_k}(h_j))$ and $t_{j,k} = rank(h_j, M_{0_k}(h_j))$. The *GS-domains* for the variables in J are defined as follows:

$$dom(x_i) = \begin{cases} \{rank(r_i, h_j) : h_j \in GS(r_i)\}, & \text{if } GS(r_i) \neq \emptyset \\ \{m + 1\}, & \text{otherwise} \end{cases}$$

$$dom(y_{j,k}) = \begin{cases} \{s \in S_j : q_{j,k} \leq s \leq t_{j,k}\}, & \text{if } 1 \leq k \leq d_j \\ \{n + k\}, & \text{if } d_j + 1 \leq k \leq c_j. \end{cases}$$

Lemma 5. *The GS-domains are arc consistent in J.*

Proof. First consider Constraint 1, and suppose that $k < d_j$. Then $\min(dom(y_{j,k+1})) = q_{j,k+1} > q_{j,k} = \min(dom(y_{j,k}))$. Now suppose that $k = d_j < c_j$. Then $y_{j,k+1} = n+d_j+1 > n \geq y_{j,k}$. Finally suppose that $d_j < k < c_j$. Then $y_{j,k+1} = n + d_j + 1 > n + d_j = y_{j,k}$.

Now consider Constraint 2 and suppose that $y_{j,k} \geq q$. Then during the execution of the HGS algorithm either (i) hospital h_j offered the resident r_i at position q its a^{th} post for some a $(1 \leq a \leq c_j)$, or (ii) the pair (r_i, h_j) was deleted, where $p = rank(r_i, h_j)$ and $q = rank(h_j, r_i)$. Now consider the two cases below:

Case (i). If h_j offered resident r_i its a^{th} post as part of the HGS algorithm, then r_i will delete all those hospitals ranked lower than h_j on his preference list, i.e. $x_i \leq p$.

Case (ii). If the pair (r_i, h_j) is deleted, then resident r_i must have received an offer from a hospital h_v that he prefers to h_j, where $rank(r_i, h_v) = s < p$. Since r_i deletes all hospitals in his preference list ranked below h_v when he receives such an offer, it follows that $x_i \leq s$. In particular $x_i \leq p$.

Consider Constraint 3, and suppose that $x_i \neq p$. Then hospital h_j has been deleted from resident r_i's preference list, where $rank(r_i, h_j) = p$, by either the RGS or HGS algorithm. The same algorithm ensures that the preference lists are consistent and removes r_i from the list of h_j, i.e. $y_{j,k} \neq q$ $(1 \leq k \leq c_j)$, where $q = rank(h_j, r_i)$.

For Constraint 4, suppose that $x_i \geq p$ and $y_{j,k-1} < q$, where $p = rank(r_i, h_j)$ and $q = rank(h_j, r_i)$. If $t_{j,k} \leq q$, then $y_{j,k} \leq q$, since $y_{j,k} \leq t_{j,k}$ by definition, as required. Now suppose for a contradiction that $t_{j,k} > q$. Then $t_{j,a} < q$ for $1 \leq a \leq k-1$, and $t_{j,a} > q$ for $k \leq a \leq c_j$. Hence r_i is not assigned to h_j in M_0, so (r_i, h_j) was deleted as part of the RGS algorithm, since either r_i is unassigned in M_0 or prefers h_j to $M_0(r_i)$. As (r_i, h_j) has been deleted, h_j must have become full during an execution of RGS algorithm with residents that it prefers to r_i. It follows that $t_{j,c_j} < q$, a contradiction. Hence $t_{j,k} \leq q$.

Finally consider Constraint 5 and suppose that $y_{j,c_j} < q$. Then resident r_i has been deleted from hospital h_j's preference list, where $rank(h_j, r_i) = q$, by either the HGS or RGS algorithm. The same algorithm ensures that the preference lists are consistent and removes h_j from the list of r_i, i.e. $x_i \neq p$, where $p = rank(r_i, h_j)$. □

The following result follows by Lemmas 3, 4 and 5, and the fact that AC algorithms find the unique maximal set of arc consistent domains [2].

