Happy Homes

Our farmer friends Karl, Otto, Jim, Laura and Martin each want to buy a new animal for their farm, but they only have certain types of food available – so they have to make sure they choose a type of animal which likes at least one of these foods. Can you help the farmers work out who should buy which animal?

Below are seven scenarios describing what the different animals like to eat, and what kinds of food each farmer has. In each case, can you find a way to match up exactly one animal with each farmer? If not, why not?

Scenario 1	The sheep only eats carrots or hay. The horse only eats carrots. The cow only eats hay or grass. The pig only eats wheat, rye or carrots and the chicken only eats wheat or rye. Jim has hay and carrots and grass. Karl has grass and carrots. Otto has carrots wheat and rye. Martin has wheat and rye and Laura has carrots, wheat and rye.
Scenario 2	The sheep only eats grass. The horse eats carrots. The cow only eats carrots or grass. The pig only eats wheat, rye or carrots and the chicken only eats wheat or rye. Jim has hay and carrots and grass. Karl has grass and carrots. Otto has carrots wheat and rye. Martin has wheat and rye and Laura has carrots, wheat and rye.
Scenario 3	The cow only eats wheat or rye and the sheep only eats hay. The horse eats carrots. The pig only eats rye and the chicken only eats grass. Jim only has hay and carrots. Karl has wheat, while Otto has wheat and rye. Laura has grass and Martin has grass, wheat and rye.
Scenario 4	The cow eats wheat, carrots or hay. The sheep eats hay or carrots. The horse eats hay, carrots or wheat. The pig eats rye or grass, and the chicken also eats rye or grass. Jim has hay, carrots and wheat. Karl has hay, carrots and wheat. Otto has grass and rye. Laura has rye and grass and Martin also has rye and grass.
Scenario 5	The cow eats wheat or carrots, and the sheep eats hay, rye or grass. The horse eats carrots or wheat. The pig eats hay or rye. The chicken eats grass, rye or hay. Jim has hay, rye and grass. Karl has carrots and wheat. Otto has wheat and carrots. Laura has wheat and carrots and Martin has wheat and carrots.
Scenario 6	The cow eats wheat or rye. The sheep eats hay. The horse eats wheat. The pig eats rye. The chicken eats grass. Jim has hay and wheat. Karl has carrots. Otto has wheat. Laura has rye and Martin has rye and grass.
Scenario 7	The cow eats grass. The sheep eats wheat or rye. The horse eats carrots. The pig eats grass and the chicken eats hay. Jim has carrots, wheat and rye. Karl has carrots and hay. Otto has carrots and grass. Laura has grass, wheat and rye. Martin has grass.

To make this activity more interactive, you can give members of the group different roles (farmers or animals), and each only has information about what foods they eat or have available. The group then has to work together to find a solution. To do this, print all the pages in the file "Happy homes – scenario cards" and cut along the dotted lines to provide a card for each member of the team (taking care to keep the cards for the same scenario together).

You could even split the group in two, and give one group the scenario cards so they have to work as a team, while the other half have all the information (from the sheet "Happy homes – scenario descriptions"). Which group can solve all the scenarios first?

Questions for discussion

- Could you find a solution in every example?
- Which examples could you not find a solution?
- Can you see a reason why these examples didn't have a solution?

The answers for each example are as follows:

Example 1	One possible solution: cow - Karl, sheep - Otto, horse - Jim, pig - Martin, chicken - Laura.
Example 2	One possible solution: sheep - Karl, cow - Jim, horse - Laura, chicken – Otto, pig – Martin.
Example 3	No solution. Both the sheep and horse can only go to Jim, but he can only take one animal.
Example 4	No solution. The cow, the sheep and the horse can only go to Jim or Karl.
Example 5	No solution. The sheep, the pig and the chicken can only go to Jim.
Example 6	No solution. No animal can go to Karl (so there are four remaining homes for five animals).
Example 7	One possible solution: sheep - Laura, horse - Jim, cow - Martin, pig – Otto, chicken - Karl.

Extension – a card game

If you have time, you can extend this activity with a two-player game. Each pair needs a printout of the file "Happy homes – card game". Cut along the dotted lines to create cards.

Lay out the large farm cards face up on the table. Shuffle the small food cards, then place two on each of the large farm cards (you may find blue-tack or similar useful for holding these in place). Now shuffle the animal cards and deal equally to the two players – players can look at their cards.

