The parameterised complexity of list problems on graphs of bounded treewidth

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Vertex colouring

- Given a graph $G = (V, E)$, $\phi : V \to \{1, \ldots, k\}$ is a *proper c-colouring* of G if, for all $uv \in E$, $\phi(u) \neq \phi(v)$.
- The *chromatic number* $\chi(G)$ of G is the smallest c such that there exists a proper c-colouring of G.

CHROMATIC NUMBER

Input: A graph $G = (V, E)$.

Question: What is $\chi(G)$?

- It is NP-complete to decide whether $\chi(G) \leq 3$.
- If G has fixed treewidth at most k, $\chi(G)$ can be computed in linear time (Arnborg and Proskurowski, 1989).

List Colouring

For graph $G(V, E)$ and a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper colouring ϕ of G such that $c(v) \in L_v$ for all $v \in V$.

LIST COLOURING

Input: A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$.

Question: Is there a proper list colouring $(G, \mathcal{L})$?

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Theorem (Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider and Thomassen, 2011)

LIST COLOURING *is W[1]-hard, parameterised by treewidth.*

List Chromatic Number

The *list chromatic number* $\mathrm{ch}(G)$ of G is the smallest integer c such that, for any assignment of lists $(L_v)_{v \in V(G)}$ to the vertices of G with $|L_v| \geq c$ for each v, there exists a proper list colouring of $(G, \mathcal{L})$.

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Theorem (Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider and Thomassen, 2011)

The List Chromatic Number *problem, parameterised by the treewidth bound k, is fixed-parameter tractable, and solvable in linear time for any fixed k.*

Edge Colouring

- Given a graph $G = (V, E)$, a *proper edge colouring* of G is an assignment of colours to the edges of G such that no two incident edges receive the same colour.
- The *edge chromatic number* $\chi'(G)$ of G is the smallest integer c such that there exists a proper edge colouring of G using c colours.
- It is NP-hard to determine whether $\chi'(G) \leq 3$ for cubic graphs (Holyer, 1981).
- $\chi'(G)$ can be computed in linear time on graphs of bounded treewidth (Zhou, Nakano and Nishizeki, 2005).

Line Graphs

- Given a graph $G = (V, E)$, the line graph $L(G)$ of G is $(E, \{ef : e, f \in E \text{ and } e, f \text{ incident in } G\})$.
- A proper edge colouring of G corresponds to a proper vertex colouring of $L(G)$.
- If G has treewidth k and maximum degree at most Δ, then $L(G)$ has treewidth at most $(k + 1)\Delta$.

List Edge Colouring

For graph $G(V, E)$ and a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper list colouring ϕ of G such that $c(v) \in L_v$ for all $v \in V$.

List Edge Colouring

Input: A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_e)_{e \in E(G)}$.

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Question: Is there a proper list edge colouring $(G, \mathcal{L})$?

Theorem (Zhou, Matsuo, Nishizeki, 2005)

LIST EDGE COLOURING *is NP-hard on series-parallel graphs.*

Theorem (Marx, 2005)

LIST EDGE COLOURING *is NP-hard on outerplanar graphs.*

Total Colouring

- Given a graph $G = (V, E)$, a *proper total colouring* of G is an assignment of colours to the vertices and edges of G such that
  - no two adjacent vertices receive the same colour
  - no two incident edges receive the same colour
  - no edge receives the same colour as either of its endpoints.
- The *total chromatic number* $\chi_T(G)$ of G is the smallest integer c such that there exists a proper total colouring of G using c colours.
- It is NP-hard to determine $\chi_T(G)$ for regular bipartite graphs (McDiarmid and Sánchez-Arroyo, 1994).
- $\chi_T(G)$ can be computed in linear time on graphs of bounded treewidth (Isobe, Zhou and Nishizeki, 2007).

Total Graphs

- Given a graph $G = (V, E)$, the total graph $T(G)$ of G has vertex set $V \cup E$ and edge set

$$\begin{aligned} &E \cup \{ef : e, f \in E \text{ and } e, f \text{ incident in } G\} \\ &\cup \{ve : v \in V, e \in E, e \text{ incident with } v\}). \end{aligned}$$

- A proper total colouring of G corresponds to a proper vertex colouring of $T(G)$.
- If G has treewidth k and maximum degree at most Δ, then $T(G)$ has treewidth at most $(k+1)(\Delta+1)$.

