Inspecting the Fences

After planting the crops, it's important to make sure all the fences are in good condition – we don't want rabbits getting into the fields and eating all of our new crops! So you need to plan a route you can walk so that you'll go along every fence (you only need to walk along one side of each fence). But you don't want to waste time by walking along the same section of fence more than once, or spending time walking when you aren't checking a fence.

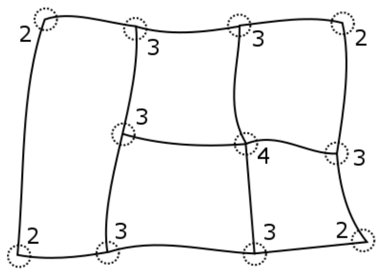
Take a look at your farm plan from *Planting Time* (or, if you didn't previously do this activity, draw a plan of some farm fields for your neighbour). Can you find a way to walk along each section of fence exactly once, and end up where you started? Another way to think about this would be to take a blank piece of paper: can you draw a copy of your farm layout (just the fence lines) without taking your pencil off the paper and without going over any lines twice? Would it help if your friend can drop you off anywhere on the farm in their tractor and then pick you up from somewhere else later?

Now try drawing a layout for your neighbour to try and do the same – can you make sure they can't walk along every fence exactly once?

Questions for discussion

- Who was able to find a good route around their fences?
- What stops you finding a good route?
- When does it help getting a lift in a tractor?

There's actually a simple way to work out whether you'll be able to walk along every fence exactly once. Look at all the places on your farm where two or more fences meet, and count up for each one how many different sections of fence you could follow from that point. In the example below, all of these points are circled, and labelled with the number of sections of fence coming out of the point.



You can only walk along every fence exactly once and end up back where you started if all of these numbers are even. Think about what happens on all the times that you visit a specific point. You will walk along one section of fence to get to the point, and another to leave… then you go somewhere else, but next time you come back to the point again you arrive along a new section of fence and leave along another. So each time you visit the point you'll use exactly two of the fences coming out of it. So if there's an odd number of fences coming out of the point, one of these must be left over at the end. (For the starting point it's slightly different: this time you can pair up fences as the one you use to leave and the one you use to come back.) In fact, it turns out that if you have an even number of fences coming into each point then you'll always be able to find a good route, but this isn't so obvious.

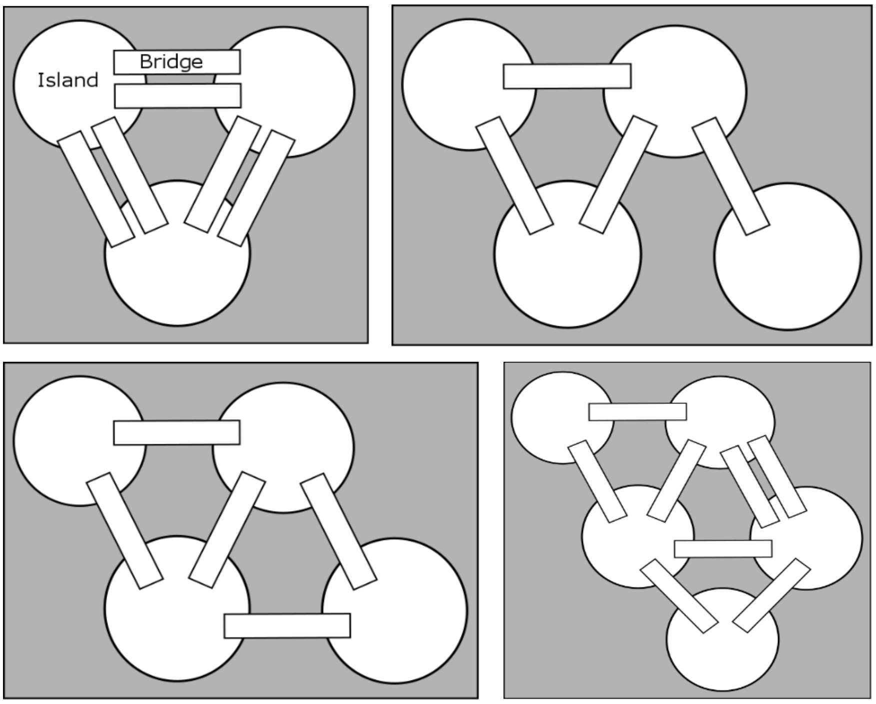
There's a small exception to this if you have a friend who can drop you off and pick you up, so that you don't have to get back to where you started: in this case it's okay for there to be an odd number of fences coming out of the points where you start and finish, so long as all the others have an even number. This is because we can use up an extra "single" fence to leave the starting point (we don't need another to get back any more) and similarly we can use a single fence to reach the finishing point without needing another edge to leave.

Extension 1 – a physical game

For this you need a collection of at least five objects to represent "islands" (e.g. gym mats, hoops) and at least eight of another type of object to represent "bridges" (e.g. smaller mats, skipping ropes).

Set up the islands and bridges in one of the configurations below. If you have enough equipment, you can set up several configurations at the same time and split the group into smaller teams who try these simultaneously (then swap to try the other configurations).

