Flood-filling Games on Graphs

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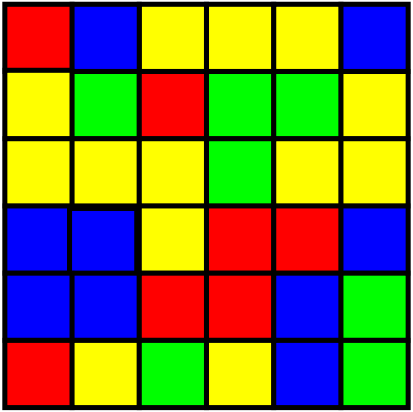
The original Flood-It game



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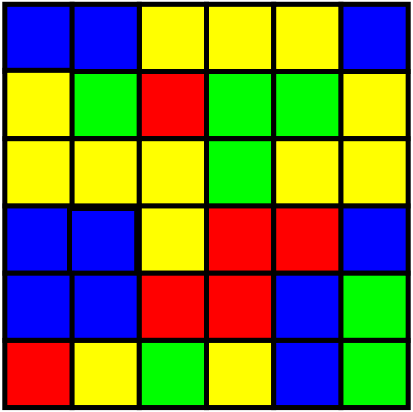

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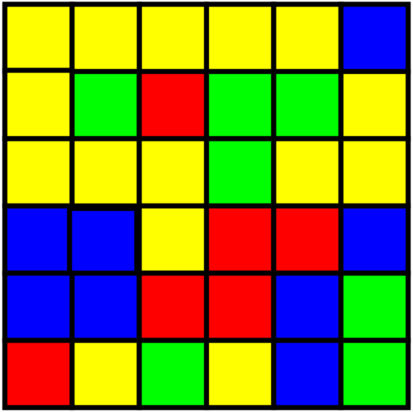

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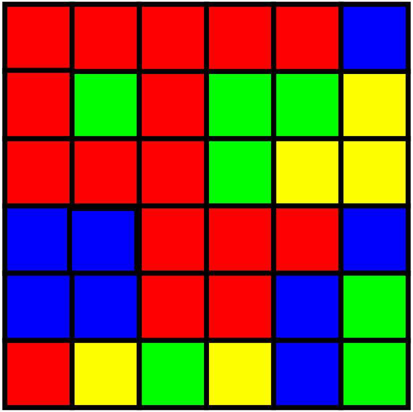

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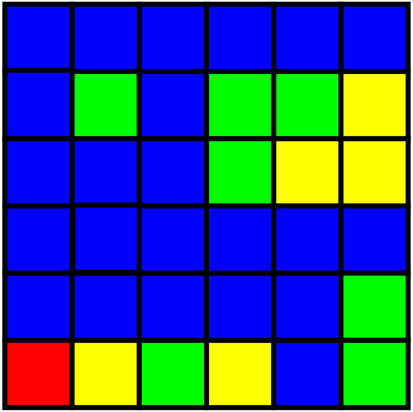

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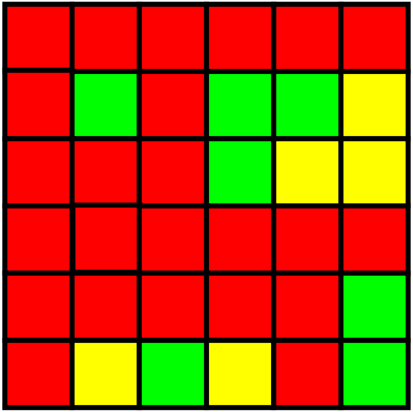

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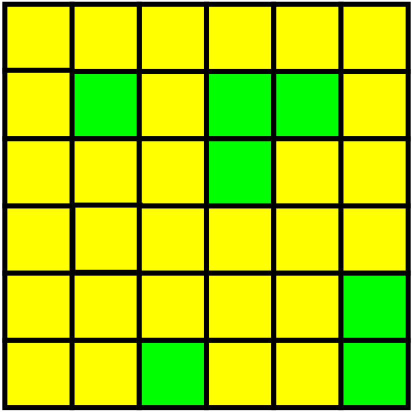

