# Analysis of Stochastic Matching Markets

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**Abstract.** Suppose that the agents of a matching market contact each other randomly and form new pairs if is in their interest. Does such a process always converge to a stable matching if one exists? If so, how quickly? Are some stable matchings more likely to be obtained by this process than others? In this paper we are going to provide answers to these and similar questions, posed by economists and computer scientists. In the first part of the paper we give an alternative proof for the theorems by Diamantoudi *et al.* and Inarra *et al.* which imply that the corresponding stochastic processes are absorbing Markov chains. Our proof is not only shorter, but also provides upper bounds for the number of steps needed to stabilise the system. The second part of the paper proposes new techniques to analyse the behaviour of matching markets. We introduce the Stable Marriage and Stable Roommates Automaton and show how the probabilistic model checking tool PRISM may be used to predict the outcomes of stochastic interactions between myopic agents. In particular, we demonstrate how one can calculate the probabilities of reaching different matchings in a decentralised market and determine the expected convergence time of the stochastic process concerned. We illustrate the usage of this technique by studying some well-known marriage and roommates instances and randomly generated instances.

## 1 Introduction

The Stable Roommates problem (SR) is a classical combinatorial problem that has been studied extensively in the literature, see e.g. [11]. An instance $I$ of SR contains an undirected graph $G(V,E)$, where $V{=}\{v_1,\dots,v_n\}$ and $m{=}|E(G)|$. We refer to $G$ as the *underlying graph* of $I$, and we interchangeably refer to the vertices of $G$ as the *agents*. If $(v_i,v_j)$ is an edge in $E(G)$, then we say that $v_i$ and $v_j$ find each other *acceptable*. Each agent $v_i$ has a linear order $>_{v_i}$ over her acceptable partners, where $v_k >_{v_i} v_j$ means that $v_i$ prefers $v_k$ to $v_j$. Let $M(v_i)$ denote the *partner* of $v_i$ in a given matching $M$. An edge $(v_i,v_j)$ is said to be *blocking* with respect to $M$ if (i) either $v_i$ is unmatched in $M$ or prefers

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$v_j$ to $M(v_i)$, and (ii) either $v_j$ is unmatched in $M$ or prefers $v_i$ to $M(v_j)$. A matching is called *stable* if it admits no blocking edge. If $G$ is bipartite, then the problem of finding a stable matching is called the Stable Marriage problem (SM). In this case, if the graph is $G(U, W, E)$, then we refer to $U = \{m_1, \ldots, m_{n_1}\}$ and $W = \{w_1, \ldots, w_{n_2}\}$ as the sets of men and women, respectively.

Note that both the Stable Roommates and the Stable Marriage problems can be seen as NTU-games, since for any SR or SM instance the set of stable matchings coincide with the core of the corresponding game. Fur further details, see for example the celebrated book by Roth and Sotomayor [29].

Gale and Shapley [10] give a linear time algorithm that finds a stable matching for any instance of SM, while also illustrating an instance of SR that does not admit a stable matching. Irving [15] gives a linear time algorithm that, for any instance of SR, finds a stable matching or reports that none exists. Both algorithms assume that the preference lists are complete (i.e., the graph $G$ is complete), although it is straightforward to extend the algorithms to incomplete lists [11].

Suppose that we are given a SR instance $I$ with underlying graph $G$. For a matching $M$, if a pair $(v_i, v_j)$ is blocking, then we may *satisfy this blocking pair* and get a new matching $M^{(v_i,v_j)}$, where $(v_i, v_j) \in M^{(v_i,v_j)}$ and for each $w \in \{v_i, v_j\}$, if $w$ is matched in $M$, then $M(w)$ is unmatched in $M^{(v_i,v_j)}$. Roth and Vande Vate [30] prove that, given an instance of SM, starting from any unstable matching we can always obtain a stable matching by successively satisfying blocking pairs.[3] Diamantoudi *et al.* [8] show that a similar result holds for the roommates problem, namely, for a given instance of SR that admits a stable matching and starting from any unstable matching, one can obtain a stable matching by successively satisfying blocking pairs. This essentially means that the corresponding stochastic processes (to be defined in Section 3) are *absorbing Markov chains* (for more details of these stochastic processes see, e.g. Chapter 3 of [17]). Since there are only finitely many matchings in any instance, the result of Roth and Vande Vate implies that, starting from an arbitrary matching, the process of allowing randomly chosen blocking pairs to match will converge to a stable matching with probability one.

The proof of Roth and Vande Vate is based on the following idea. Suppose that we have a stable matching for an instance of SM and we add a new agent to the market, then there is a natural proposal-rejection sequence (described in Section 3) that leads to a stable matching for the extended instance. If, we start with the empty matching and run this incremental algorithm, then the resulting

[3] Note that this question was originally proposed by Knuth [21] (Problem 8 from his twelve famous research problems) in a slightly different setting. In his case, the set of possible matchings was restricted to the complete matchings (as all the preference lists were supposed to be complete), and whenever a blocking pair was satisfied the left-alone agents formed a new pair immediately. The above described transition from a complete matching to another one was called *interchange*. Knuth asked whether, given an instance of SM and a starting matching $M$, there always exist a sequence of interchanges from $M$ to some stable matching? Tamura [32], and independently Tan and Su [36], answered this question negatively by giving counterexamples.

stable matching will depend on the order in which the agents arrive. This is called the *random order mechanism*. By assuming that each order is equally likely, we may calculate the probability of each stable matching being obtained. Ma [25] carried out this calculation for an instance, originally suggested by Knuth [21], and observed that not all stable matchings can be reached by this mechanism and there is a higher probability of reaching some stable matchings over others (although his calculation was not entirely correct as [19] pointed out).[4] In this paper we will also study this instance (Example 2 in Section 3) with respect to a different stochastic process.