The group chooses a runner, and a "home" island (it might help to mark this with a piece of clothing or similar as a reminder). The runner starts on the home island; each time the runner crosses a bridge, the rest of the team remove that bridge. The goal is for the runner to get back to the home island after all the bridges have been removed – and of course the runner is not allowed to jump between islands without using a bridge! If they don't succeed (either some bridges are left, or the runner doesn't make it back home), let them try again; after several failures, offer them the option to place one new bridge at the start to see if this helps.



If there is spare time after trying all of these configurations, split the group into teams (if you haven't already) and ask them to design configurations for each other to try; to make this more challenging for the group designing the configuration, give the other team the option to add one additional bridge at the start.

Questions for discussion:

- What stopped you removing all the bridges?
- When was it helpful to add a new bridge?

Extension 2 – a two-player board game

The first stage of this is essentially a seated version of Extension 1 – but this means that more complicated configurations can be used. This activity is done in pairs, and each pair needs:

- a printout of "Inspecting the Fences – game board";
- 20 "bridges" in each of two different colours – these can be cut out from a colour printout of "Inspecting the Fences – paths to cut" or some other coloured object of a similar shape (e.g. crayons) could be used;
- a token e.g. a counter or coin.

Initially, just use one colour of "bridge". Player 1 places the 20 bridges on the board, so that each bridge joins two of the green islands. Player 2 then places the token on the island marked "start" and begins moving around the token around the board by crossing bridges to get to new islands. Each time the player moves the token across a bridge, that bridge is removed from the board. Player 2 wins if they succeed in removing all the bridges and returning to the starting island; otherwise Player 1 wins. Now swap roles and play again.

For the second stage, Player 1 takes the red bridges and Player 2 the blue bridges. To setup, Player 1 places a red bridge between two islands anywhere on the board, then Player 2 places a bridge between any two islands on the board (it is allowed to have multiple bridges between the same pair of islands); this process repeats until all bridges have been placed. Place the token on the starting island.

Player 2 moves first, moving the token over some bridge from the island. If the chosen bridge is red (the colour belonging to Player 1) then Player 2 removes the bridge; if it is blue (Player 2's own colour) the bridge is left on the board. Player 1 then takes a turn to move the token along any bridge to a new island; again, if the token is blue (Player 2's colour) the bridge is removed, but not if it is red (Player 1's colour). Players continue taking turns in this way until either

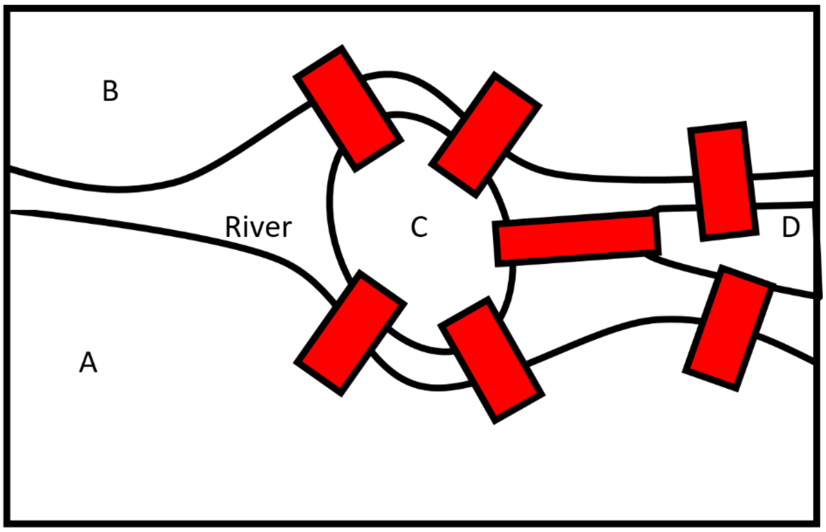
- all bridges of one colour have been removed, or
- the token is on an island with no bridges coming out (so nobody can move).

The winner is the player who has removed more of the other player's bridge (so if more red bridges have been removed then Player 2 wins).

The maths behind *Inspecting the Fences*

Inspecting the Fences is based on an even older mathematical problem than *Planting Time*: this one dates back to the 1730s. What you were trying to do in all of these problems was essentially the same: travel along every route exactly once, and end up back at your starting point. You can represent all of these examples as a graph: use a dot to represent a junction of two or more fences, or an island; draw a line between two dots if there is a fence leading directly between the two (without any junctions on the way), or if there is a bridge between the two. Then the route you are trying to find has a special name: it's called an *Euler Tour*, after the famous Swiss mathematician Leonhard Euler.

The example of crossing bridges was in fact the original inspiration for this problem. The city of Königsberg (now Kaliningrad) has four separate land-masses, separated by rivers, and the picture below shows where there were bridges in Euler's time. He wanted to know whether it was possible to go for a walk in which he would cross every bridge exactly once and end up back where he started.



As you can probably spot easily by now, this isn't possible: all of the areas of land have an odd number of bridges going out!