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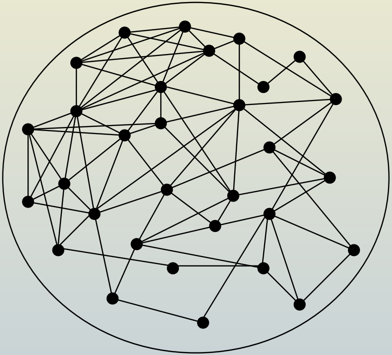
When can an efficient decision algorithm be used to find and count witnesses?

1st March 2016

Kitty Meeks

Given a graph on n vertices, we are interested in subgraphs with k vertices that have particular properties.



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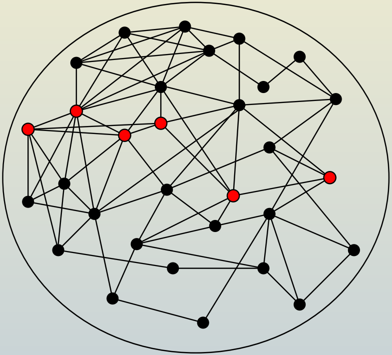

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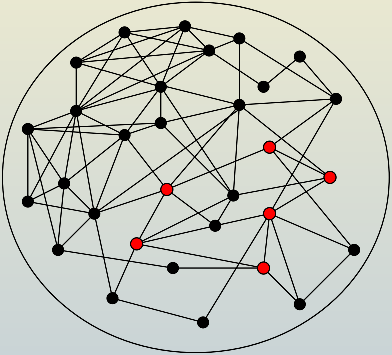

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- Paths on k vertices
- Cycles on k vertices

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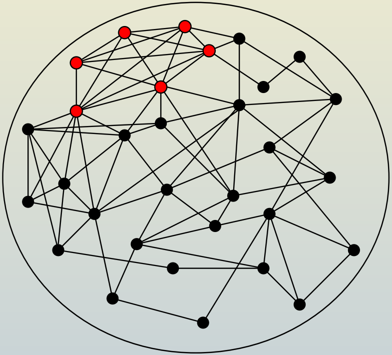

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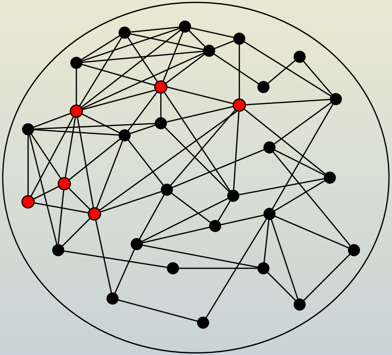

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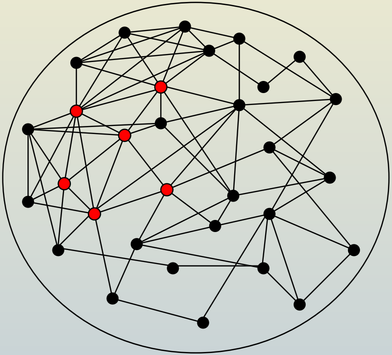

For example:

- Paths on k vertices
- Cycles on k vertices
- Cliques on k vertices
- Connected k-vertex induced subgraphs
- k-vertex induced subgraphs with an even number of edges

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Deciding, counting and enumerating

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The values of n and k

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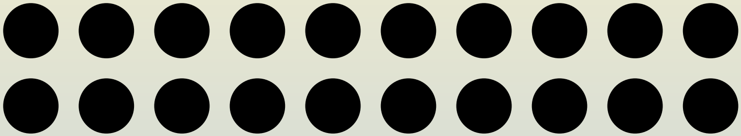
We are interested in what happens as n and k both tend to infinity, independently, with $k << n$.

- We can consider all possible k-vertex subgraphs in time $O(n^k)$.
- We would like to be able to answer questions about k-vertex subgraphs in time $f(k) \cdot n^{O(1)}$.