We may suppose that all agents are present in the market and, starting with the empty matching, the blocking pairs to be satisfied are chosen randomly (with equal probability in each step). In this case, every stable matching can be reached with positive probability (since we may satisfy all pairs involved in this matching at the beginning of the process), but still, as we will illustrate in Section 3, some stable matchings can be more likely than others.

There is also a growing literature concerning stable roommates problems that may not admit stable solutions. Tan [34] shows that a *stable half-matching* always exists for any given instance of SR. A *half-matching* is a weight function $h : E(G) \rightarrow \{0, \frac{1}{2}, 1\}$ such that $\sum_{v_i \in e} h(e) \leq 1$ for each vertex $v_i$. A half-matching is said to be stable if, for each edge $(v_i, v_j) \in E(G)$, one of its vertices, say $v_i$ satisfies $\sum_{(v_i,v_k) v_k \geq_{v_i} v_j} h((v_i, v_k)) = 1$, (otherwise the edge $(v_i, v_j)$ is said to be *blocking*). Note that if $h : E(G) \rightarrow \{0, 1\}$ is stable, then $h$ corresponds to a stable matching.[5] Tan and Hsueh [35] give a polynomial time algorithm to find a stable half-matching for a given instance of SR. This algorithm is, in fact, a generalised version of the algorithm of Roth and Vande Vate (a detailed description of this is given in Section 3).

Inarra *et al.* [13] define an *h-stable matching* $M$ relative to a stable half-matching $h$ as follows. Let $M$ contain every edge that has weight 1 in $h$, every second edge from each even half-weighted cycle of $h$ (if there were any), and $k$ (disjoint) edges from each odd half-weighted cycle of length $2k + 1$ in $h$.[6] They show that, starting from an arbitrary matching, one can get an $h$-stable matching by successively satisfying blocking pairs for a given instance of SR. Note that for every solvable instance of SR the set of $h$-stable matchings is equivalent to the set of stable matchings, thus the above result generalises the theorem of

[4] An explanation for the first observation is the result of Blum and Rothblum [7] which demonstrates that, when using the Roth-Vande Vate algorithm, the last agent to arrive always gets their best stable partner (an alternative proof of this result is given by Biró *et al.* [4]). Hence, a stable matching in which nobody gets their best partner cannot be obtained by this mechanism.

[5] The existence of a half-matching may be proved by the Lemma of Scarf [31], as Aharoni and Fleiner [2] demonstrate. The notion of *stable fractional matching* (or *fractional core*) is an extension of stable half-matching that may be defined for more general matching problems (or NTU-games) as well, see more on this theory in a recent paper by Biró and Fleiner [5].

[6] This concept was originally proposed by Tan [33] as a method to find a matching as large as possible that is stable for the matched agents in an unsolvable instance.

Diamantoudi *et al.* [8]. In Section 2, we give an alternative short proof for the theorem of Inarra *et al.* [13] by using the Tan-Hsueh algorithm, by providing also upper bounds for the steps needed to reach a desired matching.

In another paper, Inarra *et al.* [14] define the *absorbing sets* for an instance of SR as follows. Each absorbing set consists of matchings that are reachable from one another by successively satisfying blocking edges, but no other matching can be reached from this set by satisfying a blocking edge. These are in fact the *ergodic sets* of the corresponding Markov chain (see e.g. [17]), and a matching $M$ is in an ergodic set if and only if the limit probability of $M$, starting from the empty matching, is positive. Moreover, Klaus *et al.* [20] prove that the absorbing sets consist of exactly those matchings that have positive probabilities in the limit distribution of a stochastic process where, starting from any matching, the agents make mistakes with small probabilities in their myopic blocking decisions. They called this process *perturbed blocking dynamics*. Similar stochastic systems have been studied for the Stable Marriage problem in the context of network formations by Jackson and Watts [16].

Ackermann *et al.* [1] study the convergence time of the stochastic processes occurring from stable marriage problems. They refer to the stochastic process, where in each step a blocking pair is chosen uniformly at random and satisfied, as the *random better response dynamics*. They demonstrate that, although the process converges to a stable matching, the expected convergence time is exponential for a family of SM instances. Our experiments conducted for the above family of instances confirm this finding, as we describe in Section 3. However, we also demonstrate that this behaviour is unexpected in an average market, since for the randomly generated instances the expected convergence time is significantly smaller.

The dynamics of matching markets have also been in focus in some recent engineering papers on P2P systems, see, e.g, [26] for an overview. In particular, Lebedev *et al.* [24] show that the convergence is fast for systems, modelled with SR instances, where the preferences are *acyclic*, i.e., the preferences are derived from some global rank function on the pairs. This is a realistic assumption in case of some real P2P networks. Finally, Arcaute and Vassilvitskii [3] and Hoefer [12] studied similar stochastic market processes with the extra feature of an underlying social network that dynamically determine the accessible partners and the so-called locally stable matchings.

In our analysis we also suppose that the stochastic process follows the random better response dynamics. Does this model give a good description of decentralised matching markets? There are two very recent experimental studies that provide some positive evidence for that. Echenique and Yariv [9] conducted experimental tests with students who were allowed to make and accept proposals in a decentralised manner. Most of the outcomes in these games were stable matchings and when several stable matchings were possible then they recorded their distribution as outcomes. In particular, they found that, when the market had three stable matchings, then the median one emerged as the modal empirical outcome. They showed with simulations (subsection 7.1. in [9]) that this distribution of stable matchings was relatively close to those that a stochastic model, the same as ours, would predict. Pais *et al.* [28] received similar results regarding the likeliness of obtaining the median stable matching in their experiments.