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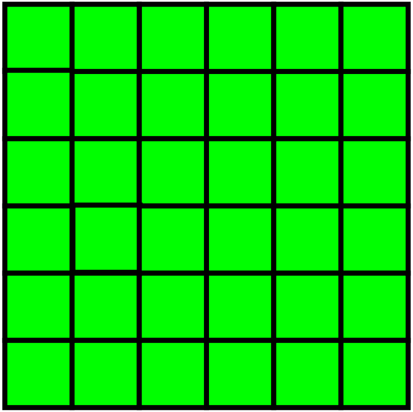

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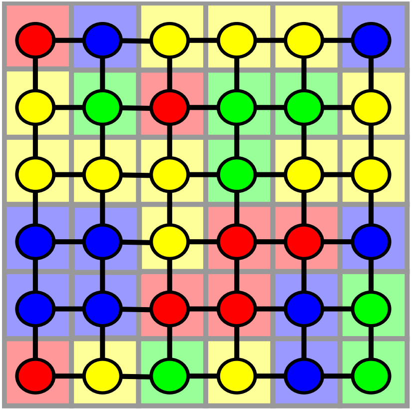

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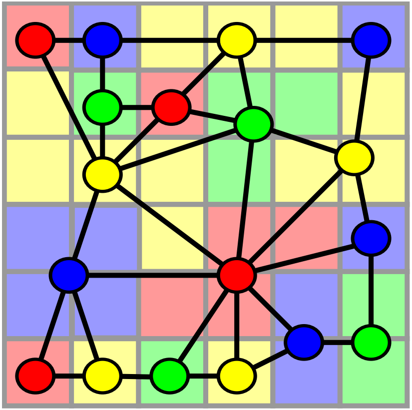

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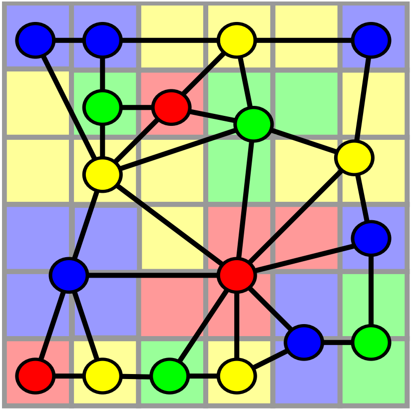

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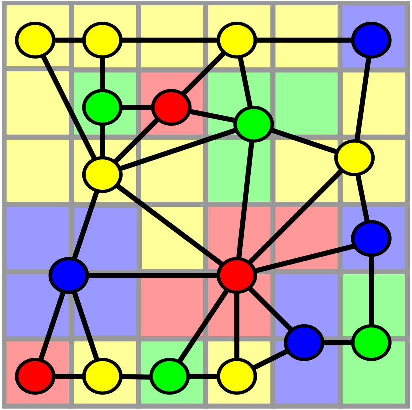

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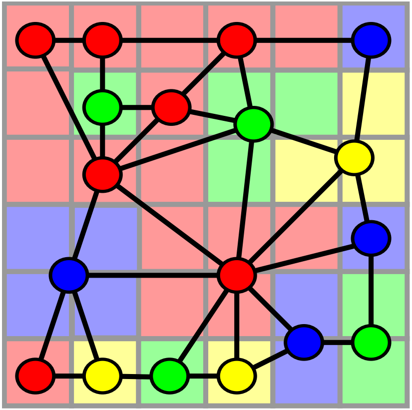

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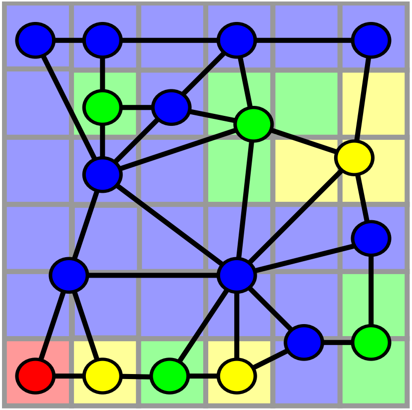