Theorem 6. *Let I be an instance of HR, and let J be a CSP instance obtained by the encoding of this section. Then the domains remaining after AC propagation in J correspond exactly to the GS-lists in I.*

For example, in the context of the HR instance given in Figure 1, the GS-domains for x_2, $y_{1,1}$ and $y_{1,2}$ are $\{1,3,4\}$, $\{1\}$ and $\{3,4\}$ respectively. In general, following AC propagation in J, matchings M_0 and M_z may be obtained as follows. Let $x_i \in X$. If $x_i = m+1$, resident r_i is unassigned in both M_0 and M_z. Otherwise, in M_0 (respectively M_z), r_i is assigned to the hospital h_j such that $rank(r_i, h_j) = p$, where $p = \min(dom(x_i))$ (respectively $p = \max(dom(x_i))$).

In the context of the time complexity function for establishing AC as mentioned in Section 3, for this encoding we have $e = O(Lc)$ and $d = O(n+m)$ (recall that L is the total length of the residents' preference lists in I). The constraints shown in Figure 3 may be revised in $O(1)$ time, assuming that upper and lower bounds for the variables' domains are maintained throughout propagation. It follows by [31] that the time complexity for establishing AC in this model is $O(Lc(n+m))$. Since the space complexity is $O(Lc)$, the model presented in this section is more efficient than the cloned model in terms of both time and space.

The next result shows that the encoding presented above can be used to enumerate all the solutions of I in a failure-free manner using AC propagation with a value-ordering heuristic. Before presenting this result, we firstly remark that if a variable x_i has two values in its domain following AC propagation, then neither value can be $m+1$. For, if $m+1 \in dom(x_i)$, then r_i is unassigned in M_z, for otherwise some hospital would have offered r_i a post during an execution of the HGS algorithm, resulting in the removal of value $m+1$ from x_i's domain. Now let $p = \min(dom(x_i))$ and suppose that $p \leq m$. Then r_i applies to some hospital during an execution of the RGS algorithm, so that r_i is assigned in M_0. This is a contradiction to Theorem 2(ii). In what follows, for any persons p and q in I, q is a *stable partner* of p if p and q are partners in some stable matching in I.

Theorem 7. *Let I be an instance of HR and let J be a CSP instance obtained by the encoding of this section. Then the following search process enumerates all solutions in I without repetition and without ever failing due to an inconsistency:*

- *AC is established as a preprocessing step, and after each branching decision including the decision to remove a value from a domain;*
- *if all domains are arc consistent and some variable x_i has two or more values in its domain then search proceeds by setting x_i to the minimum value p in its domain. On backtracking, the value p is removed from the domain of x_i;*
- *when a solution is found, it is reported and backtracking is forced.*

Proof. Let T be the search tree as defined above. We prove by induction on T that each node in T corresponds to an arc consistent CSP instance J', which in turn corresponds to the GS-lists I' for an HR instance derived from I such that any stable matching in I' is also stable in I. To prove this we first show that it holds for the root node of T, then we assume it is true at any branch node u in T and show that it is true for each child of u.

The root node of T corresponds to the CSP instance J' with arc consistent domains, where J' is obtained from J by AC propagation. Therefore by Theorem 6, J' corresponds to the GS-lists I' for the HR instance I. Using standard properties of the GS-lists [13, Lemmas 1.6.2 and 1.6.4], any stable matching in I' is also stable in I.

Now suppose that we have reached a branching node u of T. By the induction hypothesis we have, associated with u, a CSP instance J' with arc consistent domains. Furthermore, J' corresponds to the GS-lists I' for an HR instance derived from I, such that any stable matching in I' is stable in I. Then since u is a branching node, there exists a variable x_i ($1 \leq i \leq n$) such that the domain of x_i contains at least two values. Hence in T, u has two children, namely v_1 and v_2, each having an associated CSP instance J'_1 and J'_2 derived from J' in the following way. In J'_1, x_i is assigned the smallest value p (which corresponds to the rank of r_i's best stable partner h_j in I') in its domain, and in J'_2, p is removed from x_i's domain.