To summarise, the contribution of this paper is the following. In Section 2 we give an alternative proof for the theorems of Diamantoudi *et al.* [8] and Inarra *et al.* [13]. This new proof, which is based on the Tan-Hsueh algorithm, is not only shorter and simpler than the originals, but also provides upper bounds on the number of steps needed to reach a stable (or $h$-stable) matching. In Section 3 we define the Stable Marriage and Stable Roommates Automata and then we demonstrate how the probabilistic model checker PRISM [23, 38] can be used to analyse and compare the performance of difference instances. In particular, we study two well-known SM instances, a SR instance and then present a case study involving structured and random SM instances. We believe that this approach will also have applications in the study of the interaction of agents in real markets and networks for more complex settings.

## 2 Convergence to stability, an alternative proof

In this section we describe the Roth-Vande Vate and the Tan-Hsueh algorithms. We use the latter to give an alternative proof for the theorems of Diamantoudi *et al.* [8] and Inarra *et al.* [13]. That is, we show that starting from an arbitrary matching of a solvable SR instance one can always find a stable matching by successively satisfying blocking pairs; and that starting from an arbitrary matching of an instance of SR (solvable or unsolvable) one can always find an $h$-stable matching by successively satisfying blocking pairs. Note that these theorems were the main results of the above papers. Our proof is much shorter and it gives upper bounds for the number of blocking pairs that need to be satisfied to obtain a stable (or $h$-stable) matching. Also, it shows that the argument of Roth and Vande Vate for the marriage case can be extended for the roommates case in a natural way.

**The Roth-Vande Vate algorithm.** Suppose that we are given an instance $I$ of SM together with a matching $M_0 = \{(m_1, w_1), \ldots, (m_k, w_k)\}$. We shall show that we can reach a stable matching by successively satisfying blocking pairs. A variant[7] of the Roth-Vande Vate algorithm works as follows.

During the procedure we gradually extend a set $S \subseteq (U \cup W)$ and a matching $M_S$ that is stable in $S$. Initially let $S = \{\emptyset\}$ and $M_S = \{\emptyset\}$. For each index $i$ ($i = 1, \ldots, k$), if $M_S \cup \{(m_i, w_i)\}$ is stable in $S \cup \{m_i, w_i\}$, then let $M_{S'} = M_S \cup \{(m_i, w_i)\}$ and $S' = S \cup \{m_i, w_i\}$ (i.e. we simply enlarge both $S$ and $M_S$ with a new pair). Otherwise we add $m_i$ and $w_i$ to $S$ one by one as follows.

Without loss of generality suppose that $m_i$ is involved in a blocking pair with an agent of $S$ with respect to matching $M_S \cup \{(m_i, w_i)\}$, let $w_{i_1}$ be the

[7] This version of the Roth-Vande Vate algorithm has been described by Ma [25]. Note that it slightly differs from the original method described in [30], but the difference is not substantial.

woman who is the best blocking partner of $m_i$ and let $S' = S \cup \{m_i\}$. If $w_{i_1}$ is unmatched in $M_S$, then $M_{S'} = M_S \cup \{(m_i, w_{i_1})\}$ is a stable matching in $S'$. Otherwise, let $m_{i_1}{=}M_S(w_{i_1})$ and $M_{S' \setminus \{m_{i_1}\}} = (M_S \setminus \{(m_{i_1}, w_{i_1})\} \cup \{(m_i, w_{i_1})\}$ is stable for $S' \setminus \{m_{i_1}\}$. Now we let $m_{i_1}$ re-enter the market. If $m_{i_1}$ is not involved in any blocking pair, then $M_{S' \setminus \{m_{i_1}\}}$ is stable for $S'$. Otherwise we satisfy the best blocking pair $m_{i_1}$ is involved in according to his preferences, and so on. This process must terminate after satisfying at most $m$ blocking pairs, since no woman ever receives a worse partner. We can also add $w_i$ in a similar manner, reversing the role of men and women. (Note that if $m_i$ was not involved in a blocking pair with an agent of $S$ with respect to matching $M_S \cup \{(m_i, w_i)\}$ then $w_i$ must have been involved in a blocking pair, so we start by adding $w_i$ to $S$ first, followed by $m_i$.)

After processing all pairs of $M_0$, we add the remaining agents one by one in the same way. Therefore, we obtain the sequence of blocking pairs that we need to satisfy to reach a stable matching starting from $M_0$. Since we never satisfy a pair twice when adding a new agent to $S$, it follows that the number of steps in the path to stability is at most $mn$.

**The Tan-Hsueh algorithm.** The Tan-Hsueh algorithm deals with SR instances (rather than SM instances) and stable half-matchings (rather than stable matchings), and there is no starting matching $M_0$. But otherwise it is based on the same idea as the Roth-Vande Vate algorithm: we gradually extend a set $S \subseteq V(G)$ and we restore the stability of a half-matching $h_S$ in $S$.[8]

Suppose that we are given an instance $I$ of SR with an underlying graph $G(V,E)$. Initially let $S{=}\{\emptyset\}$ and $h_S(e){=}0$ for each $e \in E(G)$. Suppose that after adding $k$ agents we have $S{=}\{v_1, v_2, \ldots, v_k\}$ with a corresponding stable half-matching $h_S$ where each half-weighted cycle has odd length. Let $S' = S \cup \{v_{k+1}\}$. Now we describe how we can construct the new stable half-matching $h_{S'}$ in $S'$.