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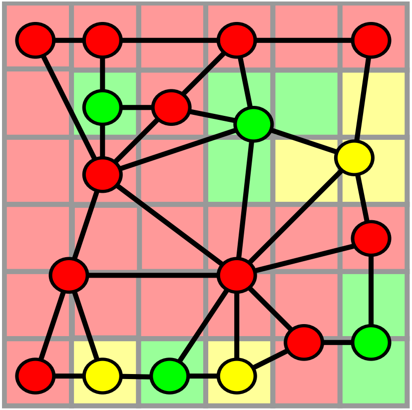

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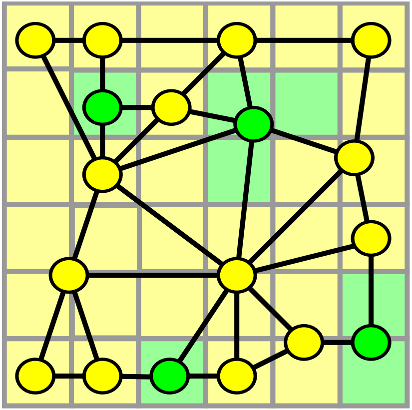

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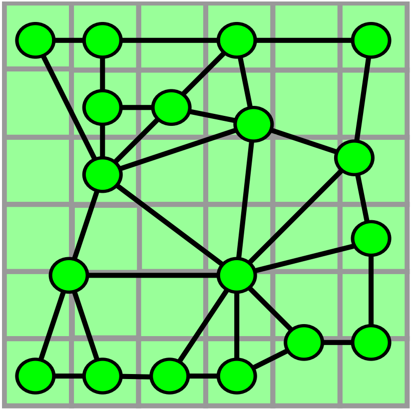

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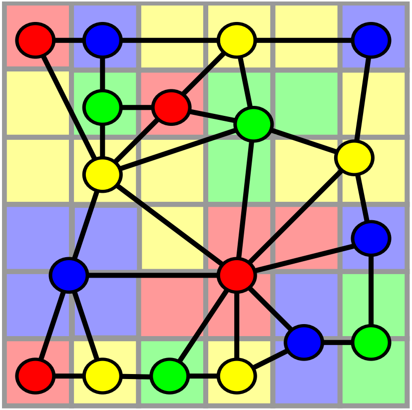

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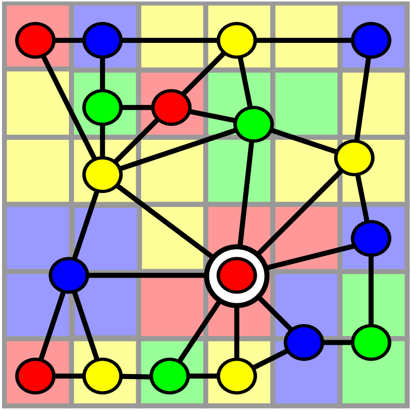

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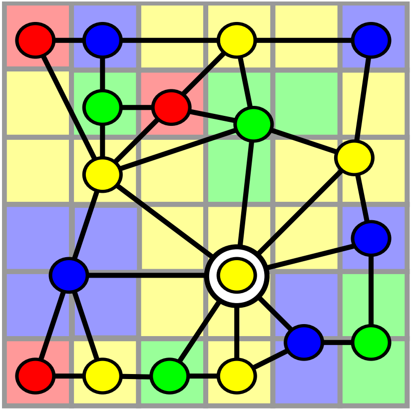

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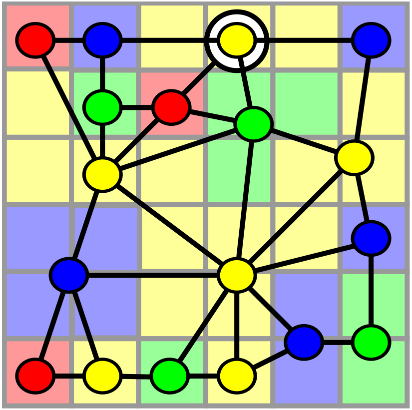

