Graph modification problems in epidemiology

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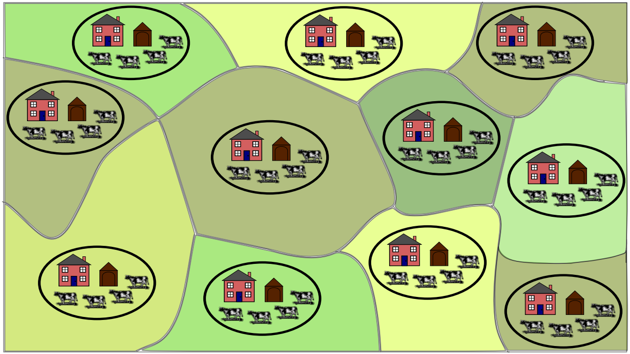
Joint work with Jessica Enright (University of Stirling)

The animal contact network



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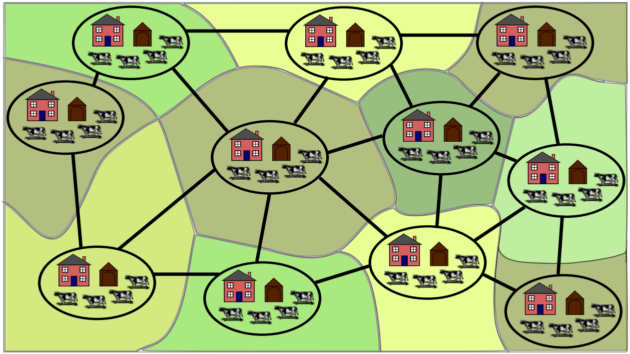

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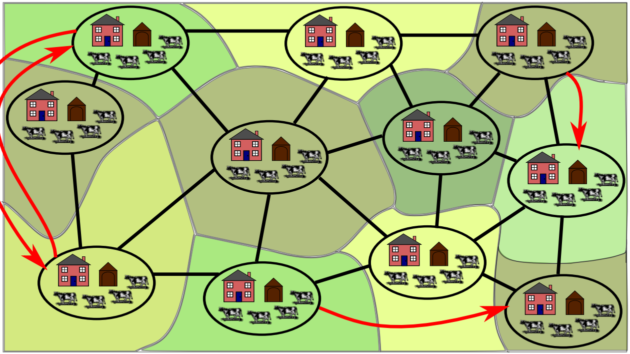

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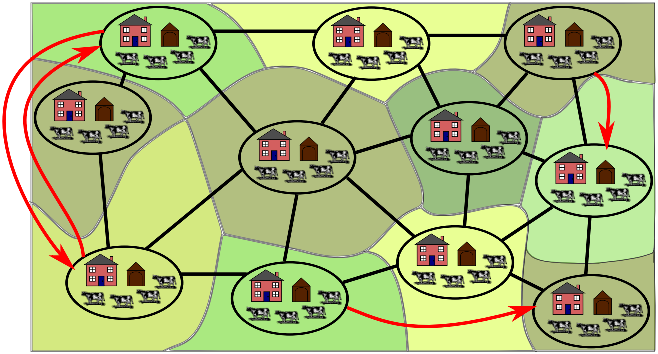

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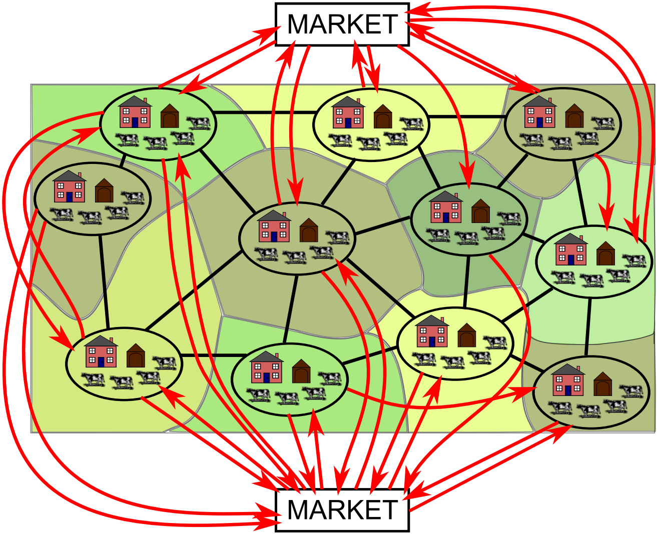

Modifying the network

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- Testing or quarantine for animals on a particular trade route

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Cost of modifications The cost of deleting individual vertices/edges may vary; this can be captured with a weight function on vertices and/or edges.