If $v_{k+1}$ is not involved in any blocking pair in $S'$, then $h_S$ remains stable in $S'$, obviously. Otherwise let $v_j$ be the best blocking partner of $v_{k+1}$ in $S'$. If $v_j$ is unmatched (i.e., not matched and not covered by a half-weighted cycle either), then by setting $h_{S'}((v_{k+1}, v_j)){=}1$ and $h_{S'}(e){=}h_S(e)$ for every other edge we obtain a new stable half-matching in $S'$. If $v_j$ is covered by a half-weighted odd cycle, say by $(v_{c_1}, v_{c_2}, \ldots, v_{c_{2l+1}})$ where $v_j{=}v_{c_1}$, then by setting $h_{S'}((v_{k+1}, v_j)) = 1$, $h_{S'}((v_{c_{2i}}, v_{c_{2i+1}})){=}1$ for $i{=}1, \ldots, l$, $h_{S'}((v_{c_{2i-1}}, v_{c_{2i}})){=}0$ for $i{=}1, \ldots, l$ and $h_{S'}((v_{c_{2l+1}}, v_{c_1})){=}0$ we obtain a new stable half-matching. The last case is when $v_j$ is matched in $h_S$ to an agent, say $v_{a_1}$. By setting $h_{S' \setminus \{v_{a_1}\}}((v_{k+1}, v_j)){=}1$ and $h_{S' \setminus \{v_{a_1}\}}((v_j, v_{a_1})){=}0$ we obtain a half-matching that is stable in $S' \setminus \{v_{a_1}\}$. Now, we restart the process with $v_{a_1}$.

In contrast with the SM context, it is possible that the latter case happens every time and the above process never ends, since in a sequence $v_{a_1}, v_{b_1}, \ldots, v_{a_l}, v_{b_l}$, $v_{b_l}$ may be the same as $v_{a_1}$. Tan and Hsueh [35] showed that, if such a repetition occurs, then a subset of these agents will be involved in a never ending

[8] For any bipartite graph the Tan-Hsueh algorithm is identical to the Roth-Vande Vate algorithm if the starting matching of the latter algorithm is $\{\emptyset\}$.

cycling and we can form a new half-weighted odd cycle on the corresponding edges resulting in a new stable half-matching $h_{S'}$ in $S'$.

**Alternative proofs of [13] and [8].** Modifying the Tan-Hsueh algorithm slightly (with $h$-stable matchings rather than with stable half-matchings), we can obtain an alternative proof for the following theorem of Inarra *et al.* [13], with an upper bound on the number of steps needed to reach an $h$-stable matching.

**Theorem 1.** *Suppose that we are given an instance of* SR *and a matching $M_0$, then one can always reach an $h$-stable matching starting from $M_0$ by successively satisfying at most $mn$ blocking pairs.*

*Proof.* Let $M_0=\{(v_1,v_2),(v_3,v_4),\ldots,(v_{2k-1},v_{2k})\}$. Just as in the proof of Roth and Vande Vate, we gradually extend a set $S \subseteq V(G)$ and a matching $M_S$ in $S$, where initially $S=\{\emptyset\}$ and $M_S=\{\emptyset\}$.

Suppose that $S=\{v_1,v_2,\ldots,v_{2i}\}$ and $M_S$ is a $h_S$-stable matching relative to a stable half-matching $h_S$. Recall that each edge $e$ of weight 1 in $h_S$ is represented in $M_S$ and each half-weighted odd cycle $C=(v_{c_1},v_{c_2},\ldots,v_{c_{2l+1}})$ is represented by $l$ disjoint edges of $C$ in $M_S$. Consider a half-matching $h^*$ in $S\cup\{v_{2i+1},v_{2i+2}\}$ where $h^*((v_{2i+1},v_{2i+2}))=1$ and the other weights are the same as in $h_S$. If $h^*$ is stable in $S\cup\{v_{2i+1},v_{2i+2}\}$, then it follows that $M_{S'}=M_S\cup\{(v_{2i+1},v_{2i+2})\}$ is an $h^*$-stable matching in $S'=S\cup\{v_{2i+1},v_{2i+2}\}$. Otherwise, if $h^*$ is not stable in $S\cup\{v_{2i+1},v_{2i+2}\}$, then we add $v_{2i+1}$ and $v_{2i+2}$ to $S$ as follows.

Without loss of generality suppose that $v_{2i+1}$ is involved in a blocking pair with an agent of $S$ with respect to matching $M_S\cup\{(v_{2i+1},v_{2i+2})\}$, let $v_j$ be the best blocking partner of $v_{2i+1}$ and let $S'=S\cup\{v_{2i+1}\}$. If $v_j$ is unmatched in $h_S$ (and so also in $M_S$), then $M_{S'}=M\cup\{(v_{2i+1},v_j)\}$ is an $h_{S'}$-stable matching in $S'$ where $h_{S'}((v_{2i+1},v_j))=1$ and otherwise it is the same as $h_S$. Note that $(v_{2i+1},v_j)$ must be a blocking pair for $M_S$ too, so we may obtain $M_{S'}$ from $M_S$ by satisfying $(v_{2i+1},v_j)$.

If $v_j$ is covered by a half-weighted odd cycle in $h_S$, say by $C=(v_{c_1},\ldots,v_{c_{2l+1}})$ where $v_{c_1}=v_j$, then we proceed as follows. Note $v_j$ may be matched to her preferred partner among her two neighbours in $C$, say to $v_{c_2}$. In this case it might be the case that $(v_{2i+1},v_j)$ is blocking for $h_S$ but it is not blocking for $M_S$. However, in this case, we can always rotate the edges of $M_S$ in $C$ by successively satisfying blocking pairs so that $v_j$ becomes unmatched. Then we can satisfy $(v_{2i+1},v_j)$ and obtain an $h_{S'}$-stable matching $M_{S'}$ where $h_{S'}$ is the stable half-matching that we would get from $h_S$ according to the Tan-Hsueh algorithm.