If we can decide, we can find a witness



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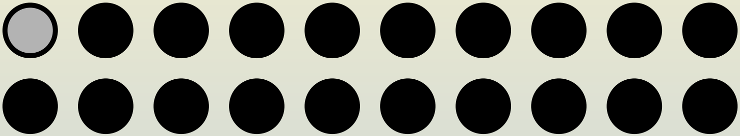

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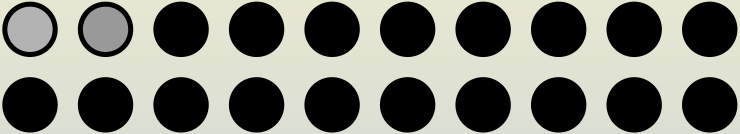

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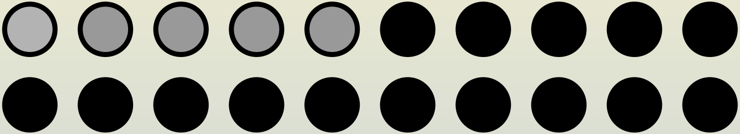

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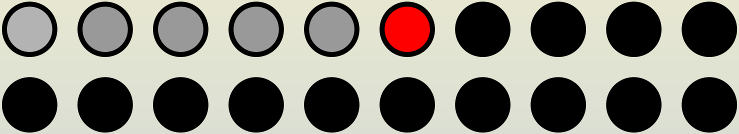

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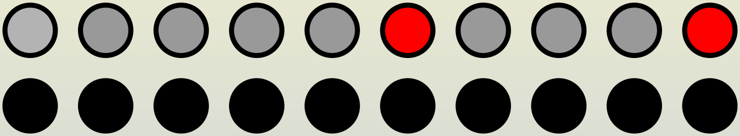

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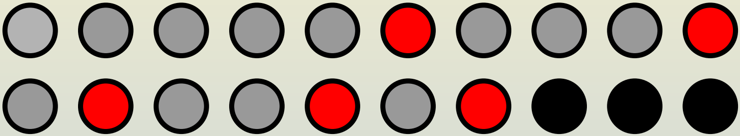

If we can decide, we can find a witness

Theorem (Björklund, Kaski and Kowalik, 2014)

There exists an algorithm that extracts a witness using at most

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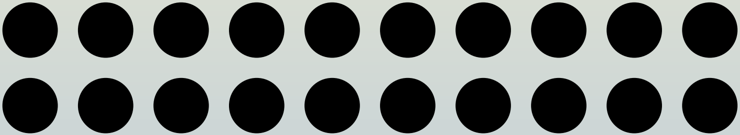

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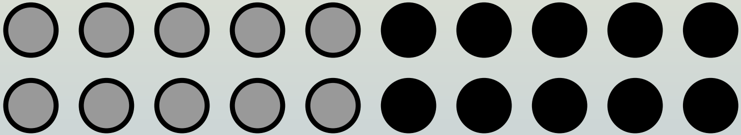

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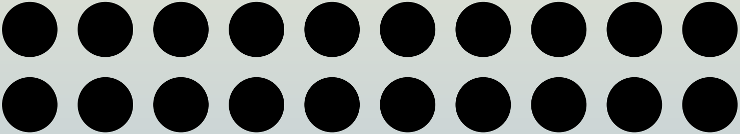

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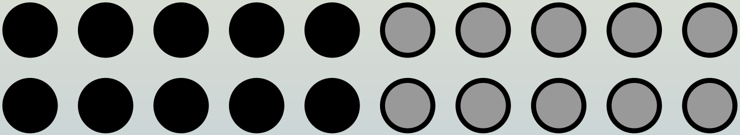

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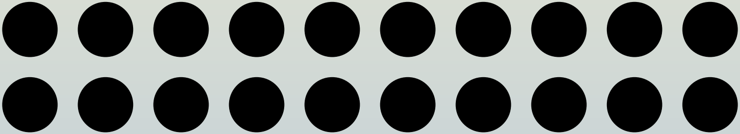

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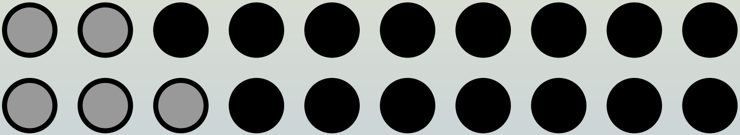

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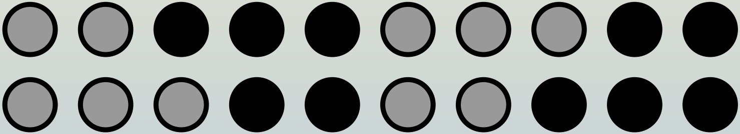