The players now take turns to place one of their animal cards on a farm which

1. Does not yet have any animal on it, and
2. Has a kind of food that the animal on the card eats.

The game ends when neither player can place another animal – this may or may not mean both players have run out of cards. You win if you have placed all your animals but your opponent still has at least one card in their hand. Otherwise the game is a draw.

Variations:

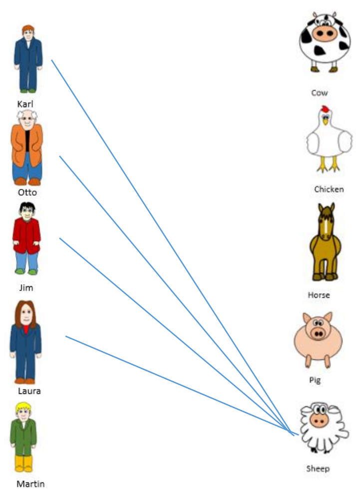
If you get bored of this game, here are two variations:

1. When it's your turn, you can move around the animals that have already been placed to create a space for your animal – but every animal that was already placed has to end up with a home.
2. Don't take turns any more – count "3, 2, 1, Go!" and then race to place your cards before the other player. The first player to find homes for all their animals wins.

The maths behind *Happy Homes*

What you saw in this activity was an example of a **matching problem**. These arise in all sorts of real-world situations (not just sending animals to farms!). For example, suppose you want to share a box of chocolates with some friends, but you each like a different collection of the chocolates – you need to match each person to a chocolate they like, so that everyone gets one chocolate. Or perhaps everyone is trying to book an appointment for a haircut with the same hairdresser, but everyone is available at different times: now we need to match each person to one time slot when they are available.

It's often helpful to think of matching problems graphically: draw a row of dots down the left-hand side of the paper representing one group (say, the farmers) and another row down the right-hand side for the other group (the animals). Now draw a line between a dot on the left and a dot on the right if they are compatible (i.e. if the animal could live on that farm). This gives you a special kind of graph or network called a **bipartite graph**.



In Example 1, the sheep eats carrots and hay. Jim has hay and carrots, and Karl, Laura and Otto have carrots. So, the sheep can go to Jim, Karl, Laura and Otto. This is indicated in the graph by drawing a line from the sheep to each of Jim, Karl, Laura and Otto.

To complete the graph, we now have to draw lines from each of the other animals to their compatible farmers – for example, we will need lines from the pig to all of the farmers, and from the horse to Jim, Karl and Otto.

Drawing out the scenario like this can make it much easier to find a solution or to see why none can exist.

By the time you've done the activity, you've probably noticed one simple thing which can prevent us finding homes for all the animals. In example three, there were two animals (the sheep and the horse) for whom the only possible home was with Jim – as Jim can only take one animal, there's no way that all the animals can find homes. Similarly, in example four, there were three animals (cow, sheep and horse) who all had to go to either Jim or Karl: there is no way we can find a home for every animal when we would have to fit three animals into two places. In general, we can see that if we have a group of n animals, and the number of farms where at least one of these animals could go is smaller than n (e.g. in example before we had 3 animals and 2 possible farms) then we can't possibly find a home for every animal.

Perhaps surprisingly, it turns out that this is the only thing that can stop us finding a home for all the animals. It's not at all obvious that nothing else could go wrong, but in 1935 Philip Hall proved that this is true. Of course, he wasn't looking at the case of animals going to farms – in fact, he phrased the problem in terms of groups of men and women who were willing to marry some collection of the others, so the result is often called **Hall's Marriage Theorem**.

In general, matching problems can become much more complicated if the individuals involved have preferences. Here, we assumed that each farmer was equally happy with any type of animal they can feed, and each animal is equally happy to eat any of the types of food that is suitable for it. But what if in fact Jim really likes horses, so would much rather buy a horse than a cow if he has the option, and although the horse will eat grass or wheat she really prefers hay, so would choose to go to a farm that has hay. In that case, we'd better make sure that we don't let Martin – who only has grass – buy the horse, while Jim – who has hay – ends up with the cow: the horse will then want to run away to live with Jim, who'd be very happy with this arrangement, but now Martin won't have any animals and the poor cow is homeless! What we need in this case is something called a **stable matching**, which means there are no pairs like this who would be better off if they ignore what they've been told to do and pair up on their own.