List Total Colouring

For graph $G(V,E)$ and a collection of colour lists $\mathcal{L} = (L_x)_{x \in V \cup E}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper total colouring ϕ of G such that $c(x) \in L_x$ for all $x \in V \cup E$.

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Input: A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_x)_{x \in V \cup E}$.

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Question: Is there a proper list total colouring $(G, \mathcal{L})$?

Theorem (Zhou, Matsuo, Nishizeki, 2005)

List Total Colouring *is NP-hard on series-parallel graphs.*

List Edge and Total Chromatic numbers

- The *list edge chromatic number* $\mathrm{ch}'(G)$ of G is the smallest integer c such that, for any assignment of lists $(L_e)_{e\in E(G)}$ to the edges of G with $|L_e| \geq c$ for each e, there exists a proper list edge colouring of $(G, \mathcal{L})$.

$$\Delta(G) \leq \chi'(G) \leq \mathrm{ch}'(G) \leq 2\Delta(G) - 1$$

- The *list total chromatic number* ch_T of G is the smallest integer c such that, for any assignment of lists $(L_e)_{e\in E(G)}$ to the edges of G with $|L_e| \geq c$ for each e, there exists a proper list edge colouring of $(G, \mathcal{L})$.

$$\Delta(G) + 1 \leq \chi_T(G) \leq \mathrm{ch}_T(G) \leq 2\Delta(G) + 1$$

Parameterised complexity of colouring problems

	General problem	Parameter treewidth	List version, parameter treewidth	List Chromatic number, parameter treewidth
Vertex colouring	NP-c	FPT	W[1]-hard	FPT
Edge colouring				
Total colouring				

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Total colouring	NP-c	FPT	W[1]-hard	

Theorem

List Edge Chromatic Number and List Total Chromatic Number are fixed parameter tractable, parameterised by the treewidth bound k, and are solvable in linear time for any fixed k.

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List Edge Chromatic Number *and* List Total Chromatic Number *are fixed parameter tractable, parameterised by the treewidth bound k, and are solvable in linear time for any fixed k.*

- If G has treewidth k and bounded maximum degree, then $L(G)$ and $T(G)$ both have bounded treewidth.
- So it is possible in this case to compute the $\mathrm{ch}(L(G))$ or $\mathrm{ch}(T(G))$ in linear time.

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LIST EDGE CHROMATIC NUMBER *and* LIST TOTAL CHROMATIC NUMBER *are fixed parameter tractable, parameterised by the treewidth bound k, and are solvable in linear time for any fixed k.*

- If G has treewidth k and bounded maximum degree, then $L(G)$ and $T(G)$ both have bounded treewidth.
- So it is possible in this case to compute the $\mathrm{ch}(L(G))$ or $\mathrm{ch}(T(G))$ in linear time.
- It remains to consider the case that $\Delta(G)$ is very large compared with the treewidth.

Bounded treewidth and large maximum degree

Theorem

Let G be a graph with treewidth at most k and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}'(G) = \Delta(G)$.

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- So we have
$$\Delta(G) = \mathrm{ch}'(G) \geq \chi'(G) \geq \Delta(G),$$
and in particular $\mathrm{ch}'(G) = \chi'(G)$.
- This is a special case of the List (Edge) Colouring Conjecture, which asserts that
$$\mathrm{ch}'(G) = \chi'(G)$$
for every graph G.

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- Sufficient to prove that, if G has treewidth at most k, then $\mathrm{ch}'(G) \leq \max\{\Delta(G), (k+2)2^{k+2}\}$.

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- Let $(G, \mathcal{L} = \{L_e : e \in E\})$ be an edge-minimal counterexample. Assume $|L_e| = \Delta_0 = \max\{\Delta(G), (k+2)2^{k+2}\}$ for each e.

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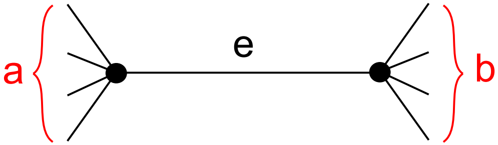
- We may assume any proper subgraph G' of G has $\mathrm{ch}'(G') \leq \Delta_0$.