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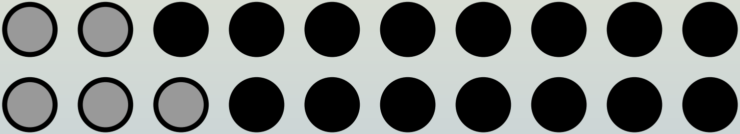

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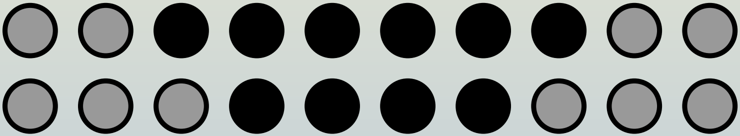

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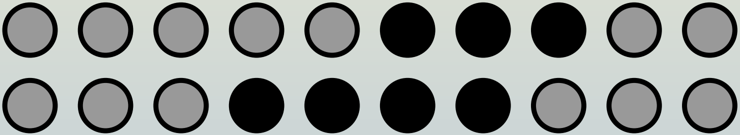

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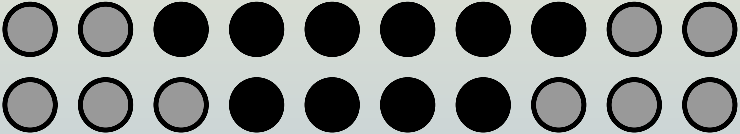

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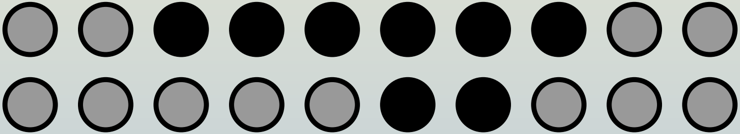

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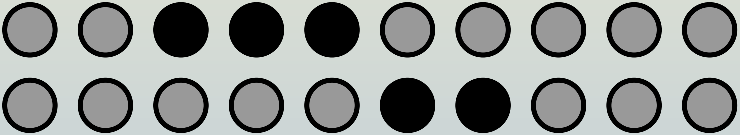

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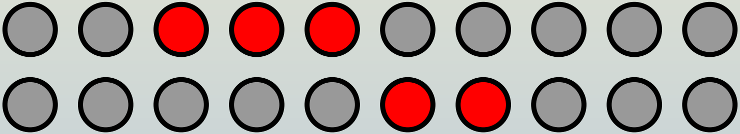

Deciding, counting and enumerating

DECISION Is there a witness?	**EXTRACTION** Identify a single witness
APPROX COUNTING Approximately how many witnesses?	**UNIFORM SAMPLING** Pick a single witness uniformly at random
EXACT COUNTING Exactly how many witnesses?	**ENUMERATION** List all witnesses

If we can count approximately, we can decide

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... at least with high probability.

An FPRAS for a counting problem Π is a randomised approximation scheme that takes an instance I of Π (with $|I| = n$), and numbers $\epsilon > 0$ and $0 < \delta < 1$, and in time $poly(n, 1/\epsilon, \log(1/\delta))$ outputs a rational number z such that

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Set $\epsilon < \frac{1}{2}$, and we will distinguish between 0 and at least 1 with probability at least $1-\delta$.

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Uniform sampling is harder than finding a witness

GENCYCLE
Input: A directed graph G.
Output: A cycle selected uniformly, at random, from the set of all directed cycles of G.

Theorem (Jerrum, Valiant, Vazirani, 1986)

Suppose there exists a polynomial time bounded Probabilistic Turing Machine which solves the problem GENCYCLE. *Then NP = RP.*

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Self-reducibility

A relation $R \subseteq \Sigma^* \times \Sigma^*$ is *self-reducible* if and only if:

