Reed’s conjecture and strong edge coloring

Marthe Bonamy, Thomas Perrett, Luke Postle

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Vertex coloring

\[ \chi : \text{Minimum number of colors to ensure that } a \neq b. \]

\[ a \setminus b \Rightarrow \begin{cases} a \neq b, \\ a \in L(x), \\ b \in L(y) \end{cases} \]

\[ \chi^\ell : \text{Minimum size of every } L(v) \text{ such that } \omega \leq \chi \leq \chi^\ell \leq \Delta + 1. \]

\[ \omega : \text{Maximum size of a clique.} \]
Vertex coloring

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\[ a \neq b \Rightarrow a \in L(x) \quad \text{and} \quad b \in L(y) \]

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\[ \chi : \text{Minimum number of colors to ensure that} \]
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\[ \chi_\ell : \text{Minimum size of every } L(v) \text{ such that} \]
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Vertex coloring

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\[ a \quad \Rightarrow \quad \begin{cases} 
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  b \in L(y)
\end{cases} \]

\[ \omega : \text{Maximum size of a clique.} \]
\[ \omega \leq \chi \leq \chi_l \]
Vertex coloring

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\[ \Delta: \text{Maximum degree.} \]
\[ \omega \leq \chi \leq \chi_\ell \leq \Delta + 1. \]
Beyond a trivial bound

**Theorem (Folklore)**

*Every graph* \( G \) *satisfies* \( \chi(G) \leq \Delta(G) + 1 \).
Beyond a trivial bound

**Theorem (Reed '98)**

\[ \exists \epsilon > 0, \forall G, \chi(G) \leq \left( 1 - \epsilon \right) \cdot \left( \Delta(G) + 1 \right) + \epsilon \cdot \omega(G) \]
Beyond a trivial bound

Theorem (Reed '98)

\[ \exists \epsilon > 0, \forall G, \chi(G) \leq \lceil (1 - \epsilon) \cdot (\Delta(G) + 1) + \epsilon \cdot \omega(G) \rceil. \]

Max/sup such \( \epsilon \)?
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Max/sup such \( \epsilon \)? False for \( \epsilon = 1 \).

**Theorem (Mycielski '55)**

\( \forall k, \text{ there is a graph } H_k \text{ with } \chi(H_k) \geq k \text{ and } \omega(H_k) = 2. \)
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Theorem (Reed '98)

There exists $\epsilon > 0$, such that for all $G$, $\chi(G) \leq \lceil (1 - \epsilon) \cdot (\Delta(G) + 1) + \epsilon \cdot \omega(G) \rceil$.

Max/sup such $\epsilon$? False for $\epsilon = 1$.

Theorem (Mycielski '55)

For all $k$, there is a graph $H_k$ with $\chi(H_k) \geq k$ and $\omega(H_k) = 2$.

\[ \Delta(G_p) = (p - 1) + 2p \]
\[ \omega(G_p) = 2p \]
\[ \chi(G_p) = \lceil \frac{5p}{2} \rceil \]
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\[ (1 - \epsilon) \cdot 3p + \epsilon \cdot 2p \geq \frac{5p}{2} \]
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\[ \Rightarrow \epsilon \leq \frac{1}{2} \]

**Conjecture (Reed '98)**

\( \forall G, \chi(G) \leq \lceil \frac{1}{2} \cdot (\Delta(G) + 1) + \frac{1}{2} \cdot \omega(G) \rceil. \)
A short proof of Reed’s theorem

Proof from King and Reed ’12 (yields $\epsilon \approx \frac{1}{130,000}$).

Case 1: $\omega(G) > \frac{2}{3}\Delta(G)$.

Case 2: $\omega(G) \leq \frac{2}{3}\Delta(G)$ and $\exists v$ such that $N(v)$ is very dense.

