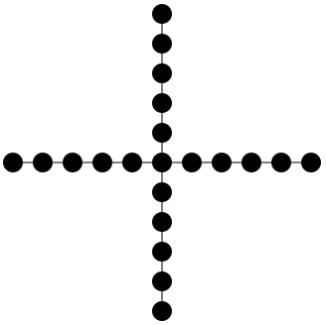
Which cow coughed first?

Some of our favourite farmers' cows have a cough – so we need to call the vet to come and work out which cows need medicine, and which cow caught the cough first and infected all the others. In this activity a few people (at most five, but fewer is fine) are going to play the role of vets, and the rest cows – you can change who has which role between games.

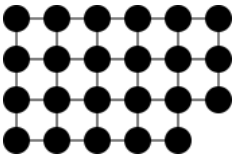
First, everyone needs to know the rules about how the cough spreads. One cow starts off having the cough. We then run a fixed number of "infection rounds" (a good number of infection rounds is about one quarter of the number of cows, so if we have 20 cows we could choose to have 5 infection rounds). In an infection round, each cow looks to see whether any of its neighbours is ill at the start of the round – if so, that cow now catches the cough (but doesn't pass it onto any of its neighbours until the next round). After our chosen number of infection rounds, the vets will have to try to work out which cows have the cough and who had the cough first.

You should run a game with various different arrangements of the cows – here are some suggested arrangements:

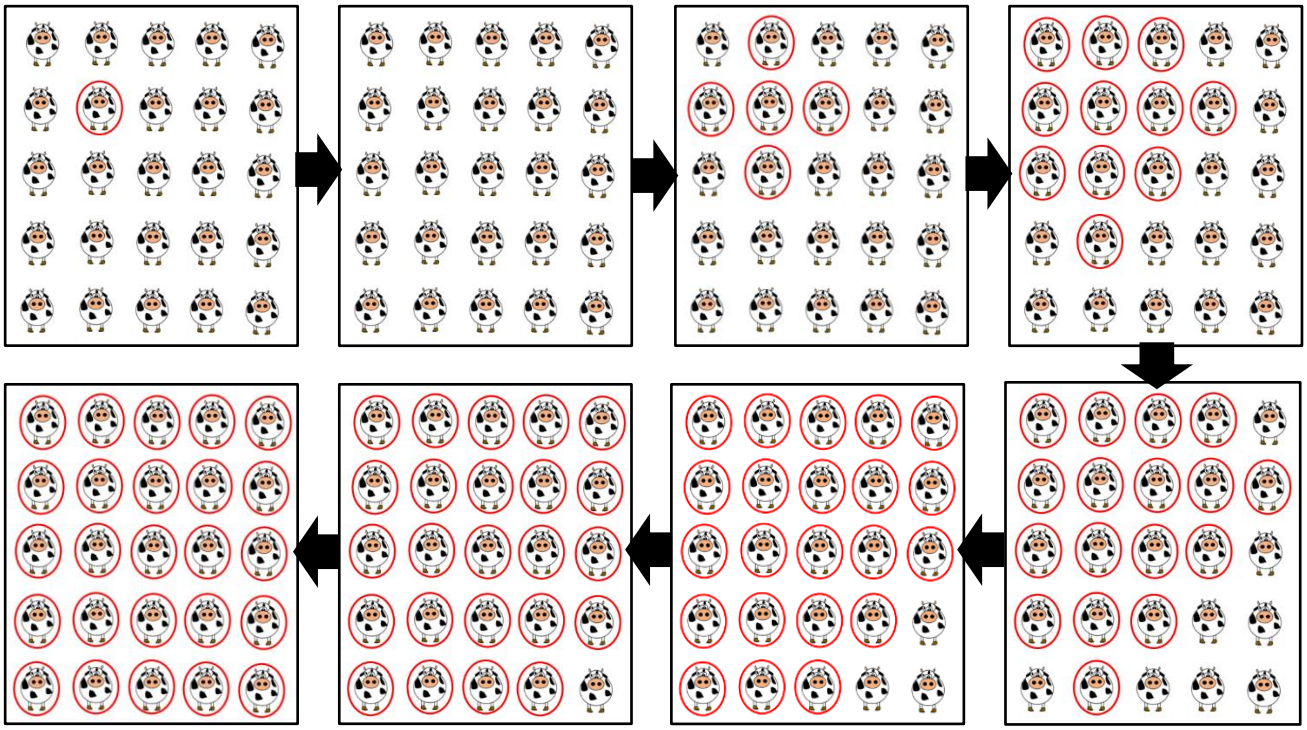
- All the cows sit or stand in one long line – a cow's neighbours are the two cows on either side of them in the line (except for the cows at each end who each have only one neighbour);
- The cows form two separate lines;
- The cows form a cross-shape with two long lines that have one cow in common (as illustrated below) – most cows have two neighbours, except for the four at the ends of the lines who have one neighbour each and the one cow that belongs to both lines and has four neighbours.