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- consider the total number of animals in e.g. a connected component, rather than just the number of animal holdings
- place more or less strict restrictions on individual animal holdings

Bounding the component size by deleting edges

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$\mathcal{C}_h$-Free Edge Deletion
Input: A Graph $G = (V, E)$ and an integer k.
Question: Does there exist $E' \subseteq E$ with $|E'| = k$ such that $G \setminus E$ does not contain any $H \in \mathcal{C}_h$ as an induced subgraph?

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This problem has also been called:

- Min-Max Component Size Problem
- Minimum Worst Contamination Problem
- Component Order Edge Connectivity

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Theorem (Cai, 1996)

$\mathcal{C}_h$-Free Edge Deletion can be solved in time $O(h^{2k} \cdot n^h)$, where n is the number of vertices in the input graph.

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Theorem (Gross, Heinig, Iswara, Kazmiercaak, Luttrell, Saccoman and Suffel, 2013)

$\mathcal{C}_h$-FREE EDGE DELETION *can be solved in polynomial time when restricted to trees.*

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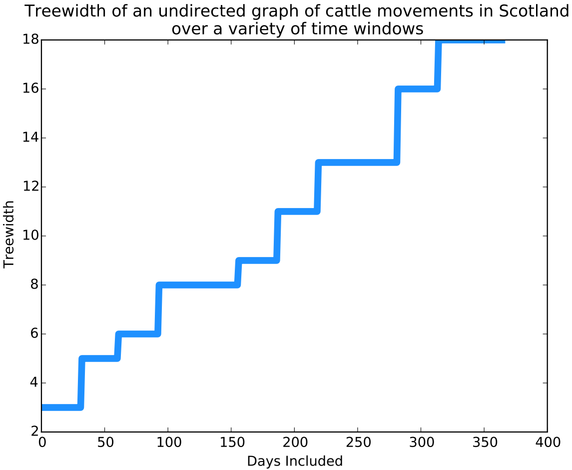

A plot of the treewidth of the largest component in an undirected version of the cattle movement graph in Scotland in 2009 over a number of different days included: all day sets start on January 1, 2009.

New results

Theorem (Enright and M., 2015+)

There exists an algorithm to solve $\mathcal{C}_h$-FREE EDGE DELETION *in time* $O((wh)^{2w}n)$ *on an input graph with* n *vertices whose treewidth is at most* w*.*

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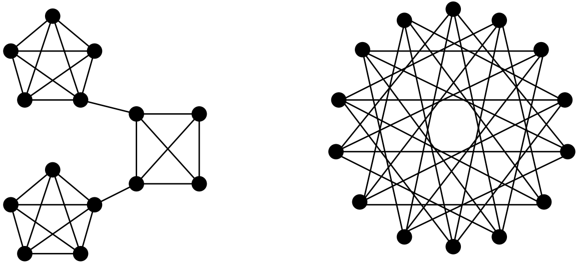
We recursively compute the minimum number of edges we delete, for each combination of:

- a partition of the vertices in the bag, indicating which are allowed to belong to the same component, and
- a function from blocks of the partition to $[h]$, indicating the maximum number of vertices allowed so far in the component containing the block in question.

New results



New results

Theorem (Enright and M., 2015+)

Let Π *be a monotone graph property defined by the set of forbidden subgraphs* $\mathcal{F}$*, where* $\max\{|F| : F \in \mathcal{F}\}$ *exists and is equal to* h*. Then* EDGE DELETION TO Π *can be solved in time* $f(h, w) \cdot n$ *on an input graph with* n *vertices whose treewidth is at most* w*, where* f *is an explicit computable function.*

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- More complicated or less costly goals

Thank you.