Case 3: $\omega(G) \leq \frac{2}{3}\Delta(G)$ and every $N(v)$ is somewhat sparse.
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Case 3: $\omega(G) \leq \frac{2}{3}\Delta(G)$ and every $N(v)$ is somewhat sparse.
Case 1: Large Clique Number

We use the following theorem.

**Theorem (King 2010)**

If $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$, then there exists an independent set $I$ hitting every maximum clique in $G$. 

$\Delta(G) \leq \Delta(G \setminus I) + 1,$

$\omega(G) = \omega(G \setminus I) + 1,$

$\chi(G) \leq \chi(G \setminus I) + 1.$
Case 1: Large Clique Number

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Case 1: Large Clique Number

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If \( \omega(G) > \frac{2}{3}(\Delta(G) + 1) \), then there exists a *maximum* independent set \( I \) hitting every maximum clique in \( G \).

\[
\begin{align*}
\Delta(G) &\leq \Delta(G \setminus I) + 1, \\
\omega(G) &\leq \omega(G \setminus I) + 1, \\
\chi(G) &\leq \chi(G \setminus I) + 1.
\end{align*}
\]
Case 2: Small Clique Number and a Dense Neighborhood

$k$-critical graph: not $(k - 1)$-colorable but every proper subgraph is.

**Lemma**

∀α > 0 and 0 < ε < \(\frac{1}{6} - 2\sqrt{\alpha}\), if G is \((1 - \epsilon)\Delta(G)\)-critical and \(\omega(G) \leq \frac{2}{3}\Delta(G)\), then ∀v, \(N(v)\) has at most \((1 - \alpha)\left(\frac{\Delta}{2}\right)\) edges.

**Proposition**

Every graph H has antimatching M of size at least \(\frac{|V(H)| - \omega(H)}{2}\).
Case 2: Small Clique Number and a Dense Neighborhood

Idea: Find an antimatching $M$ in $H = G[N(v)]$.
If $G[M]$ is dense, then color $G - M$ by criticality and extend to $M$. 
Case 3: Sparse Neighborhoods

**Lemma**

\[ \exists \Delta_0 \text{ such that if } \Delta(G) \geq \Delta_0 \text{ and } \alpha > \left( \log \Delta \right)^3 / \Delta, \text{ and every } N(v) \text{ contains at most } (1 - \alpha) \left( \frac{\Delta}{2} \right) \text{ edges, then} \]

\[ \chi(G) \leq (1 - \frac{\alpha}{2e^6})(\Delta + 1). \]
Case 3: Sparse Neighborhoods

Lemma

$\exists \Delta_0$ such that if $\Delta(G) \geq \Delta_0$ and $\alpha > (\log \Delta)^3/\Delta$, and every $N(v)$ contains at most $(1 - \alpha)(\Delta/2)$ edges, then

$$\chi(G) \leq (1 - \frac{\alpha}{2e^6})(\Delta + 1).$$

Wasteful Coloring Procedure:

- Color each vertex $v$ with a random color.
- Uncolor any vertex $v$ receiving the same color as a neighbor.
- Complete the coloring by using Greedy.

Idea: Every vertex $v$ sees many repeated colors in $N(v)$. 
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\[ \geq \mathbb{E}(P_v) - \mathbb{E}(T_v) \text{ saved colors around } v. \]
Case 3: Sparse Neighborhoods

Lemma

There exists $\Delta_0$ such that if $\Delta(G) \geq \Delta_0$ and $\alpha > (\log \Delta)^3/\Delta$, and every $N(v)$ contains at most $(1 - \alpha)(\Delta/2)$ edges, then

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Wasteful Coloring Procedure:

- Regularize the graph
- Color each vertex $v$ with a random color.
- Uncolor any vertex $v$ receiving the same color as a neighbor.
- Complete the coloring by using Greedy.

Idea: Every vertex $v$ sees many repeated colors in $N(v)$.

$$\geq \mathbb{E}(P_v) - \mathbb{E}(T_v)$$ saved colors around $v$. 