- All the cows form a large circle – every cow now has two neighbours (those on either side in the circle)
- The cows form two separate circles – each cow has two neighbours in its own circle.
- The cows are arranged in a grid (as illustrated below), and a cow's neighbours are those immediately to the left, right, in front or behind (so a cow has up to four neighbours). [ASIDE: ask the group to work out how they could arrange themselves in a grid so that they form a perfect rectangle.]



Here's an illustration of how the cough would spread on a rectangular grid.



Once everybody understands the rules, we need to stop the vets seeing who gets infected: they could leave the room, or just be asked to close their eyes or face the wall as it's possible to run the infection stage so that the vets can't hear anything useful. Now, hand every cow a small piece of paper; one of these pieces of paper should have a large "X" on it, in a circle. Whoever receives this circled "X" is the first cow to have the cough. Now, for each infection round, each cow looks at their neighbours' pieces of paper; if any of their neighbours has an "X", the cow puts an "X" on their own piece of paper (but with no circle – the circle is to identify the first cow). At the end of the final infection round, all cows should turn their pieces of paper over so the vets can't see whether they are infected, and the vets are invited to open their eyes or come back into the room.

Now, the vets get to check cows. Each time they check a cow, they point at the cow in question; if the cow is ill they must cough, otherwise they stay silent. Keep track of how many checks the vets perform. This continues until the vets think they have worked out (1) which cows have the cough, and (2) which cow had the cough first – but the vets want to make as few checks as possible. Make a note of what the vets think, and then ask the cows to reveal the truth to see if the vets were right.

You could now try the same layout with a different group of vets, to see if they manage to work out the correct answer with fewer guesses, before moving onto the next layout.

Questions for discussion:

- Did the vets have to check every cow to work out who had the cough? If not, what methods did they use?
- Could the vets have used fewer checks in any of the examples?
- Which layouts were easiest/hardest for the vets? Why?

Extension 1 – introducing randomness

To make things a bit more realistic we no longer assume that if one of your neighbours is ill you will automatically catch the cough. Instead, at each round, if one of your neighbours is ill, you toss a coin to decide whether you become ill too: if you get heads you stay healthy, but if you get tails you become ill. If none of your neighbours is ill you don't need to toss a coin – you still can't catch the cough.

Try each of the layouts above again with this variation. You could also vary the probability that the disease spreads by using dice instead of coins: to make it unlikely that the disease spreads, use the rule that you only catch the cough if one of your neighbours is ill and you roll a six, or to make it very likely that the disease spreads use the rule that you catch the cough if one of your neighbours is ill and you roll any number except one.

Questions for discussion:

- Was this version easier or harder for the vets? Why?
- How did the number of cows with the cough change?
- If you tried with different dice-rolling rules, was it easier for the vets when the disease was more or less likely to spread?