First consider instance J'_1. During AC propagation in J'_1 we consider the revisions made by Constraint 3 when x_i is assigned the value p. Let h_v be a hospital such that $rank(r_i, h_v) > p$. Then AC propagation in J'_1 forces $y_{v,k} \neq t$ ($1 \leq k \leq c_v$), where $t = rank(h_v, r_i)$. After such revisions, J'_1 corresponds to an HR instance I'_1 obtained from I' by deleting the pairs (r_i, h_v), where $v \neq j$. Now let M be any stable matching in I'_1. Suppose that the pair (r, h) blocks M in I'. If $h \in PL(r)$ in I'_1, then (r, h) blocks M in I'_1, so (r, h) must have been deleted in I'_1. Hence $(r, h) = (r_i, h_v)$ for some h_v such that $rank(r_i, h_v) > p$. Now suppose that M_0 denotes the resident-optimal stable matching in I'. In M_0, each resident obtains his best possible stable partner in I', hence $(r_i, h_j) \in M_0$. It can be easily verified that M_0 is also stable in I'_1. Theorem 2(ii) applied to I'_1 therefore implies that r_i is matched in M. In particular, $(r_i, h_j) \in M$, as h_j is the only hospital on r_i's list in I'_1. Thus (r_i, h_v) cannot block in M in I' after all, as r_i prefers h_j to h_v. Therefore M is stable in I', and hence by the induction hypothesis M is also stable in I. So at node v_1, AC is established in J'_1 giving instance J''_1 which we associate with this node. By Theorem 6, J''_1 corresponds to the GS-lists I''_1 of HR instance I'_1. Using the properties of the GS-lists given in [13, Lemmas 1.6.2 and 1.6.4], we have that any stable matching in I''_1 is stable in I'_1, which in turn is stable in I by the preceding argument.

We now consider J'_2. Let $q = rank(h_j, r_i)$. Then during AC propagation in J'_2 we consider the revisions made when p is removed from the domain of x_i. Propagation of Constraint 3 forces $y_{j,k} \neq q$ ($1 \leq k \leq c_j$). Then propagation of Constraint 4 gives $y_{j,1} \leq q$. However $y_{j,1} \neq q$, so $y_{j,1} < q$. Hence further propagation of Constraint 4 gives $y_{j,2} \leq q$, and hence $y_{j,2} < q$. Continuing in this way we obtain $y_{j,k} < q$, for $1 \leq k \leq c_j$. Hence after such revisions J'_2 corresponds to an HR instance I'_2 obtained from I' by deleting the pairs (r_u, h_j), where $rank(h_j, r_u) \geq q$. Now let M be any stable matching in I'_2. Suppose that (r, h) blocks M in I'. Then $(r, h) = (r_u, h_j)$, for some r_u where $rank(h_j, r_u) \geq q$, for otherwise (r, h) blocks M in I'_2. Consider M_z, the hospital-optimal stable matching in I', where each resident obtains his worst possible stable partner in I' [13, Theorem 1.6.1]. Then M_z is a matching in I'_2, since $(r_i, h_j) \notin M_z$, and hence h_j is full in M_z and prefers its worst assignee to r_i, for otherwise (r_i, h_j) blocks M_z in I'. Clearly M_z is stable in I''_2. By Theorem 2 applied to I'_2, h_j must be full in M. Also by construction of I'_2, h_j prefers its worst assignee in M to r_i. Hence (r_u, h_j) does not block M in I' after all. Thus M is stable in I', and hence by the induction hypothesis M is also stable in I. Now at node v_2, AC is established in J'_2 giving instance J''_2 which we associate with this node. The rest of the proof is similar to that used for instance J'_1 above. Hence by induction the claim is true for all nodes in T.

We can now see that the branching process never fails due to an inconsistency – setting the variable x_i to p leaves the resident-optimal stable matching, while excluding p always leaves the hospital-optimal stable matching. Also, since we explore all areas of the search space with the branching process, all possible stable matchings for an HR instance I are listed. We can also prove that there are no repeated solutions. First observe that the leaf nodes of T correspond to the stable matchings in I. Suppose for a contradiction that leaf nodes l_1 and l_2 correspond to the same stable matching M in I. Let b be the lowest common ancestor of l_1 and l_2 in T. Without loss of generality, assume l_1 is reached by taking the path from the left child of b, and l_2 is reached by taking the path from the right child of b. We know that node b corresponds to the GS-lists I' for a particular HR instance derived from I, such that some variable x_i has at least two values in its domain. This means that in I' there exists some resident r_i who has a GS-list of length greater than one. Then the left child of b is obtained by forcing r_i to be assigned to the hospital h_j at the head of his list in I', and similarly the right child of b is obtained by removing h_j from r_i's list. So l_1 corresponds to a stable matching M_1 where $(r_i, h_j) \in M_1$, and l_2 corresponds to a stable matching M_2 where $(r_i, h_j) \notin M_2$, i.e. $M_1 \neq M_2$. Therefore we have that each leaf node corresponds to a unique stable matching. □

