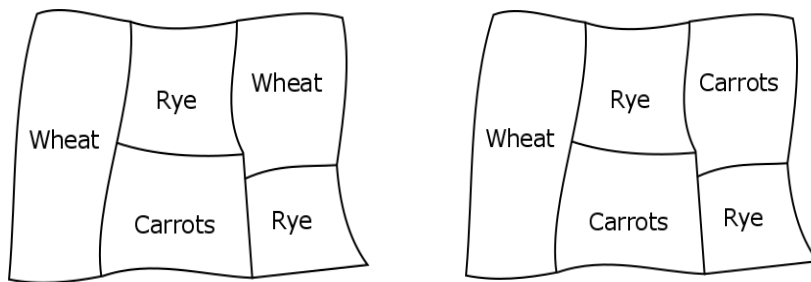


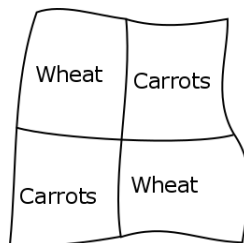
Planting Time

It's time to plant your crops – but first you have to decide what to plant where! You could choose to grow any of wheat, rye, corn, carrots and cabbages, but you'd like to plant as few different crops as possible – it's expensive to buy the equipment to harvest all the different plants. (Also, you really don't like cabbage, so would prefer to avoid growing cabbages if possible!).

Why would you need to grow more than one type of crop? Well, the problem is that pests that damage the crop easily spread from field to field if two adjacent fields contain the same kind of crop. So you want to make sure that you don't plant the same crop on both sides of any fence, otherwise one bug might destroy all of your crops. So the layout on the left is okay, but the one on the right is not because we have carrots on both sides of one fence.



To make life a bit easier, we've decided it's okay to risk having the same crop in fields that only meet at the corner, so the layout below is also allowed.



You've asked a friend to help you design the layout of your farm: your friend will decide where to put the fences to create fields, and then you can choose what to plant in each field. However, your friend might not be as helpful as they seem...

The activity

Split into pairs. First of all, on a blank sheet of paper, design a layout of fields for your friend – but secretly you want to force them to grow cabbages, so you're trying to come up with a design where your friend will need to plant five different kinds of crops. Once you're satisfied with your design, swap with your partner – now it's your turn to try and find a way to plant the fields so that you grow as few different crops as possible.

When you're planning where to plant your crops, you might like to use a different coloured pen for each crop, and colour in the field when you've decided what to plant there; alternatively, you could use different coloured post-it notes,¹ which makes it easier if you want to change your mind later!

Questions for discussion

- Did anyone have to plant cabbages?
- What is the largest number of crops anybody had to grow?

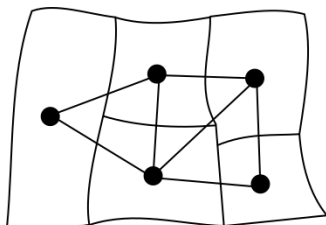
In fact, it's impossible to draw a layout of fields on a sheet of paper (unless you cheat and draw bridges!) that needs more than four different crops. So if you think your friend forced you to grow cabbages, have another look – you will be able to change things around to avoid this!

The maths behind *Planting Time*

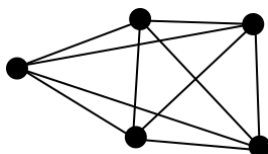
The ideas you saw in this activity date back to the 1850s: the problem mathematicians were interested in then wasn't about planting crops, though. They wanted to know, given a map of countries, how many colours might you need to use so that you colour in each country with some colour, and if two countries have a common border they have different colours (otherwise it might look as if they formed one big country). Nobody could find an example that needed more than four different colours, but proving that there would never be an example that needed five colours was a very different matter.

In fact, the question wasn't resolved until 1976, when Kenneth Appel and Wolfgang Haken proved the famous "**Four Colour Theorem**": you will never need more than four colours to colour a map in this way (or, in other words, you never *have* to plant cabbages). The proof was very controversial, because it was the first time that a proof had relied on using a computer to check a number of possibilities far too large to do by hand. Of course, the computer couldn't consider all possible maps – there are infinitely many possibilities, so it would never finish – so some very clever maths was required to work out that it would be enough to check a specific collection of maps (which contained well over 1000 examples).

This problem is actually a special example of a more general topic called **graph colouring**. Suppose you take your field layout, draw a dot in the middle of each field, and then draw a line between two dots if the fields the dots are in meet at a fence, you have drawn an example of a graph, as in the example below.



Now, the rule is that we want to give each dot a colour, and if there is a line between the two dots then they have to have different colours. Graphs we draw in this way, based on a map, are a very special kind of graph called planar graphs. But in general we can draw a graph by putting dots anywhere on the page and drawing lines between any pairs of dots. For general graphs, we might need a lot more than four colours to give every dot a colour so that dots with a line between them have different colours. For example, the graph below needs five different colours – can you see why?



Can you think of an example of a graph that needs 10 colours? 20? 100?

¹ The idea of using post-it notes for colouring activities is due to Fran Rosamond and Mike Fellows.