Finally, if $v_j$ is matched in $h_S$ (and also in $M_S$) to an agent $v_{j_1}$, then we satisfy $(v_{2i+1},v_j)$ obtaining a matching $M_{S'\setminus\{v_{j_1}\}}=(M_S\setminus\{(v_j,v_{j_1})\})\cup\{(v_{2i+1},v_j)\}$ which is an $h_{S'\setminus\{v_{j_1}\}}$-stable matching in $S'\setminus\{v_{j_1}\}$, where $h_{S'\setminus\{v_{j_1}\}}$ is a stable half-matching in $S'\setminus\{v_{j_1}\}$ that we would obtain in the Tan-Hsueh algorithm. Again, we continue the same process with $v_{j_1}$.

If a repetition occurs for the first time, namely, when a left-alone agent gets a proposal later in the process, then in the Tan-Hsueh algorithm we would form a new half-weighted odd cycle from the agents involved in the cycling, resulting in a new stable half-matching $h_{S'}$. But regarding the matching $M_{S'}$, we can just stop after seeing the first repetition, and $M_{S'}$ will be an $h_{S'}$-stable matching.

Note that if a repetition occurs, then we have to satisfy at most $m$ blocking pairs (since each left-alone agent keeps getting worse partners, so no pair occurs twice as a blocking pair). Otherwise, if we have no repetition, then we also reach a new $h$-stable matching within $m$ steps, since even if we have to rotate edges along a half-weighted odd cycle, the agents of this cycle could not be involved in any blocking pair satisfied before invoking this rotation. Thus we can obtain the final $h$-stable matching in $mn$ steps. □

This result implies the theorem of Diamantoudi *et al.* [8] with an upper bound on the number of steps needed to reach a stable matching.

**Corollary 1.** *Suppose that we are given a solvable instance of* SR *and a matching $M_0$, then one can always reach a stable matching starting from $M_0$ by successively satisfying at most $mn$ blocking pairs.*

## 3 Analysing the market behavior with automata

If the input is random, then the automaton may simulate the dynamics of a matching market where two agents meet with each other randomly and behave in a myopic way (i.e. they form a new pair if they both would be better off). This is called the *better response dynamics* by Ackermann *et al.* [1]; whilst, Klaus *et al.* [20] refer to it as *unperturbed blocking dynamics*. What is the expected outcome of a matching market with myopic agents? To answer this question first we define the stable marriage and roommates automata as follows.

**Definition 1.** *Let $I$ be a* SR (SM) *instance with underlying graph $G$. The* stable roommates automaton (stable marriage automaton) *of $I$, denoted* SRA($I$) (SMA($I$)) *is the automaton $(\mathcal{M}(G), M_0, E(G), \delta, S_I)$ where:*

- *the set of states is the set of all matchings $\mathcal{M}(G)$ of $G$;*
- *the initial state $M_0$ is any matching (e.g. the empty matching $\{\emptyset\}$);*
- *the set of symbols is the set of edges $E(G)$ of $G$;*
- *the transition function $\delta : \mathcal{M}(G)\times E(G) \to \mathcal{M}(G)$ is given by:*

$$\delta(M, (v_i, v_j)) = \begin{cases} M^{(v_i,v_j)} & \text{if } (v_i, v_j) \text{ blocks } M \\ M & \text{otherwise} \end{cases}$$

- *the set of accepting states equals the set $S_I$ of stable matchings of $I$.*

Recall, for a matching $M$ and blocking pair $(v_i, v_j)$, $M^{(v_i,v_j)}$ is the matching such that $(v_i, v_j) \in M^{(v_i,v_j)}$ and for each $w \in \{v_i, v_j\}$, if $w$ is matched in $M$, then $M(w)$ is unmatched in $M^{(v_i,v_j)}$.

Suppose that in each step of the process each blocking edge is chosen with equal probability, then starting from an arbitrary matching (e.g. the empty matching $\{\emptyset\}$) we can calculate the exact probabilities of particular matchings occurring after certain rounds. To be more precise, we will calculate these probabilities in the following absorbing Markov chain.

**Definition 2.** *Let $I$ be a* SR *instance with corresponding automaton* $\mathrm{SRA}(I) = (\mathcal{M}(G), M_0, E(G), \delta, S_I)$*. The Markov chain of $I$ is given by $(\mathcal{M}(G), M_0, \mathbf{P})$ where the set of states and initial state are taken from* $\mathrm{SRA}(I)$ *and the probability transition matrix $\mathbf{P} : \mathcal{M}(G) \times \mathcal{M}(G) \to [0,1]$ is such that for $M, M' \in \mathcal{M}(G)$ :*

- *if $M$ is stable, then $\mathbf{P}(M, M')$ equals 1 if $M{=}M'$ and 0 otherwise;*
- *if $M$ is not stable, then*

$$\mathbf{P}(M, M') = \frac{|\{(v,v') \in E(G) \,:\, (v,v') \textit{ blocks } M \textit{ and } \delta(M,(v,v')) = M'\}|}{|\{(v,v') \in E(G) \,:\, (v,v') \textit{ blocks } M\}|}\,.$$

For any SM instance or solvable SR instance, the stochastic process is an absorbing Markov chain where the absorbing states are the stable matchings.