5 A specialised n-ary constraint

We now present a specialised n-ary constraint HRN for the Hospitals / Residents problem. A model based on HRN requires only one constraint for the whole problem. We assume that this constraint will be processed by an HDT92 [31] type arc consistency algorithm. That is, the algorithm has a stack of calls to revise constraints, and if a variable v loses a value then a call to all constraints involving v will be added to the stack along with the removed value.

5.1 Preliminaries

Our model involves a constrained integer variable x_i corresponding to each resident $r_i \in R$, where the domain values represent ranks, as in Section 4. In addition, we associate a single constrained integer variable y_j corresponding to each hospital $h_j \in H$ with similar meanings for the domain values. In this model only the x variables are search variables, meaning that a solution consists of a single value being assigned to each x variable, but the y variables may have multiple values remaining in their associated domains.

We assume that we have the following functions, each being of $O(1)$ complexity, that operate over constrained integer variables:

- $getMin(v)$ delivers the smallest value in $dom(v)$.
- $getMax(v)$ delivers the largest value in $dom(v)$.
- $getValue(v, a)$ returns the a^{th} smallest value in $dom(v)$, if $|dom(v)| < a$ then $getMax(v)$ is returned.
- $setMax(v, a)$ removes all values greater than a from $dom(v)$.
- $remVal(v, a)$ removes the value a from $dom(v)$.
- $PL(r_i, k)$ returns the k^{th} entry in r_i's preference list.
- $swap(a, b)$ swaps the values of the variables a and b.

The HRN constraint also requires the following data structures:

- $\check{x}$ is an array of n reversible integer variables containing the previous lower bounds of all x variables. All are initially set to $min(x) - 1$. On backtracking the values in $\check{x}$ are restored by the solver.
- $\check{y}$ is an array of m reversible integer variables containing the value that represents y's least favourite resident to be offered a post at y. For hospital h_j, $\check{y}_j$ will equal the c_j^{th} lowest value in $dom(y_j)$. All are initially set to $min(y) - 1$. On backtracking the values in $\check{y}$ are restored by the solver.
- $post$: an $m \times c$ matrix of reversible integer variables which stores applications for hospital posts. Each array element is initialised to ∞ (i.e. the largest integer). Row $post_j$ stores the applications for hospital h_j and entry $post_{j,k}$ stores the k^{th} best application received by hospital h_j.

To implement a constraint we require two methods: one that is called at the head of search to initialise the constraint and one that is called when a value is removed from a constrained variable. We now give the first of these methods:

The *init* method (Figure 4) is called at the head of search. Each resident applies to their favourite hospital (lines 2-4) via the *apply*(i) function (details given later), then each hospital makes an offer to their c favourite residents (lines 5-7) via the *offer*(j) function (details given later).

As HRN constrains two sets of variables we require two different method to call when a value is removed from one of the variable's domains. These methods are given below:

The *deltaX* method, shown in Figure 5(a), is called when some value a, where $a < m+1$, is removed from $dom(x_i)$. The index j of the hospital a represents is found (line 2),

init()
  for $i := 1$ to n **loop**
    apply(i);
  end loop;
  for $j := 1$ to m **loop**
    offer(j);
  end loop;

Figure 4: Method *init*.

and r_i is then removed from the domain of h_j (line 3). If a represents the last hospital r_i applied to (line 4), then r_i will make a new application to its new favourite via the *apply*(i) function (line 5). Note that either the deletion on line 3 or an indirect deletion via a call to the *apply*(i) function (details given later) could cause a reduction in the domain of some y variable and thus a call to *deltaY* will be placed on the call stack.

The *deltaY* method, shown in Figure 5(b), is called when some value a, where $a < n+1$, is removed from $dom(y_j)$. The index i of the resident a represents is found (line 2) and h_j is then removed from the domain of r_i (line 3). If a represents a resident h_j that made an offer to (line 4), then h_j will make a new set of offers via the *offer*(j) function (line 5). Note that either the deletion on line 3 or an indirect deletion via a call to the *offer*(j) function (details given later), could cause a reduction in the domain of some x variable and thus a call to *deltaX*. Therefore the propagation of this constraint results from the mutual recursion between methods *deltaX* and *deltaY*.