Bounded treewidth and large maximum degree



$$a + b < \Delta_0$$

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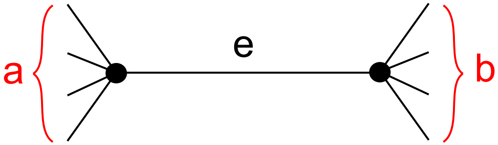

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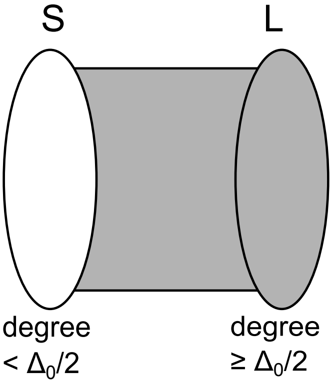
- We may assume every edge is incident with at least Δ_0 others.

Bounded treewidth and large maximum degree

- Every edge is incident with at least one vertex in L.



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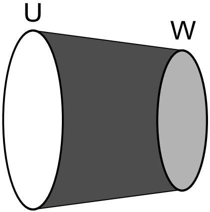
We want

- $\Gamma(u) = W \ \forall u \in U$
- $|U| \geq |W|$
- U independent



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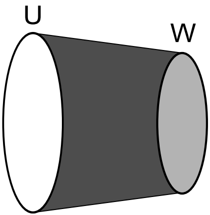

Theorem (Galvin,1995)

If G is a bipartite graph then $\mathrm{ch}'(G) = \Delta(G)$.

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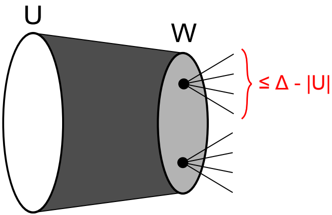

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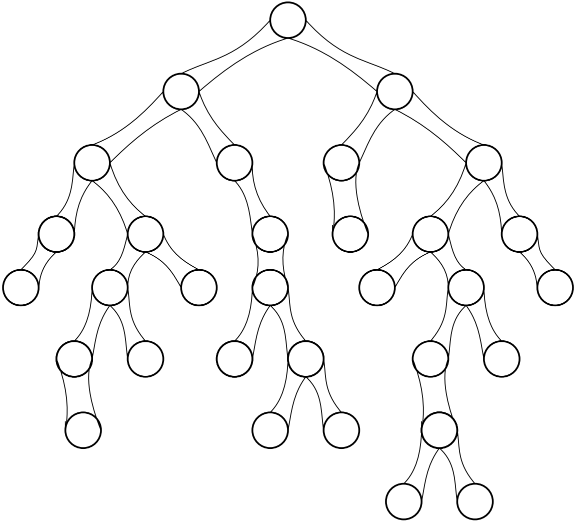

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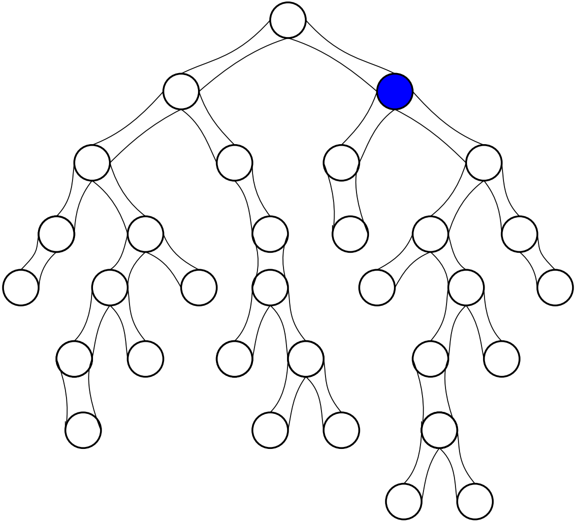