Extension 2 – an interactive version

In this version the vets have the chance to interact with the cows while the disease is spreading. Setup the game as before (and use the same layouts). The difference now is that, before the first infection round, the vets get to choose a cow to check. If the cow they choose is ill then, as before, the cow coughs. If the chosen cow is currently healthy, the vets vaccinate this cow: this means that the chosen cow will *never* catch the cough or pass it onto its neighbours, even if one or more of its neighbours becomes ill.

Now run an infection round, which proceeds as normal except for any vaccinated cows who ignore the fact their neighbours may be infected. After the infection round, the vets choose another cow to check – again, if the cow is not yet infected, it is vaccinated.

Continue in this way, alternating infection and checking rounds, until the vets think they have identified all of the sick cows. The cows can now reveal who did indeed have the cough, and we find out whether the vets got it right.

It's best to run this version first without using coins or dice, but to make things more challenging you could try a game where you combine both of the extensions.

Questions for discussion:

- Did this make the game easier or harder for the vets? Why?
- What was a good choice of cow to check?
- How did the total number of cows with a cough change?
- Were some layouts easier or harder than before in this version? Why?

The maths behind *Which cow coughed first?*

The problem you were looking at in *Which cow coughed first?* is actually very important in real life, although we might more often look at whether a particular farm has any diseased animals rather than considering individual cows: if a farmer finds that some of their animals are ill, vets need to try and figure out where the disease first came from so that they know which other farms might be at risk. To do this, they can look at who bought and sold animals recently (when animals move from one farm to another they can spread disease) and also which farms are geographically close to each other (if two animals from different farms have a chat over the fence, one might make the other ill). The idea that we should look at contact between individuals to understand how a disease spreads seems very obvious, but is actually relatively new: for a long time people trying to model the spread of the disease based their models on the assumption that everybody mixed with everybody else. The study of how disease spreads based on contact between individuals is called **network epidemiology**.

When everybody stood in a line, it was relatively easy to work out which animals were infected: once you had identified the left-most and right-most infected animal, you knew that everyone in between must also be infected because that was the only way the disease could have been passed around (and, except in the extension, you knew that the cow in the middle of this infected section must have coughed first – unless it was the cow at the very end of the row). This is because, if we turn the layout into a graph by putting a dot for each person, and drawing a line between two dots if the people are next to each other (so can pass the cough to each other), we get a very special kind of graph called a **tree**. In a tree, if we pick any two dots, there is exactly one way to get between them by following lines. Lots of problems that are difficult to solve on general graphs are much easier on trees. The third layout you saw was a slightly more complicated example of a tree.

As soon as our graph has a **cycle** (a sequence of lines we can follow to get back to where we started), things become more complicated. For example, when everybody is in a circle and we know two cows that are infected, there are two possibilities for how the disease might have spread between the two (going either way around the circle). Generally, the problem becomes harder if we have lots of cycles, for example in the "grid" layout.

It was also important whether the graph was **connected**. A graph is connected if we can get between any pair of dots by following lines. In the examples where we had two lines of cows or two circles of cows, the graph was not connected: this meant that if the cough started off in one line (or circle) it could never get to the other one.

Of course, in the real world, things don't behave quite as nicely as in this example: animals move around so the layout changes over time. Also, it isn't certain that if a diseased animal has contact with another it will pass on the disease: the extension made things a bit more realistic in this way, because there was something random going on to decide whether we actually passed on the disease. Some of the same reasoning still worked in this situation – if two cows had the cough, then all of the animals on some path between them must also have the cough – but we don't know any more that the cough must have started off in the middle as it could spread more quickly in one direction than the other.

The ideas you saw in this activity don't just apply to understanding the spread of disease. More generally, mathematicians and physicists like to look at how simple "local" rules (e.g. to work out whether you had the cough, you just had to look at your neighbours, not what was happening far away) can result in interesting behaviour in a large system. When you sat in a "grid" layout, you actually acted out an example of a **cellular automaton**. It's incredible what complicated things can happen based on simple rules: to get a taste of what is possible, try looking up **Conway's Game of Life** online.