The *apply*(i) function of Figure 6(a) is called either at the head of search (via the *init* method) or when the lower bound of x_i changes (via the *deltaX* method). Resident r_i will apply to each hospital that it prefers to any other in its domain, and to which it has not previously applied to (line 2). First the hospital h_j to be applied to is found (line 3), then resident r_i makes an application to hospital h_j via a call to the *apply*(j, a) function(line 4). If c_y applications have been made to hospital h_j (line 5) then h_j must not consider any resident worse then its c_j^{th} favourite applicant (line 6). $\check{x}_i$ is then updated with the current lower bound of x_i (line 8). As the runtime of this function is dependent on the number of domain reductions made since the previous call to this function, it therefore has $O(1)$ complexity per deletion.

The *apply*(j, a) function of Figure 6(b) is called only by the *apply*(i) function when hospital h_j receives an application from its a^{th} choice resident. The hospital's preference for this applicant is placed in the list of applicants in ascending order. If more than c_j applications have been received then the worst applicant will drop off the end of the array and will effectively be removed from the list. This function runs in $O(c)$ time.

Figure 7 gives the *offer*(j) function which can be called either at the head of search (via the *init* method) or when a resident that was previously offered a place has been removed from $dom(y_j)$ (via the *deltaX* method). Hospital h_j will offer a post to r_i, the

1. deltaX(i,a)
2. $j := PL(r_i, a)$;
3.   remValue(y_j,rank(h_j, r_i));
4.   **if** $a = \check{x}_i$ **then**
5.   apply(i);

Figure 5: (a) Method *deltaX*.

1. deltaY(j,a)
2. $i := PL(h_j, a)$;
3.   remValue(x_i,rank(r_i, h_j));
4.   **if** $a \leq \check{y}_j$ **then**
5.   offer(j);

(b) Method *deltaY*.

apply(i)
for $k := \check{x}_i + 1$ to $\min(x_i)$ **loop**
  $j := PL(r_i, k)$;
  apply(j,rank(h_j, r_i));
  if $post_{j,c_j} < \infty$ **then**
    setMax(y_j,$post_{j,c_j}$);
end loop;
$\check{x}_i := \min(x_i)$;

Figure 6: (a) Function *apply*(i).

apply(j,a)
for $k := 1$ to c_j **loop**
  if $post_{j,k} = a$ **then**
    $a := n + 1$;
  if $post_{j,k} > a$ **then**
    swap$(post_{j,k}, a)$;
end loop;

(b) Function *apply*(j, a).

c_j^{th} favourite resident still in its domain, and to all other residents that it prefers to r_i to which it has not yet offered a place to. $\check{y}_j$ is then updated to its preference for r_i. This function contains one loop which cycles at most c_j times, therefore it runs in $O(c)$ time.

5.2 Complexity

The *deltaX* and *deltaY* methods contains no loops, but each calls a function which runs in $O(c)$ time. Thus *deltaX* and *deltaY* both have a complexity of $O(c)$. The *deltaX* method can be called at most once for each value in the domain of an x_i variable, and similarly *deltaY* can be called at most once for each value in the domain of the y_j variable. Therefore we have a time complexity of $O(Lc)$. Hence the time complexity for the HRN constraint improves those of the models presented in earlier sections. The space complexity of this encoding is dominated by the ranking arrays, and is $O(nm)$. However, if preference lists are short we may economically trade time for space, or use some sparse data structure, or a hash table to map preferences to indices.

Table 1 summarises the time and space complexities for the HR models in this paper (the columns refer respectively to the models in Sections 3, 4 and 5).

5.3 Searching for all solutions

Arc consistency processing on the HRN constraint yields the *GS-domains* as defined in Section 4. A search process need only consider the resident variables (and need not instantiate the hospital variables), following a similar process to that outlined in Theorem 7.