M. Bonamy, T. Perrett, L. Postle
Randomly coloring sparse graphs
Let $G'$ be the graph induced in $G$ by uncolored vertices.

- "Every $N_{G'}(v)$ contains at most $(1 - \alpha)(\Delta(G'))$ edges"?
Why not iterate?

Let $G'$ be the graph induced in $G$ by uncolored vertices.

- "Every $N_{G'}(v)$ contains at most $(1 - \alpha)(\Delta(G'))$ edges"?
- List coloring?
Correspondence Coloring

Let $L$ be a $k$-list-assignment for $G$.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Note that we could as well assume that $L(v) = [k]$ for all $v$ by permuting the matchings as necessary.
Correspondence Coloring

Let $L$ be a $k$-list-assignment for $G$.

\begin{center}
\begin{tikzpicture}
\node (v1) at (0,0) {1};
\node (v2) at (1,0) {2};
\node (v3) at (2,0) {3};
\node (v4) at (3,0) {4};
\node (v5) at (2,-1) {2};
\node (v6) at (3,-1) {3};
\node (v7) at (4,-1) {4};
\draw (v1) -- (v2);
\draw (v2) -- (v3);
\draw (v3) -- (v4);
\draw (v5) -- (v6);
\end{tikzpicture}
\end{center}
Let $L$ be a $k$-list-assignment for $G$. 

![Diagram with edges and labels: 1-2, 2-3, 3-4, 4-3, 2-4, 1-3]
Let $L$ be a $k$-list-assignment for $G$.

\begin{figure}
\centering
\begin{tikzpicture}
\node (A) at (0,0) [shape=circle] {};
\node (B) at (1,0) [shape=circle] {};
\node (C) at (0.5,1) [shape=circle] {};
\node (D) at (1.5,1) [shape=circle] {};
\draw (A) -- (B);
\draw (C) -- (D);
\draw (A) -- (C);
\draw (B) -- (D);
\end{tikzpicture}
\end{figure}

**Definition (Dvořák, Postle '15)**

$k$-correspondence-assignment $C$: a set $(C_e : e \in E(G))$ where for $e = uv$, $C_e$ is a matching from $L(u)$ to $L(v)$.

$C$-correspondence-coloring $\phi$: an assignment $\phi(v) \in L(v)$ such that for every edge $e = uv$, $\phi(u)$ is not matched to $\phi(v)$ in $C_e$.

Note that we could as well assume that $L(v) = [k]$ for all $v$ by permuting the matchings as necessary.
### Definition

The *correspondence chromatic number*, $\chi_c(G)$, is the minimum $k$ such that for all $k$-correspondence assignments $C$, $G$ has a $C$-coloring.