- there exists a polynomial time computable function $g \in \Sigma^* \to \mathbb{N}$ such that $xRy \implies |y| = g(x)$;
- there exist polynomial time computable functions $\psi \in \Sigma^* \times \Sigma^* \to \Sigma^*$ and $\sigma \in \Sigma^* \to \mathbb{N}$ satisfying:
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  - $g(x) > 0 \implies \sigma(x) > 0 \quad \forall x \in \Sigma^*$
  - $|\psi(x, w)| \leq |x| \quad \forall x, w \in \Sigma^*$,

  and such that, for all $x \in \Sigma^*$, $y = y_1 \ldots y_n \in \Sigma^*$,

$$\langle x, y_1 \ldots y_n \rangle \in R \iff \langle \psi(x, y_1 \ldots y_{\sigma(x)}), y_{\sigma(x)+1} \ldots y_n \rangle \in R.$$

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Theorem (Jerrum, Valiant, Vazirani, 1986)

For self-reducible problems, approximate counting and almost-uniform sampling are polynomial-time inter-reducible.

Self-reducibility and equivalent notions

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2. We can decide whether there is at least one **multicoloured** witness in time $f(k) \cdot n^{O(1)}$.
3. We can decide whether there is at least one witness that is an **extension** of a given partial solution in time $f(k) \cdot n^{O(1)}$.

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A special case

Theorem (Arvind and Raman (2002); Jerrum and M. (2015); M. (2016))

If there is an efficient ($f(k) \cdot n^{O(1)}$) algorithm for the decision version of a self-reducible subgraph problem, and adding edges cannot decrease the number of witnesses, then we can count witnesses approximately.

Many witnesses

Proposition

Suppose that, for each k and any graph G on n vertices, the number of k-vertex witnesses is either

1. *zero, or*
2. *at least*

$$\frac{1}{g(k)p(n)}\binom{n}{k}.$$

Then there is an FPTRAS to count witnesses.

Theorem

Suppose that we have an $f(k) \cdot n^{O(1)}$ decision algorithm for the multicolour version of the problem. Then we can enumerate (and hence count) all witnesses in time $g(k) \cdot n^{O(1)} \cdot N$, where N is the total number of witnesses.

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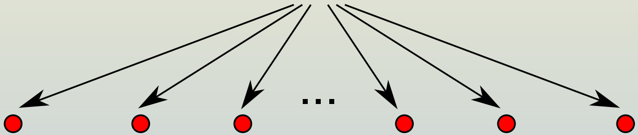

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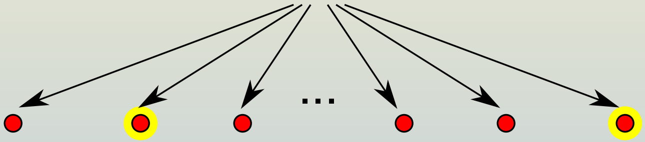

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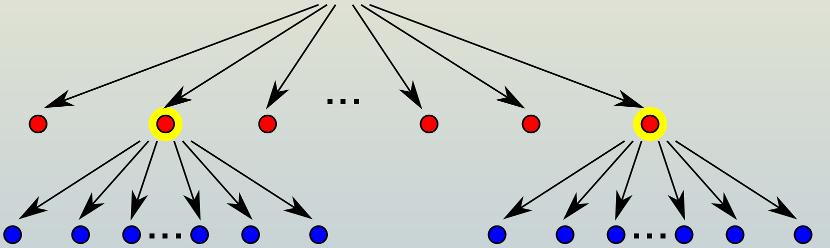

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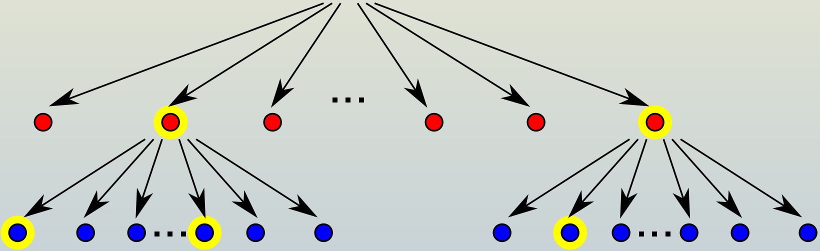

Not all subgraph problems are self-reducible

Suppose that a k-vertex subset is a witness if it either induces a clique or an independent set.