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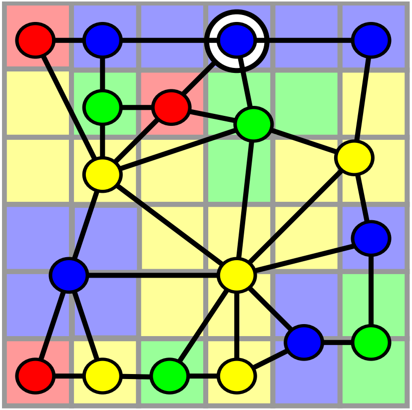

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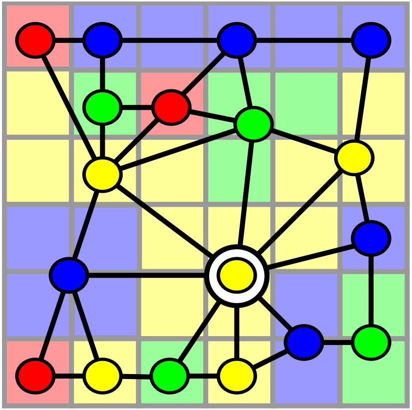

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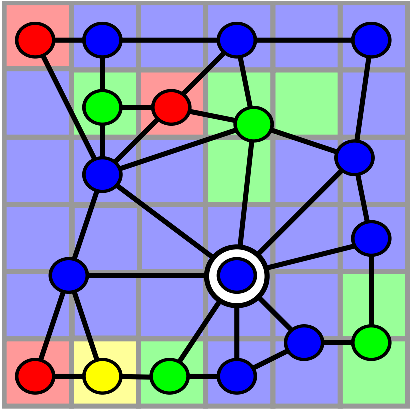

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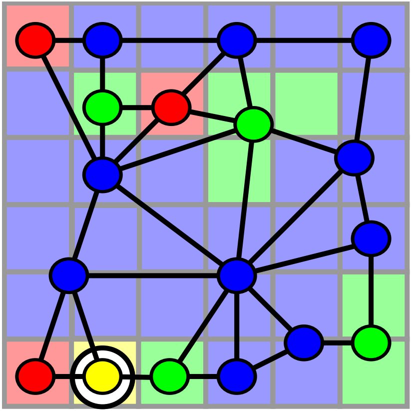

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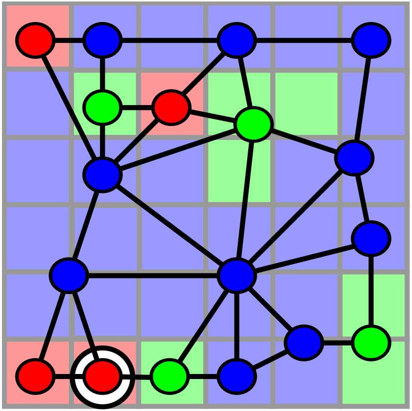

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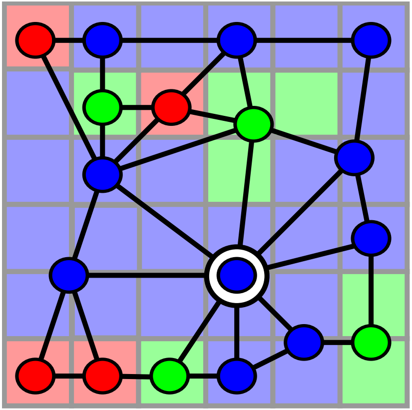

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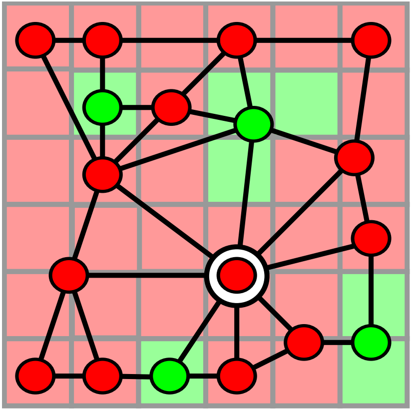

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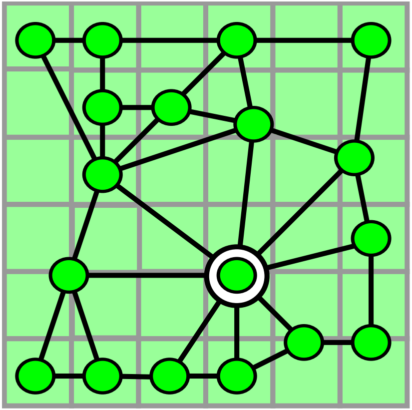

Outline

- Problems considered

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- Background

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- Open Problems

Problems considered: fixed version

Definition

Given a coloured connected graph G and a vertex $v \in V$, we define $m^{(v)}(G)$ to be the minimum number of moves required to make G monochromatic, if we always play at the vertex v.