$$\chi_c(G) \geq \chi_\ell(G).$$
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They can also differ, e.g. $\chi_c(C_4) = 3$. 

```
2 1
1 1
2
```

```
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1 1
2
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![Diagram of a triangle with vertices labeled 1, 1, 2, 2, 1, 1, 2, 2, and edges colored with the same numbers, illustrating a $C_4$-coloring.]
Why not iterate? (2)

Let $G'$ be the graph induced in $G$ by uncolored vertices.

- "Every $N_{G'}(v)$ contains at most $(1 - \alpha)(\Delta(G'))$ edges"?
- List coloring? ✓
Let $G'$ be the graph induced in $G$ by uncolored vertices.

- "Every $N_{G'}(v)$ contains at most $(1 - \alpha)(\Delta(G'))^2$ edges"?
- List coloring? ✓

Note that $\mathbb{E}(N_{G'}(v_1) \cap N_{G'}(v_2)) = p \cdot |N(v_1) \cap N(v_2)|$. 
Application to Reed’s conjecture

\[ \text{Saved}(G) = \Delta(G) + 1 - \chi(G), \quad \text{Gap}(G) = \Delta(G) + 1 - \omega(G) \]

**Definition**

A graph is *k-list-critical* if it is not \( k \)-choosable but all proper subgraphs are.

**Lemma**

*If \( G \) is k-list-critical where \( k = \Delta(G) - \text{Saved}(G) \), then \( \forall v, \ N(v) \) has at most \( (1 - \alpha) \binom{\Delta}{2} \) edges, where*

\[ \alpha \binom{\Delta}{2} \geq \frac{\text{Gap}(G)^2}{4} - \text{Gap}(G) \text{Saved}(G). \]
Application to Reed’s conjecture

\[ \text{Saved}(G) = \Delta(G) + 1 - \chi(G), \quad \text{Gap}(G) = \Delta(G) + 1 - \omega(G) \]

**Definition**

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**Lemma**

If \( G \) is *k*-list-critical where \( k = \Delta(G) - \text{Saved}(G) \), then \( \forall v, N(v) \) has at most \((1 - \alpha)\left(\frac{\Delta}{2}\right)\) edges, where

\[ \alpha\left(\frac{\Delta}{2}\right) \geq \frac{\text{Gap}(G)^2}{4} - \text{Gap}(G)\text{Saved}(G). \]

**Theorem**

*For \( \Delta(G) \) large enough, \( \text{Saved}(G) \geq \frac{\text{Gap}(G)}{25} \) (\( \chi(G) \leq \lceil (1 - \frac{1}{25}) \cdot (\Delta(G) + 1) + \frac{1}{25} \cdot \omega(G) \rceil \).*
Edge coloring

\[ \chi': \text{ Minimum number of colors to ensure that} \]
\[ a \neq b. \]
χ': Minimum number of colors to ensure that
\[ a \neq b. \]

χ'_s: Minimum number of colors to ensure that
\[ a \neq b \text{ and } a \neq c. \]
A subcase of vertex coloring

Vertex coloring of \textit{Squares of Line graphs}.

\[ \chi'(G) = \chi(L^2(G)). \]
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Note that $\Delta(L^2(G)) \leq 2\Delta(G)^2$. 
Upper bounds

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Conjecture (Erdos and Nesteril 1985)

$\chi(L^2(G)) \leq 1.25\Delta(G)^2$

This is tight for a blown-up 5-cycle: $\omega(L^2(G)) = 1.25\Delta(G^2)$. 
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Upper bounds

**Lemma (Molloy and Reed ’00)**

Let $H = L^2(G)$. For every $e \in E(G)$,

$$|E_H(N_H(e))| \leq (1 - \frac{1}{36}) \left( \frac{2\Delta(G)^2}{2} \right)$$

Combined with the Wasteful Coloring Procedure:

**Theorem (Molloy and Reed ’00)**

$$\chi(L^2(G)) \leq 1.9987\Delta(G)^2$$
Upper bounds

Lemma (Bruhn and Joos ’15)

Let $H = L^2(G)$. For every $e \in E(G)$,

$$|E_H(N_H(e))| \leq \frac{3}{4} \binom{2\Delta(G)^2}{2}$$

Combined with the Wasteful Coloring Procedure:

Theorem (Bruhn and Joos ’15)

$$\chi(L^2(G)) \leq 1.93\Delta(G)^2$$
Lemma (Bruhn and Joos ’15)

Let $H = L^2(G)$. For every $e \in E(G)$,

$$|E_H(N_H(e))| \leq \frac{3}{4} \left( \frac{2\Delta(G)^2}{2} \right)$$

Combined with the Wasteful Coloring Procedure:

Theorem (BPP ’15)

$$\chi(L^2(G)) \leq 1.876\Delta(G)^2$$
Upper bounds

Lemma (Bruhn and Joos ‘15)

Let $H = L^2(G)$. For every $e \in E(G)$,

$$|E_H(N_H(e))| \leq \frac{3}{4} \left( \frac{2\Delta(G)^2}{2} \right)$$

Combined with the Wasteful Coloring Procedure:

Theorem (BPP ‘15)

$$\chi(L^2(G)) \leq 1.835 \Delta(G)^2$$
Conclusion

Thank you!
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