6 Computational experience

The three encodings presented in this paper were implemented using the JSolver toolkit, i.e. the Java version of ILOG Solver, in order to carry out an empirical analysis. The objective was to compare the runtimes for these models as applied to randomly-generated and real-world data. Our studies were carried out using a 2.8Ghz Pentium 4 processor

offer(j)
for $k := \check{y}_i + 1$ to getValue(h_j,c_j) **loop**
  $i :=$ PL(h_j, k);
  setMax(x_i,rank(r_i, h_j))
end loop;
$\check{y}_j :=$ getValue(h_j,c_j);

Figure 7: Function *offer*(j).

Model:	Cloned	CBM	HRN
Time:	$O((n+m)^4 c)$	$O(Lc(n+m))$	$O(Lc)$
Space:	$O((nmc)^2)$	$O(Lc)$	$O(nm)$

Table 1: Summary of time and space complexities for the HR models of this paper.

with 512 Mb of RAM, running Microsoft Windows XP Professional and Java2 SDK 1.4.2.6 with an increased heap size of 512 Mb.

Random problem instances were generated with varying number of residents n, number of hospitals m, capacity c (uniform for each hospital), and a fixed residents' preference list size of 10. Hence we classify problems via the triple $n/m/c$. Instances were generated as follows. First, a uniformly random preference list of length 10 was produced for each resident, then a preference list was produced for each hospital by randomly permuting their acceptable residents. A sample size of 100 was used for each value of $n/m/c$.

Table 2 shows the mean time in seconds to construct the model and find all solutions, for the each of the four models applied to random instances with varying $n/m/c$ attributes. A table entry of $-$ signifies that there was insufficient space to create the model of that size using the specified encoding. Table 3 shows the time to establish AC (shown as "AC") and find all solutions (shown as "ALL") to three anonymised HR instances arising from SFAS [14]. The first column indicates $n/m/c$, where c is the average hospital capacity; also $l_i^r \leq 5$ in each case. (For each instance, the Cloned model ran out of memory.)

The results indicate that the HRN model was typically able to handle larger problem instances than the other models, and the average runtime was faster than for the other models in all cases. The HRN model was also applied to instances as large as $500k/11.8k/85$, finding all solutions on average in 35 seconds. As mentioned in the Introduction, instances of the NRMP typically involve around 31,000 residents and 2,300 hospitals, with residents' preference lists of size between 4 and 7 [22]. The HRN model finds all solutions to problems of size $200k/3k/67$ in 22 seconds on average. This leads us to believe that Constraint Programming is indeed a suitable technology for the HR problem.

7 Motivation: side-constraints

It is natural to build additional constraints on top of the constraint models of HR presented in this paper, in order to cope with generalisations of HR for which the RGS and HGS algorithms are inapplicable. In this section we present several variants of HR that are either NP-hard or for which no polynomial-time algorithm is currently known. In the first three cases we suggest additional side-constraints that can be added to any of our base

	50/13/4	100/20/5	500/63/8	1k/100/10	5k/250/20	20k/550/37	50k/1.2k/42
Cloned	5.84	–	–	–	–	–	–
CBM	0.24	0.36	1.69	4.75	–	–	–
HR2	0.15	0.18	0.42	0.88	9.91	112	–
HRN	0.12	0.15	0.19	0.22	0.53	1.42	4.2

Table 2: Average computation times in seconds to find all solutions to 100 randomly-generated HR instances with attributes $n/m/c$.

	# Solutions	CBM		HRN	
		AC	ALL	AC	ALL
502/41/13.2	1	1.61	1.64	0.17	0.17
510/43/11.5	1	1.64	1.7	0.17	0.17
245/34/3.9	1	0.26	0.26	0.12	0.12

Table 3: Time taken to establish AC and find all solutions to three SFAS instances.

models in order to cope with the more general problem, providing additional motivation for our approach.

Resident-exchange-stable HR. During a previous run of the SFAS matching scheme, two residents complained that, had they swapped their given hospitals, they could each have been better off. Such a swap would not have been permitted by the hospitals, of course, as it would have violated the stability criterion. However it would be desirable to avoid such a situation arising if possible, and this leads to the problem of finding a *resident-exchange stable matching* given an instance I of HR. This is a stable matching M in I such that there are no two assigned residents r_i, r_j such that r_i prefers $M(r_j)$ to $M(r_i)$, and r_j prefers $M(r_i)$ to $M(r_j)$. It is known that a such a matching need not exist in I, and indeed the problem of deciding whether such a matching exists in I is NP-complete [15, 20], even if each hospital has capacity 1. For any two residents r_i, r_j and for any two hospitals h_k, h_l such that r_i prefers h_l to h_k and r_j prefers h_l to h_k, the additional constraint $x_i = p_1 \Rightarrow x_j \neq p_2$ should be added, where $rank(r_i, h_k) = p_1$ and $rank(r_j, h_l) = p_2$.