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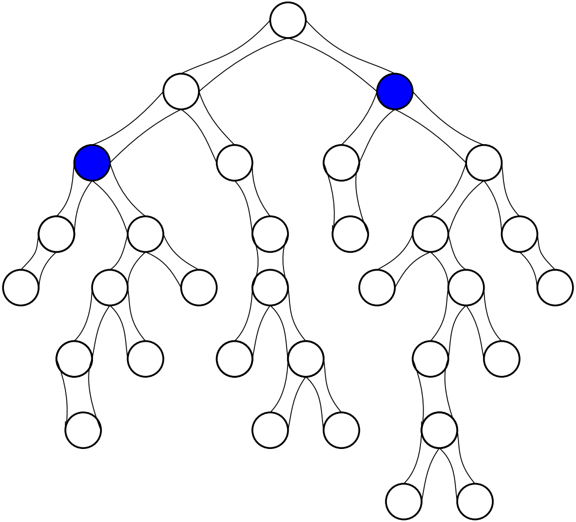

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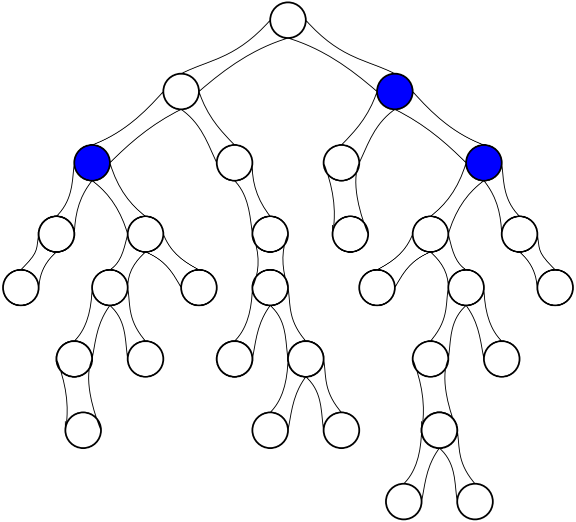

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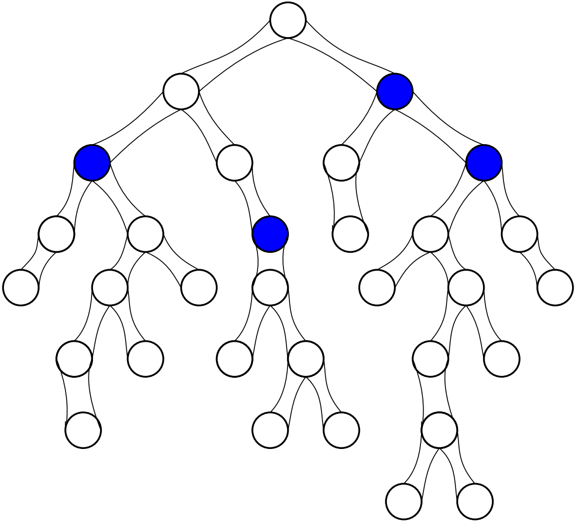

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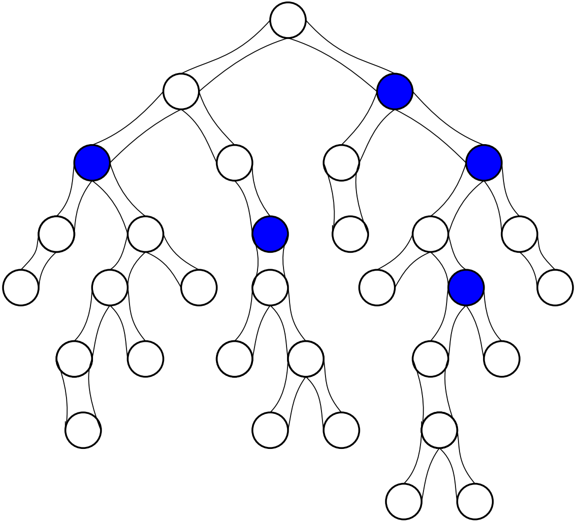

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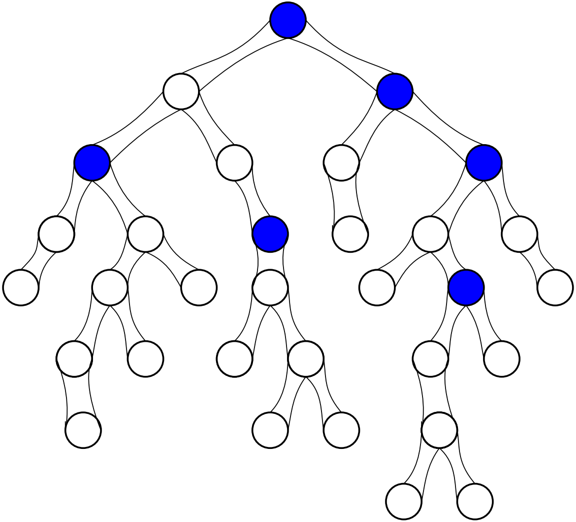