We now report on our experiments to construct and analyse the Markov chain of a number of different instances with the probabilistic model checking tool PRISM [23, 38]. PRISM has both efficient solution engines for performing analysis of Markov chains and a high-level formal modelling language for modelling the instances. Further details of our experiments are available from the PRISM website.[9] For small instances, we also exported the PRISM models to the symbolic solver Maple [27] and computed the exact rational values.

*Example 1.* We start with a classical instance by Gale and Shapley [10] with three men and three women and the following preferences:

$$\begin{array}{lll} m_1 : w_1, w_2, w_3 & m_2 : w_2, w_3, w_1 & m_1 : w_1, w_2, w_3 \\ w_1 : m_2, m_3, m_1 & w_2 : m_3, m_1, m_2 & w_3 : m_1, m_2, m_3 \end{array}$$

Here, the Markov chain has 34 states and 123 transitions, and the following three absorbing states (stable matchings):

$$\begin{array}{rll} M_m = & \{(m_1,w_1),(m_2,w_2),(m_3,w_3)\} & \text{(man-optimal)} \\ M_w = & \{(m_1,w_3),(m_2,w_1),(m_3,w_2)\} & \text{(woman-optimal)} \\ M_e = & \{(m_1,w_2),(m_2,w_3),(m_3,w_1)\} & \text{(egalitarian)} \end{array}$$

Calculating the absorption probabilities we find:

$$x^*(M_m) = x^*(M_w) = \tfrac{299}{1362} \sim 0.2195301028 \;\text{ and }\; x^*(M_e) = \tfrac{382}{681} \sim 0.5609397944\,.$$

The egalitarian stable matching is therefore more likely than both the extreme solutions together. This differs from using the random order mechanism, since in this case the egalitarian stable matching is not achievable (as nobody gets their best stable partner) and the remaining stable matchings have probability $\frac{1}{2}$.

*Example 2.* The following classical instance was proposed by Knuth [21] with four men and four women and the following preferences:

$$\begin{array}{llll} m_1{:}w_1, w_2, w_3, w_4 & m_2{:}w_2, w_1, w_4, w_3 & m_3{:}w_3, w_4, w_1, w_2 & m_4{:}w_4, w_3, w_2, w_1 \\ w_1{:}m_4, m_3, m_2, m_1 & w_2{:}m_3, m_4, m_1, m_2 & w_3{:}m_2, m_1, m_4, m_3 & w_4{:}m_1, m_2, m_3, m_4 \end{array}$$

[9] http://www.prismmodelchecker.org/casestudies/stable_matching.php

In this case, the Markov chain has 209 states, 1280 transitions, and the following 10 absorbing states (stable matchings):

$$\begin{aligned}
M_1 &= \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\} \\
M_2 &= \{(m_2, w_1), (m_1, w_2), (m_3, w_3), (m_4, w_4)\} \\
M_3 &= \{(m_1, w_1), (m_2, w_2), (m_4, w_3), (m_3, w_4)\} \\
M_4 &= \{(m_2, w_1), (m_1, w_2), (m_4, w_3), (m_3, w_4)\} \\
M_5 &= \{(m_3, w_1), (m_1, w_2), (m_4, w_3), (m_2, w_4)\} \\
M_6 &= \{(m_2, w_1), (m_4, w_2), (m_1, w_3), (m_3, w_4)\} \\
M_7 &= \{(m_3, w_1), (m_4, w_2), (m_1, w_3), (m_2, w_4)\} \\
M_8 &= \{(m_4, w_1), (m_3, w_2), (m_1, w_3), (m_2, w_4)\} \\
M_9 &= \{(m_3, w_1), (m_4, w_2), (m_2, w_3), (m_1, w_4)\} \\
M_{10} &= \{(m_4, w_1), (m_3, w_2), (m_2, w_3), (m_1, w_4)\}
\end{aligned}$$

and calculating the absorption probabilities we find:

$$\begin{gathered}
x^*(M_1) = x^*(M_{10}) = \tfrac{549582018404187049}{9518428268802561564} \sim 0.0577387362 \\
x^*(M_2) = x^*(M_3) = x^*(M_8) = x^*(M_9) = \tfrac{1582747100504304809}{19036856537605123128} \sim 0.0831412002 \\
x^*(M_4) = x^*(M_7) = \tfrac{61576717268573787}{528801570489031198} \sim 0.1164457912 \\
x^*(M_5) = x^*(M_6) = \tfrac{253084017443076793}{1586404711467093594} \sim 0.1595330722 \ .
\end{gathered}$$

Using the random order mechanism we find $M_4, \ldots, M_7$ are not achievable, whilst in our case it is more likely one of these matchings will be reached.[10]

*Example 3.* In this example we consider the roommates instance from [20, Example 3, page 25], provided by Elena Molis. This instance concerns eight agents with the following preferences:

$$\begin{aligned}
&a_1 : a_2, a_3, a_4, a_6, a_5, a_7, a_8 \\
&a_2 : a_3, a_1, a_4, a_5, a_6, a_8, a_7 \\
&a_3 : a_1, a_2, a_4, a_5, a_6, a_7, a_8 \\
&a_4 : a_6, a_3, a_5, a_1, a_2, a_7, a_8 \\
&a_5 : a_4, a_7, a_1, a_2, a_3, a_6, a_8 \\
&a_6 : a_7, a_4, a_2, a_3, a_1, a_5, a_8 \\
&a_7 : a_5, a_6, a_1, a_2, a_3, a_4, a_8 \\
&a_8 : a_3
\end{aligned}$$

It is an unsolvable instance, it admits two stable half-matchings (with no even cycles), namely $h_1$ and $h_2$, where

$$h_1((4,5)) = h_1((6,7)) = 1 \quad \text{and} \quad h_1((1,2)) = h_1((2,3)) = h_1((3,1)) = \tfrac{1}{2}$$

$$h_2((4,6)) = h_2((5,7)) = 1 \quad \text{and} \quad h_2((1,2)) = h_2((2,3)) = h_2((3,1)) = \tfrac{1}{2}$$

[10] The probabilities of getting these six matchings by the random order mechanism are as follows [19]: $p(M_1){=}p(M_{10}){=}\frac{9600}{40320}$ and $p(M_2){=}p(M_3){=}p(M_8){=}p(M_9){=}\frac{5280}{40320}$.