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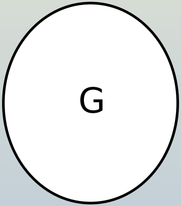

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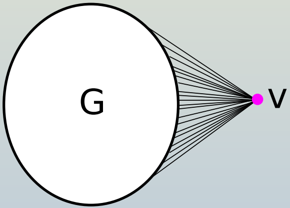

Theorem (Alon, Yuster, Zwick, 1995)

For all $n, k \in \mathbb{N}$ there is a k-perfect family $\mathcal{F}_{n,k}$ of hash functions from $[n]$ to $[k]$ of cardinality $2^{O(k)} \cdot \log n$. Furthermore, given n and k, a representation of the family $\mathcal{F}_{n,k}$ can be computed in time $2^{O(k)} \cdot n \log n$.

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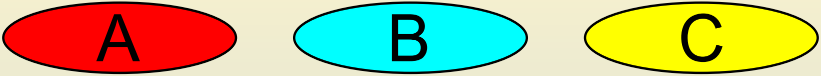
- IDEA: create many coloured instances, and enumerate the colourful copies in each (omitting duplicates)
- PROBLEM: although we're now looking for colourful witnesses, we still only have a decision algorithm for the uncoloured version...

A randomised approach



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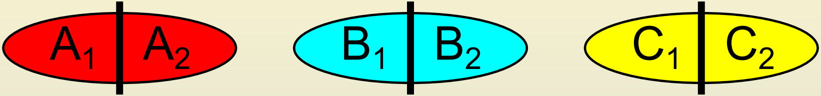

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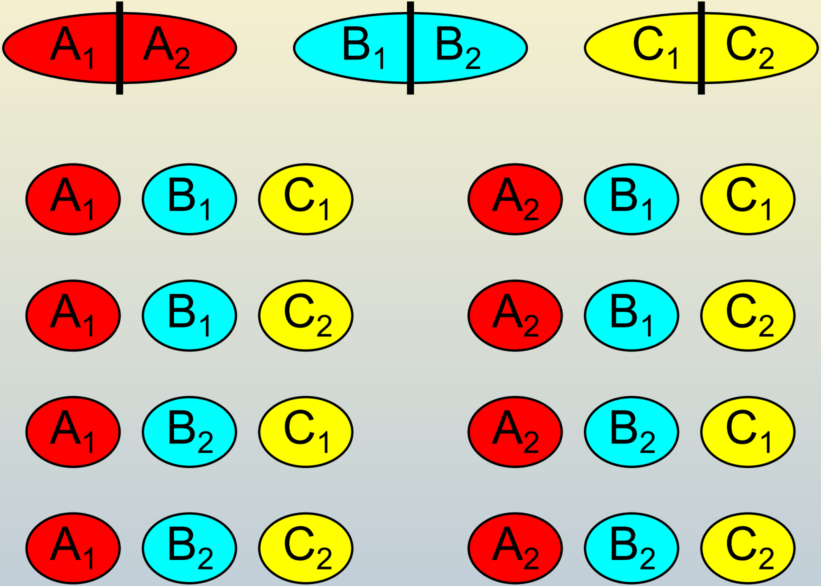

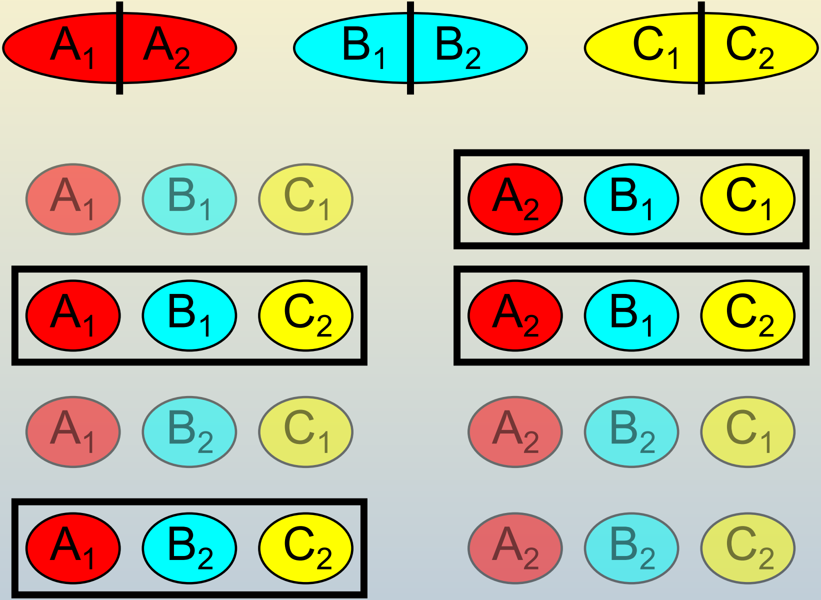
A1 A2
B1 B2
C1 C2
A1 B1 C1
A2 B1 C1
A1 B1 C2
A2 B1 C2
A1 B2 C1
A2 B2 C1
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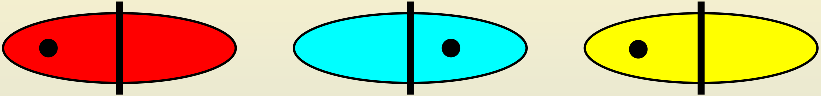
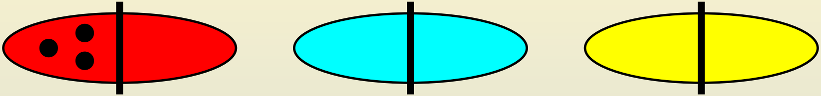