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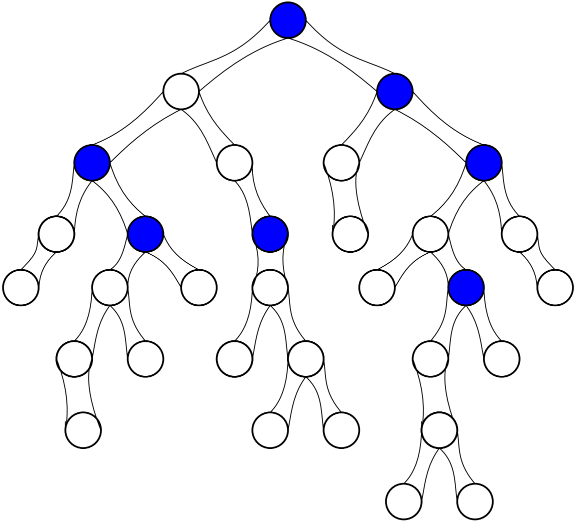

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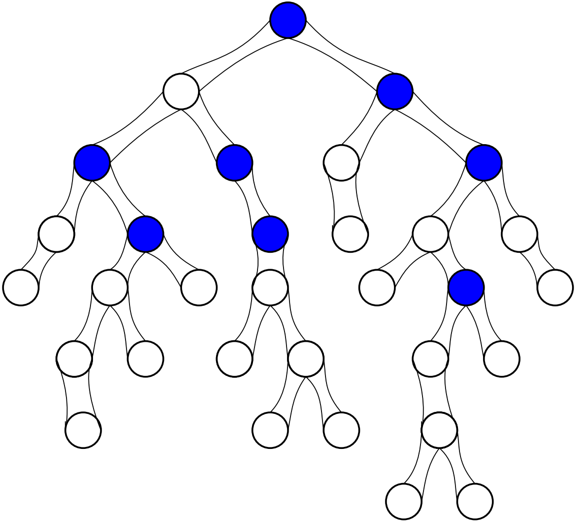

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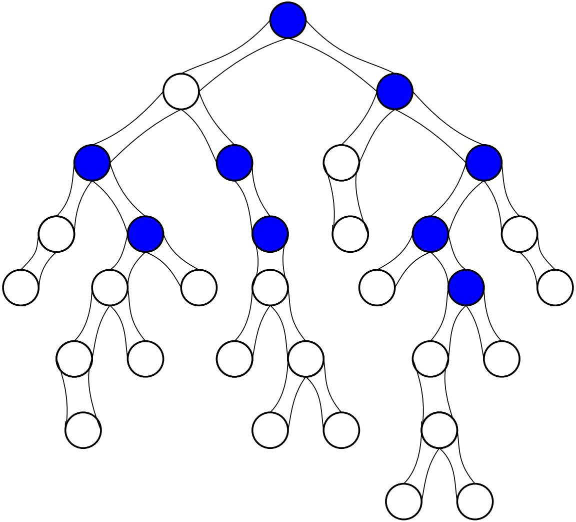

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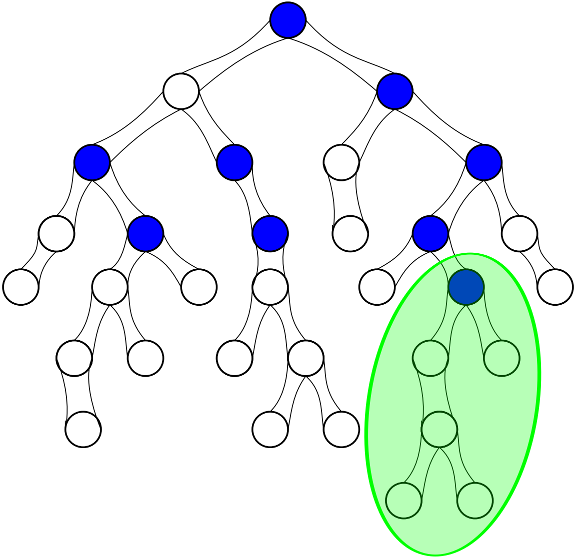