The $h_1$-stable matchings are:

$$M_1 = \{(2,3),(4,5),(6,7)\}$$
$$M_2 = \{(1,2),(4,5),(6,7)\}$$
$$M_3 = \{(1,3),(4,5),(6,7)\}$$

while the $h_2$-stable matchings are:

$$M_4 = \{(2,3),(4,6),(5,7)\}$$
$$M_5 = \{(1,2),(4,6),(5,7)\}$$
$$M_6 = \{(1,3),(4,6),(5,7)\}$$

Theorem 1 states that, starting from any matching, we can always reach one of these matchings by successively satisfying blocking pairs. This implies that any ergodic set (which is called *absorbing set* in [14] and [20]) of the corresponding Markov chain must contain some of the above matchings. Constructing this instance in PRISM, we find there are 308 matchings and a single ergodic set which consists of the matchings $\{M_4, M_5, M_6, M_7\}$, where $M_7 = \{(1,2),(3,8),(4,6),(5,7)\}$. This corresponds to the results presented in [20]. Computing the long-term likelihood of being in any one of the matching (i.e. the steady state probabilities of the Markov chain) we find:

$$x^*(M_4) = x^*(M_5) = x^*(M_6) = \tfrac{2}{7} \sim 0.285714 \ \text{ and } \ x^*(M_7) = \tfrac{1}{7} \sim 0.142857\,.$$

*Case study.* We now compare the performance characteristics of a number of different instances of the SM problem, as the number of men and women $k(=n/2)$ varies between 4 and 8.

– *Symmetric:* in this instance the preferences of the men and women are of the form $m_j : w_j, \ldots, w_k, w_1, \ldots, w_{j-1}$ and $w_j : m_j, \ldots, m_k, m_1, \ldots, m_{j-1}$.
– *Uncoord:* this instance is used in [1] to show an exponential lower bound for the convergence time. The preference lists in this instance are given by $m_j : w_{j+1}, \ldots, w_k, w_1, \ldots, w_j$ and $w_j : m_j, m_{j+1}, \ldots, m_k, m_1, .., m_{j-1}$.
– *Uniform:* in this case the preference lists of all men and all women are the same and equal $w_1, w_2, \ldots, w_k$ and $m_1, m_2, \ldots, m_k$ respectively.

In our experiments we consider both the case when we start with a random (complete) matching and the empty matching. Tables 1 and 2 report on the model statistics (states and transitions) of the Markov chains generated with PRISM. Table 1 includes both the average and the maximum expected time to reach a stable matching when starting from a complete matching, while Table 2 the expected time when starting from the empty matching and number of stable matchings. For comparison, the tables also includes the minimum, average and maximum values obtained from a sample of 1,000 random instances.[11]

[11] Since for $k$=8 each instance takes over 20 minutes to analyse, it was not feasible to study 1,000 different instances.

| Model | $k$ | states | transitions | expected time av. | expected time max. |
|---|---|---|---|---|---|
| *Symmetric* | 4 | 208 | 1,433 | 3.595 | 5.713 |
| | 5 | 1,545 | 15,901 | 5.456 | 7.919 |
| | 6 | 13,326 | 189,691 | 7.692 | 11.05 |
| | 7 | 130,921 | 2,450,001 | 10.30 | 14.09 |
| | 8 | 1,441,728 | 34,194,273 | 13.27 | 17.97 |
| *Uncoord* | 4 | 208 | 1,268 | 18.97 | 25.22 |
| | 5 | 1,545 | 14,205 | 84.23 | 93.74 |
| | 6 | 13,326 | 170,886 | 399.2 | 413.5 |
| | 7 | 130,921 | 2,222,745 | 2,197 | 2,216 |
| | 8 | 1,441,728 | 31,209,032 | 14,361 | 14,385 |
| *Uniform* | 4 | 87 | 369 | 6.822 | 9.160 |
| | 5 | 665 | 4,746 | 12.04 | 14.92 |
| | 6 | 5,972 | 64,341 | 19.08 | 22.73 |
| | 7 | 61,215 | 926,095 | 28.18 | 32.42 |
| | 8 | 702,311 | 14,175,310 | 39.61 | 44.56 |
| *1000 random samples* (min) | 4 | 102 | 461 | 4.735 | 6.932 |
| | 5 | 993 | 8,524 | 8.082 | 11.21 |
| | 6 | 9,272 | 119,035 | 11.61 | 15.59 |
| | 7 | 130,884 | 2,378,889 | 15.93 | 20.89 |
| *1000 random samples* (average) | 4 | 193 | 1,247 | 8.032 | 10.65 |
| | 5 | 1,562 | 15,618 | 13.83 | 17.34 |
| | 6 | 13,317 | 192,465 | 22.84 | 27.28 |
| | 7 | 130,918 | 2,524,157 | 37.34 | 42.74 |
| *1000 random samples* (max) | 4 | 208 | 1,460 | 17.25 | 20.30 |
| | 5 | 1,545 | 16,660 | 46.28 | 50.68 |
| | 6 | 13,326 | 202,560 | 115.8 | 121.1 |
| | 7 | 130,921 | 2,657,024 | 164.9 | 170.7 |

