Struct: Finding Structure in Permutation Sets

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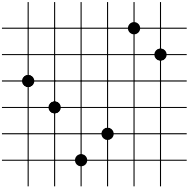
Classical Patterns

What is a permutation?

$\pi = 431265 =$ 

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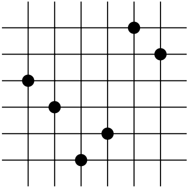
$\pi = 431265 =$ 

The property we will look at is avoidance of *classical patterns*.
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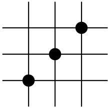
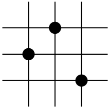
$123 =$ 

$231 =$ 

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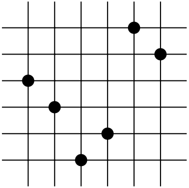
$\pi = 431265 =$ 

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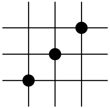
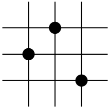
$123 =$  $231 =$ 

So π *contains* 123 but π *avoids* 231.

Permutation Classes

Let B be a set of patterns then define the set $\mathrm{Av}(B)$ to be all permutations that avoid each $\pi \in B$.

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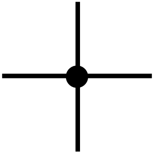
Question: Given a basis B can we find a structure for $\mathrm{Av}(B)$?

Avoiding 21

How many permutations of length n are in Av(21)?



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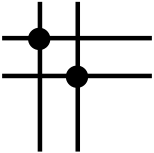

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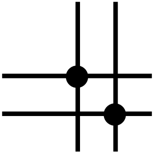

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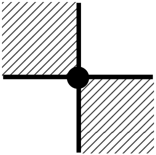

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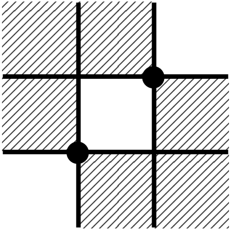

Avoiding 21

Only the increasing permutation $12\ldots n$ avoids 21 and so there is exactly one permutation of each length n avoiding 21. The generating function is therefore

$$\sum_{n\geq 0} a_n x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

What does Struct give?

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What does Struct give?

So we can read the generating function as

$$F = 1 + x \cdot F$$

which upon rearranging gives

$$F = \frac{1}{1-x}.$$

Stack-Sortable Permutations

Knuth [4] showed that a permutation is stack-sortable if and only if it avoids 231.

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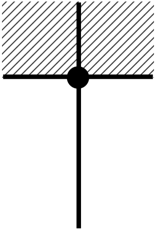

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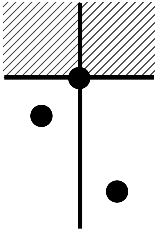

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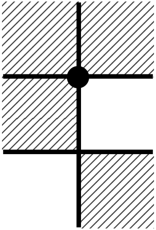

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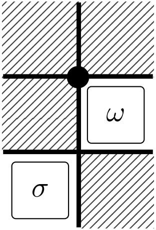

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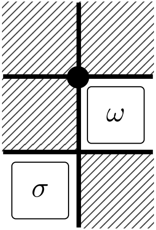

Hence these are counted by the Catalan numbers and have the generating function $C = 1 + x \cdot C^2$.

Implementation

Although still under development, the algorithm is available at GitHub: https://github.com/PermutaTriangle/PermStruct

The algorithm consists of four stages.

- Find building sets.
- Generate rules.
- Generate permutation sets from rules.
- Find a cover.

Building Sets

What are our building sets for $\mathrm{Av}(B)$?

Define the set A_π to be the set of all patterns contained in a permutation π. If we take a subset, $S \subseteq \bigcup_{\pi \in B} A_\pi$ that satisfies the condition that $S \cap A_\pi$ is non-empty for each $\pi \in B$, then we see that $\mathrm{Av}(S) \subseteq \mathrm{Av}(B)$.

These subsets $\mathrm{Av}(S)$ are the building sets used by Struct.

Generate Rules and Sets For Rules

A rule is an $n \times m$ grid with entries from our building sets.

A_1	A_2	A_3	A_4
A_5	A_6	A_7	A_8
A_9	A_{10}	A_{11}	A_{12}

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Where each A_i is a building set.
We generate the permutations by inflating.

Wilf-Equivalent

Sometimes permutation classes are enumerated by the same numbers. For example

$$|\mathrm{Av}_n(123)| = |\mathrm{Av}_n(231)| .$$

We say that these permutation classes are *Wilf-Equivalent*.

Big Bases

Given a basis $B \subseteq \mathcal{S}_4$ that is "big", we run Struct on B.

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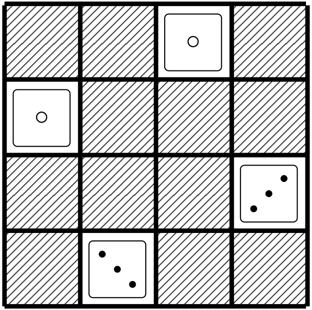
For all such bases such that $|B| > 12$, Struct found a structure. These covers were verified for length 10 permutations.

Peg Permutations

For example $3^{\circ}1^{-}4^{\circ}2^{+}$ is given by the struct rule



Polynomial Classes

By combining results from Huczynska and Vatter [3] and Albert et al. [1] we get the following theorem

Theorem (Homberger and Vatter [2])

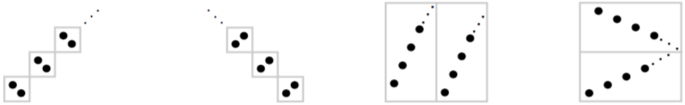
For a permutation class $\mathcal{C}$ the following are equivalent:

(1) *The number of length n permutations, $|\mathcal{C}_n|$, is given by a polynomial for all sufficiently large n,*

(2) $|C_n| < F_n$, *the n^{th} Fibonacci number, for some n,*

(3) $\mathcal{C}$ *does not contain arbitrary long permutation of any of the forms shown below (or any symmetries),*

(4) $\mathcal{C} = Grid(G)$ *for a finite set G of peg permutations.*



Bibliography

[1] Michael H. Albert, M. D. Atkinson, Mathilde Bouvel, Nik Ruškuc, and Vincent Vatter, *Geometric grid classes of permutations*, Trans. Amer. Math. Soc. **365** (2013), no. 11, 5859–5881.

[2] Cheyne Homberger and Vincent Vatter, *On the effective and automatic enumeration of polynomial permutation classes*, J. Symbolic Comput. **76** (2016), 84–96.

[3] Sophie Huczynska and Vincent Vatter, *Grid classes and the Fibonacci dichotomy for restricted permutations*, Electron. J. Combin. **13** (2006), no. 1, Research Paper 54, 14 pp. (electronic).

[4] D.E. Knuth, *The art of computer programming. Vol. 3*, Addison-Wesley, Reading, MA, 1998. Sorting and searching, Second edition.