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Given a coloured connected graph G and a vertex $v \in V(G)$, what is $m^{(v)}(G)$? The number of colours may be unbounded.

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FIXED FLOOD IT
: Given a coloured connected graph G and a vertex $v \in V(G)$, what is $m^{(v)}(G)$? The number of colours may be unbounded.

c-FIXED FLOOD IT
: The same as FIXED FLOOD IT, except that only colours from some fixed set of size c are used.

Problems considered: free version

Definition

Given a coloured connected graph, we define $m(G)$ to be the minimum number of moves required to make G monochromatic if, at each move, we can choose to play at any vertex in G.

FREE FLOOD IT

Given a coloured connected graph G, what is $m(G)$? The number of colours may be unbounded.

c-FREE FLOOD IT

The same as FREE FLOOD IT, except that only colours from some fixed set of size c are used.

Background

Theorem (Arthur, Clifford, Jalsenius, Montanaro, Sach 2010)

3-FIXED FLOOD IT and 3-FREE FLOOD IT are NP-hard on $n \times n$ grids (and the decision versions are NP-complete).

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- Both proved by means of a reduction from SHORTEST COMMON SUPERSEQUENCE (SCS).

Connecting pairs of vertices

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Let G be a connected coloured graph, and let $u, v \in V(G)$. Then $m(u, v)$ is equal to the minimum, taken over all u-v paths P, of the number of moves required to flood the path P.

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Theorem (M., Scott 2011)
Let $G = (V, E)$ be a connected graph, coloured with c colours. Then we can compute the number of moves required to link every pair $(u, v) \in V^{(2)}$ in time $O(|V|^3|E||C|^2)$.

Applications: FREE FLOOD IT on paths

Corollary

For any path P, FREE FLOOD IT *can be solved in time $O(|P|^6)$, and c-*FREE FLOOD IT *can be solved in time $O(|P|^4)$.*

Applications: approximating c-Free Flood It on $k \times n$ boards

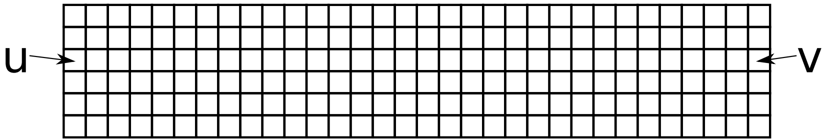
Corollary

For any fixed k, we can compute a constant additive approximation to c-Free Flood It, restricted to $k \times n$ boards, in time $O(n^4)$.

Applications: approximating c-Free Flood It on $k \times n$ boards

Let B be the coloured graph corresponding to a $k \times n$ board. Then

$$m(u,v) \leq m(B) \leq m(u,v) + c(k-1).$$



Moves:

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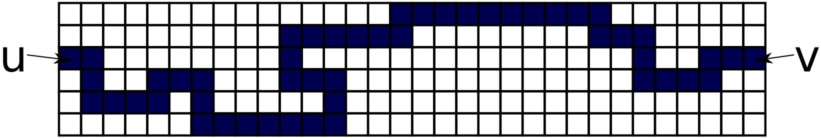

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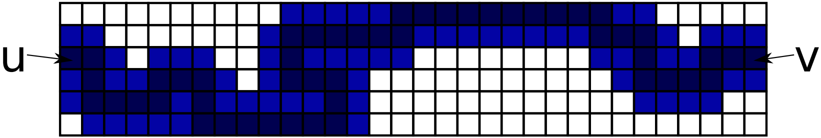

