Market Day

It's time to sell your cows! But to do this, you first have to get them all safely to the market...

The market isn't too far away, and it's a nice day, so you are going to walk the cows to market - the only trouble is that the cows are going to get hungry along the way! Luckily, cows like to eat grass and there's lots of nice grass along the paths that lead from your farm to the market... but if you take too many cows along the same path then they'll eat all of the grass and there'll be none left for the next cow. And if the cows is hungry, you won't be able to persuade it to keep walking...

Give everyone in the group a printout of the file "Market Day - Scenarios". Jim is going to walk his cows from his farm to the market, but each time a cow walks along a path it eats one of the clumps of grass along the path (cross out the clump of grass the cow eats). If there is no grass left along a particular path, he can't use that path anymore because the cows will be too hungry. In each example, how many cows can Jim take to the market?

Questions for discussion

- How many cows could you get to the market in each example?
- Does it matter which route you choose?
- Did you ever want to go back and change what you'd done in the past (i.e. take an earlier cow along a different route)?
- How did you know that you'd found the best possible solution?

The number of cows you should be able to get to market in each example is as follows:

Example	Number of cows to market	Example	Number of cows to market
1	6	6	5
2	7	7	7
3	6	8	7
4	6	9	8
5	5	10	7

Extension - outwitting your rival

Unfortunately, Martin is jealous of Jim’s cows: they are much more beautiful than his own cows, so he’s worried that nobody will want to buy his cows once they see Jim’s. To solve this problem, Martin has come up with an evil plan: first thing in the morning on market day, he goes along the paths between Jim’s farm and the market, hiding sleeping pills in some of the clumps of grass. If a cow eats one of the clumps of grass that contains a sleeping pill, it will fall asleep as soon as it’s finished eating, and Jim has no chance of getting that particular cow to market today.

Luckily for Jim, he guessed that Martin might do something like this, so he followed him and saw where he hid the pills. This means he knows which clumps of grass his cows have to avoid, but unfortunately the pills are too well hidden for him to remove them. The question now is how many cows he can still get to the market without them eating any sleeping pills?

Split the group into pairs, and give each pair a new printout of the file "Market Day – Scenarios" and a blob of blu-tack or similar (this will be used to mark where Martin has put the sleeping pills). In each pair, one person is going to take the role of Jim, and the other Martin.

For each scenario, Martin starts by placing a marker (e.g. a small piece of blu-tack) on 10 of the clumps of grass. Jim then has to try and find a way of getting as many of his cows as possible to market. If he can still get at least one cow to market, we say that Jim wins, otherwise Martin wins. Now try the same example again, but change the number of pills Martin is allowed to place: if Martin won the last game, he gets one pill less this time, whereas if Jim won he's allowed one more pill. Keep doing this – increasing or decreasing the number of pills – until the other farmer wins. Make a note of the smallest number of pills with which Martin managed to stop Jim getting all his cows to market, before moving onto the next example (again starting with 10 pills).

Questions for discussion

- How many sleeping pills did Martin need in each example?
- Did it every make sense to hide a pill in some of the clumps along a particular path, but not all of them?
- How did the number of pills Martin needed compare with the number of cows Jim would have been able to get to market without Martin interfering?

You should find that the number of pills Martin needs to win in each example is the same as the number of cows that Jim could take to market before Martin started interfering. Also, the best strategy for Martin never involves hiding pills in some but not all of the clumps of grass along a particular path.

The maths behind *Market Day*

What you were dealing with in this activity is something called *network flows*. A "flow" of cows may seem like a slightly strange concept, but this name seems more natural if you think about a flow of say water along pipes, or traffic along roads. In any of these examples, your goal is to transfer as much of whatever it is you're transporting from one location in the network (often called the *source*) to another location (often called the *sink*). The limit to how much you can transfer comes from the fact that each edge has a *capacity*, or an upper limit on how much you can transport along that edge (in this activity, the capacity of an edge was the number of clumps of grass along it).

How do we know that we've moved as many cows as possible to the market? One way we could be sure that we had done this is to consider how we could block edges so that there's no path left from the source to the sink – this is what Martin was trying to do in the extension, to prevent you getting any cows to market. We call a collection of edges which achieves this a *cut*. If you add up all the capacities of the edges in a cut, and the answer is c, we know for sure that we can't move more than c cows from the source to the sink. Why not? Well, every cow must cross one of the edges in the cut, so by the time we've moved c cows to market we must have used up every clump of grass along the edges in the cut. Once all the grass is gone on these edges it's as if they don't exist (we can't use them any more) so there's no way for any more cows to move from the source to the sink.

In fact, so long as we're careful with how we add up the capacity of edges in the cut (if we had an edge pointing the "wrong" way – going from the side with the sink to the side with the source, e.g. from the market back towards the far – we don't include it, because it's no use to us), it turns out this is the only thing that can stop you getting more cows to market. So if you think that you've found the best possible flow and it involved m cows getting to the market, you should be able to find a cut somewhere whose edges have total capacity equal to *m*. This fact was first proved in the 1950s by Ford and Fulkerson, and for obvious reasons is often called the “Max-Flow Min-Cut Theorem”. They first started looking at the problem for the US military, who were interested in finding the easiest way to damage parts of the Soviet railway network so that it would no longer be possible to get from one end to the other.

In *Market Day*, the maximum flow in each is the number of cows that Jim could take to market in the first activity (before Martin started interfering), and the best strategy for Martin is to hide pills in all the clumps of grass on a cut with the smallest capacity – in which case the number of pills he needs is equal to the capacity of the cut. So the Max-Flow Min-Cut Theorem tells us that the minimum number of pills Martin needs to win should always be equal to the maximum number of cows that Jim could take to market in the first activity.

In contrast with a lot of the problems you may have seen in the other activities, there is an efficient algorithm to find the maximum flow in this kind of directed network – it involves repeatedly finding a path from the source to the sink along which we can increase the flow (or send another cow), and may be quite similar to some of the strategies you came up with to find the maximum flow in the examples. Believe it or not, similar methods can also be used to pair up as many people as possible in *matching problems* (see *Happy Homes* to learn more about matching problems).