If a witness is colourful:

- It will always survive in exactly one combination

A randomised approach



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If a witness contains vertices of only $\ell < k$ colours:

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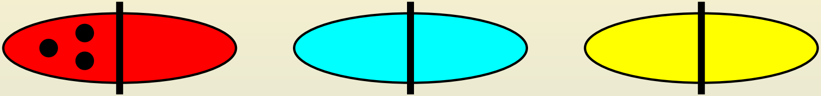

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It can then be shown that, for **any** witness, the **expected** number of combinations in which it survives at each level is at most one.

Few witnesses, revisited

Theorem

Suppose that we can decide if there is at least one witness in time $f(k) \cdot n^{O(1)}$. Then there is a randomised algorithm which enumerates all witnesses in expected time $f(k) \cdot n^{O(1)} \cdot N$, where N is the total number of witnesses in the instance.

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Corollary

Suppose that we can decide if there is at least one witness in time $f(k) \cdot n^{O(1)}$ and that, for each k and any graph G on n vertices, the total number of witnesses is at most $f(k)n^{O(1)}$. Then there exists an FPTRAS to count witnesses.

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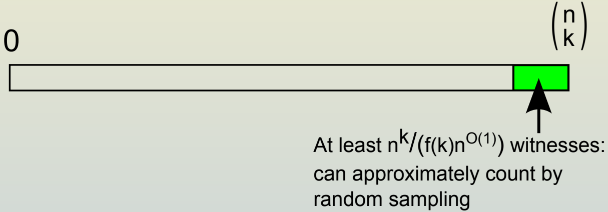

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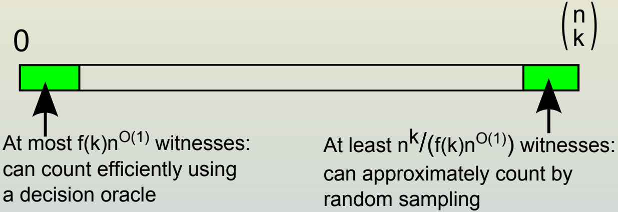

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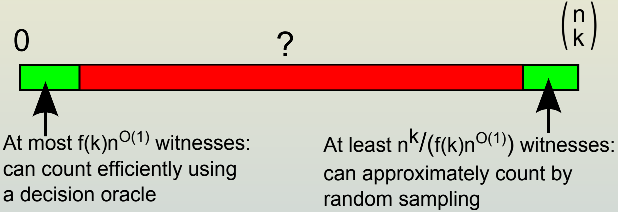

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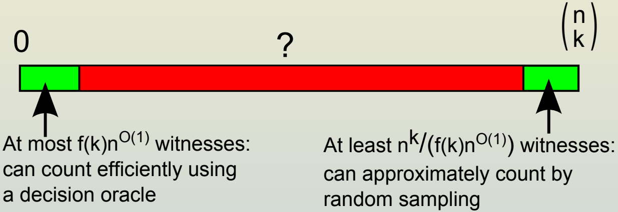

Thank you