Moves: $m(u, v) + c$

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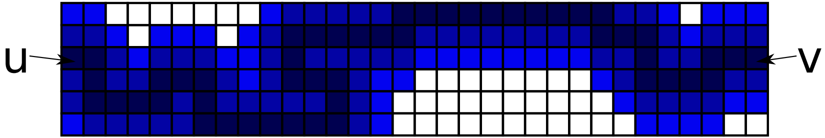

Moves: $m(u, v) + 2c$

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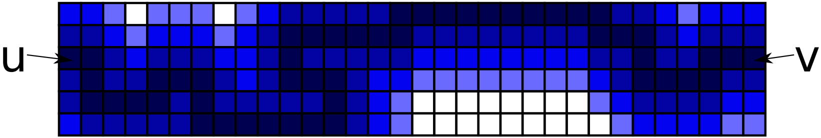

Moves: $m(u,v) + 3c$

Applications: approximating c-Free Flood It on $k \times n$ boards

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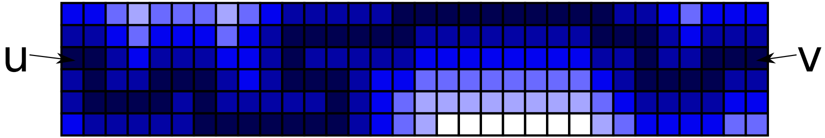

Moves: $m(u, v) + 4c$

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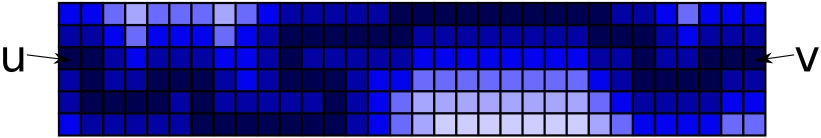

Moves: $m(u, v) + 5c$

Solving the problems exactly for $k \times n$ boards

	$1 \times n$	$2 \times n$	$3 \times n$	$n \times n$
$c = 2$				
$c = 3$				NP-h
$c = 4$				NP-h
c unbounded				NP-h

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	$1 \times n$	$2 \times n$	$3 \times n$	$n \times n$
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$3 \times n$ boards

Theorem (M.,Scott 2011)

*4-*Fixed Flood It *and 4-*Free Flood It *NP-hard on* $3 \times n$ *boards.*

Proved by a reduction from SCS.

$3 \times n$ boards

	$1 \times n$	$2 \times n$	$3 \times n$	$n \times n$
$c = 2$	P	P	P	P
$c = 3$	P		?	NP-h
$c = 4$	P		NP-h	NP-h
c unbounded	P		NP-h	NP-h

$2 \times n$ boards

	Fixed	Free
c fixed	P	
c unbounded	P	

Theorem (Clifford, Jalsenius, Montanaro and Sach 2010)

Fixed Flood It *can be solved in time $O(n)$ on $2 \times n$ boards.*

c-Free Flood It on $2 \times n$ boards

	Fixed	Free
c fixed	P	P
c unbounded	P	

Theorem (M.,Scott 2011)

When restricted to $2 \times n$ boards, c-Free Flood It is fixed parameter tractable, with parameter c.

c-Free Flood It on $2 \times n$ boards

	Fixed	Free
c fixed	P	P
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When restricted to $2 \times n$ boards, c-Free Flood It is fixed parameter tractable, with parameter c.

- Dynamic programming
- Split board into sections and consider the number of moves required to create a monochromatic path through each section, subject to certain further conditions.

Free Flood It on $2 \times n$ boards

	Fixed	Free
c fixed	P	P
c unbounded	P	NP-h

Theorem (M., Scott, 2011)

Free Flood It *remains NP-hard when restricted to* $2 \times n$ *boards.*

- Reduction from Vertex Cover.

Open Problems

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- Complexity of detemining the number of moves required to link a given set of $k \geq 3$ points, for fixed k.

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- Extremal problems:
  - What is the worst possible colouring of a $k \times n$ board with c colours?
  - Given a graph G,
    1. what is the worst possible colouring with c colours?
    2. what is the best possible proper colouring?

QUESTIONS?