**Table 1.** Expected time to reach a stable matching from a complete intial matching

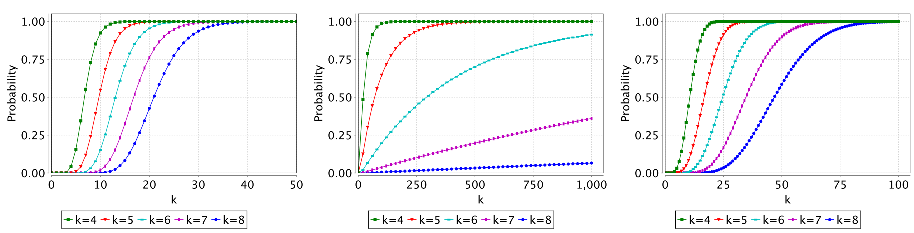
The number of states reported in Tables 1 and 2 demonstrate that, when starting from a randomly chosen complete matching, the number of reachable matchings is dependent on the particular instance. We also see that for the *Symmetric* and *Uncoord* instances all matchings (except the empty matching) are reachable. The results for the *Uncoord* instances are far slower than for the other instances, corresponding to the fact that [1] uses this instance to demonstrate a exponential lower bound on the convergence time. Considering the random sample results, we see that the performance of the *Uncoord* instance is unlikely to be seen in practice. These results also indicate that the number of stable matches does not seem to be cause of the slow convergence time demonstrated by the *Uncoord* instance. To further demonstrate how PRISM can be used to analyse instances, Figure 1 plots the probability of reaching a stable matching within $R$ rounds when starting from the empty configuration.

## 4 Further remarks

As an extension of the approach presented in this paper, it would be interesting to study stochastic processes occurring in more general settings, for example, in *coalition formation games*, where the size of possible coalitions can be larger than two. However, in this case the existence of a stable solution does not guarantee that there is a convergence to a stable solution when starting from any unstable

| Model | $k$ | states | transitions | expected time | no. of stable matchings |
|---|---|---|---|---|---|
| *Symmetric* | 4 | 209 | 1,449 | 7.469 | 1 |
| | 5 | 1,546 | 15,926 | 10.51 | 1 |
| | 6 | 13,327 | 189,727 | 13.95 | 1 |
| | 7 | 130,922 | 2,450,050 | 17.78 | 1 |
| | 8 | 1,441,729 | 34,194,337 | 21.99 | 1 |
| *Uncoord* | 4 | 209 | 1,284 | 28.04 | 4 |
| | 5 | 1,546 | 14,230 | 97.16 | 5 |
| | 6 | 13,327 | 170,922 | 416.5 | 6 |
| | 7 | 130,922 | 2,222,794 | 2,220 | 7 |
| | 8 | 1,441,729 | 31,209,096 | 14,388 | 8 |
| *Uniform* | 4 | 209 | 1,421 | 11.31 | 1 |
| | 5 | 1,546 | 15,926 | 17.66 | 1 |
| | 6 | 13,327 | 192,862 | 25.82 | 1 |
| | 7 | 130,922 | 2,525,804 | 36.03 | 1 |
| | 8 | 1,441,729 | 35,686,961 | 48.56 | 1 |
| *1000 random samples* (min) | 4 | 209 | 1,421 | 7.851 | 1 |
| | 5 | 1,546 | 15,926 | 11.53 | 1 |
| | 6 | 13,327 | 192,862 | 15.77 | 1 |
| | 7 | 130,922 | 2,525,804 | 21.04 | 1 |
| *1000 random samples* (average) | 4 | 209 | 1,421 | 11.02 | 1.506 |
| | 5 | 1,546 | 15,926 | 17.54 | 1.657 |
| | 6 | 13,327 | 192,862 | 27.39 | 1.961 |
| | 7 | 130,922 | 2,525,804 | 42.80 | 2.187 |
| *1000 random samples* (max) | 4 | 209 | 1,421 | 20.35 | 5 |
| | 5 | 1,546 | 15,926 | 50.61 | 5 |
| | 6 | 13,327 | 192,862 | 121.2 | 7 |
| | 7 | 130,922 | 2,525,804 | 170.8 | 9 |

**Table 2.** Expected time to reach a stable matching from the empty initial matching




(a) *Symmetric* (b) *Uncoord* (c) *Uniform*

**Fig. 1.** Probability of reaching a stable matching within $R$ rounds.

state, as illustrated by Klaus *et al.* [20]. So in this case absorbing states and ergodic sets may appear together in the Markov chain. Yet, one could investigate the structure of absorbing and ergodic states for special classes of coalition formation games, and analyse particular games in a similar framework as we did here, using PRISM. Furthermore, the same questions can be asked for cooperative games with transferable utilities as well, such as the stable matching problem with payments [6], where the agents who are matched together may share the value of their cooperation between themselves.[12] Finally, more gen-

[12] Note that for TU-games there are some results on the *accessibility of the core* and the number of blocks needed to access the core (or some other desired set of imputations),

eral network formation games and matching problems could also be analysed with this technique. An example for this is the resident allocation problem with couples, where the existence of a stable matching is not guaranteed in general. However, for particular preference structures Klaus and Klijn [18] show that not only the existence of a stable solution can be guaranteed but also the path to stability from any starting matching.

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see e.g. [22] and [37]. These can be seen as the counterparts of our theorem about the number of steps needed to obtain a stable (or $h$-stable) matching.

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