HR with forbidden pairs. Let F be a set of (resident,hospital) pairs in an instance I of HR. An adminstrator of a matching scheme may wish to exclude the pairs in F from any matching. Hence a matching M in I must not include any member of F, however a pair in F could still form a blocking pair (hence we cannot simply delete pairs in F from the preference lists). The task is to find a matching in I that is stable in the usual sense. Clearly a stable matching need not exist, given an instance of HR with forbidden pairs. However given an instance of SMI with forbidden pairs, there exists a linear-time algorithm to find a stable matching or report that none exists [6], and it is straightforward to extend this algorithm to HR. However no polynomial-time algorithm is currently known for the problem of finding a matching M in I (in the usual sense) with the fewest number of forbidden pairs. One possibility for modelling this problem is to add new variables $T = \{t_{i,p} : 1 \leq i \leq n \wedge 1 \leq p \leq l_i^r\}$, each with domain $\{0, 1\}$, and a constraint $x_i = p \Rightarrow t_{i,p} = 1$, for each $(r_i, h_j) \in F$, where $rank(r_i, h_j) = p$, and then minimise the sum of the values of the variables in T.

HR with groups. An extension of HR that has practical relevance arises when residents may form groups, and may decide that they are only prepared to be matched to a given hospital if the whole group is matched to it. More formally, each hospital $h_j \in H$ may have one or more associated groups $G_j \subseteq R$. A matching M must satisfy the additional property that if $(r_i, h_j) \in M$ for some $r_i \in G_j$, then $(r_k, h_j) \in M$ for all $r_k \in G_j$. No polynomial-time algorithm for this problem is currently known. However this variant can be modelled as follows. For any group $G_j = \{r_{i_1}, \ldots, r_{i_k}\}$, add the constraint $x_{i_a} = p_{i_a} \Rightarrow x_{i_b} = p_{i_b}$ $(1 \leq a \neq b \leq k)$ where $rank(r_{i_a}, h_j) = p_{i_a}$ and $rank(r_{i_b}, h_j) = p_{i_b}$. A particular case of this problem is the Hospitals / Residents problem with Couples (HRC), described below.

Other generalisations of HR. The Hospitals / Residents problem with Ties (HRT) arises when ties are permitted in the preference lists of hospitals and/or residents. For example, a popular hospital may be indifferent among several applicants. The SFAS scheme [14] already permits ties in the hospitals' lists. However it is known [18] that, in the presence of ties, stable matchings can be of different sizes, and the problem of finding a maximum stable matching is NP-hard, even for very restricted instances of SMI with ties. It has already been demonstrated [10, 11] that the earlier encodings of [9] can be extended to the case where preference lists in a given SMI instance may involve ties. We have begun to consider the corresponding extension of the models presented in Sections 4 and 5 to the HRT case, and further details will appear elsewhere.

HRC (in which couples submit joint preference lists over pairs of hospitals) is another generalisation of HR. Again it is possible that an instance need not admit a stable matching (where the stability definition is extended to the couples case), and the problem of deciding whether such a matching exists is NP-complete [23]. A constraint-based solution to this problem is motivated by the NRMP, which permits couples to submit joint preference lists.

8 Conclusions and future work

In this paper we have presented three CP models of an HR instance. The empirical results for the models as presented in Section 6 are broadly in line with what may be expected, given the summary of time and space complexities presented in Table 1. Our results indicate that, as is the case for SMI [9], CSP encodings of HR are "tractable", a notion that has been explored in detail by Green and Cohen [12]. However it remains open as to whether there exists a CSP encoding of HR that gives rise to the GS-lists, for which AC may be established in $O(L)$ time and using $O(nm)$ space. The time complexity of $O(L)$ is optimal, since SM is a special case of HR, and a lower bound of $\Omega(L)$ holds for the problem of finding a stable matching, given an instance of SM [21].

Acknowledgement

The authors are grateful to ILOG SA for providing access to the JSolver toolkit via an Academic Grant Licence.

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