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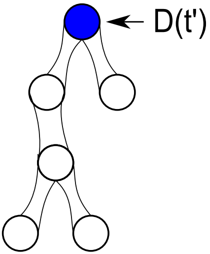

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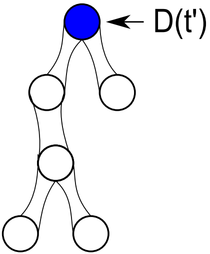

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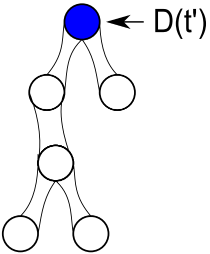
Bounded treewidth and large maximum degree



- At most $k + 1$ vertices from L
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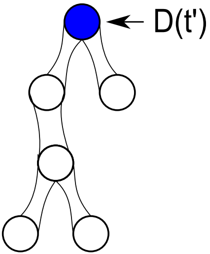

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- At most 2^{k+1} different neighbourhoods for these vertices

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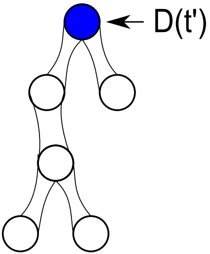

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- At least $\Delta_0/2 - k$ vertices not in $D(t')$, all from S
- At most 2^{k+1} different neighbourhoods for these vertices
- So there exists a subset U with $|U| \geq k+1$ and every vertex in U having the same neighbourhood W ($|W| \leq k+1$)

Total Colouring

Theorem

Let G be a graph with treewidth at most k and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}_T(G) = \Delta(G) + 1$.

Summary of Algorithms

Suppose we are given G together with a tree decomposition $(T, \mathcal{D})$ of width k.

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1. Determine whether $\Delta(G) \geq (k+2)2^{k+2}$.
2. If $\Delta(G) \geq (k+2)2^{k+2}$ we know $\mathrm{ch}'(G) = \Delta(G)$ and $\mathrm{ch}_T(G) = \Delta(G) + 1$.

Summary of Algorithms

Suppose we are given G together with a tree decomposition $(T, \mathcal{D})$ of width k.

1. Determine whether $\Delta(G) \geq (k+2)2^{k+2}$.
2. If $\Delta(G) \geq (k+2)2^{k+2}$ we know $\mathrm{ch}'(G) = \Delta(G)$ and $\mathrm{ch}_T(G) = \Delta(G) + 1$.
3. Otherwise, $L(G)$ and $T(G)$ have bounded treewidth.
   - Compute a bounded width tree decomposition for $L(G)$ or $T(G)$.
   - Solve List Chromatic Number for $L(G)$ or $T(G)$ in linear time.

Parameterised complexity of colouring problems - again!

	General problem	Parameter treewidth	List version, parameter treewidth	List Chromatic number, parameter treewidth
Vertex colouring	NP-c	FPT	W[1]-hard	FPT
Edge colouring	NP-c	FPT	W[1]-hard	FPT
Total colouring	NP-c	FPT	W[1]-hard	FPT

List Hamilton Path

- Determining whether a graph has any Hamilton path is NP-hard, even when restricted to
  - planar, cubic, 3-connected graphs (Garey, Johnson and Tarjan, 1976)
  - bipartite graphs (Krishnamoorthy, 1975).
- Hamilton Path can be solved in linear time on graphs of bounded treewidth (Arnborg and Proskurowski, 1989).

List Hamilton Path

Input: A graph $G = (V, E)$ and a set of lists $\mathcal{L} = \{L_v \subseteq \{1, \ldots, |V|\} : v \in V\}$ of permitted positions.

Question: Does there exist a path $P = P[1] \ldots P[|G|]$ in G such that, for $1 \leq i \leq |G|$, we have $i \in L_{P[i]}$.

List Hamilton Path

Theorem

List Hamilton Path, parameterised by pathwidth, is W[1]-hard.

THANK YOU