# Spanning trees and the complexity of flood-filling games

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Mathematical Institute
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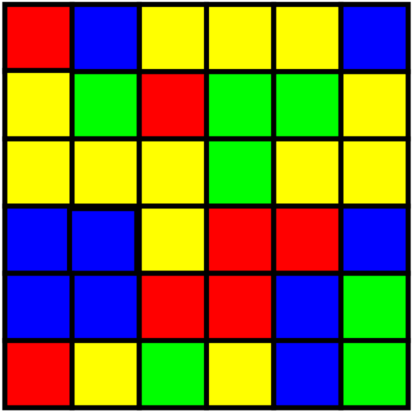
FUN 2012, Venice

# The original Flood-It game



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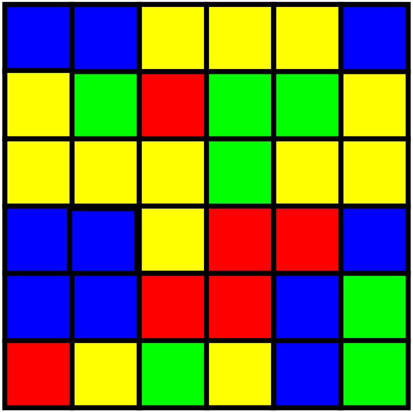

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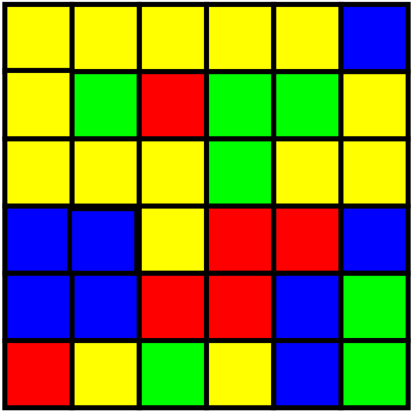

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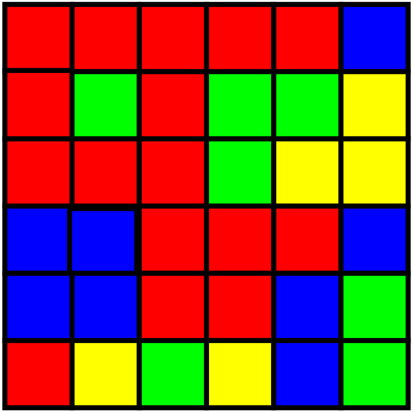

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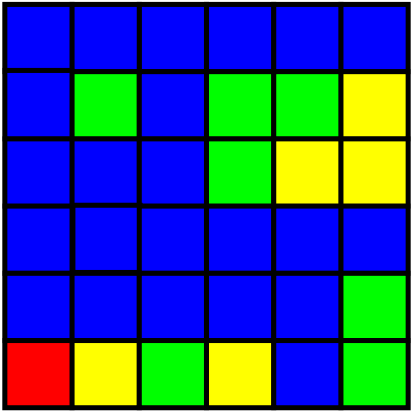

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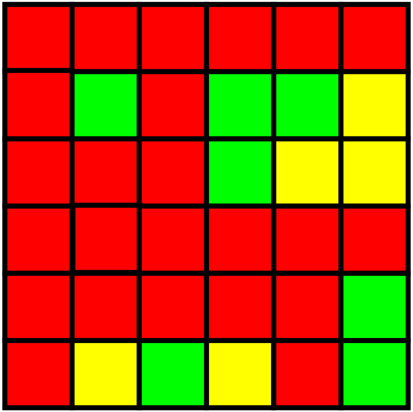

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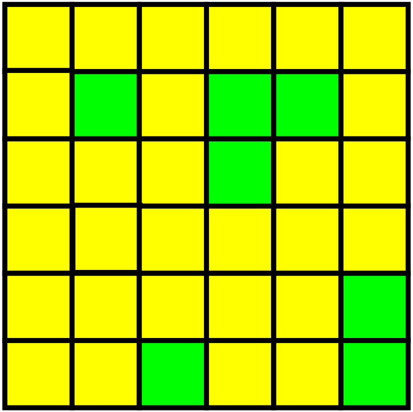

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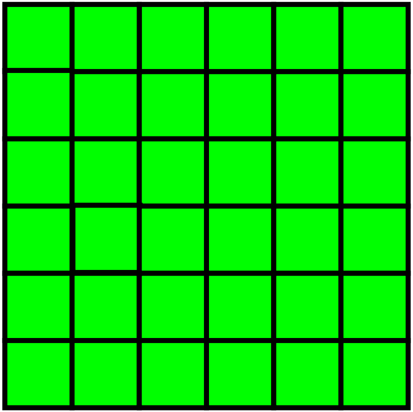
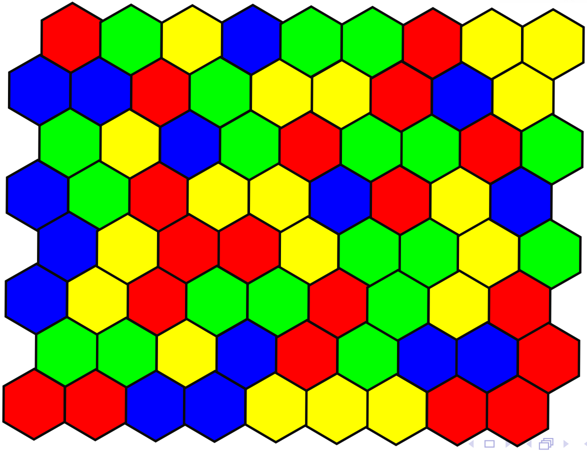


# The Honey-Bee game



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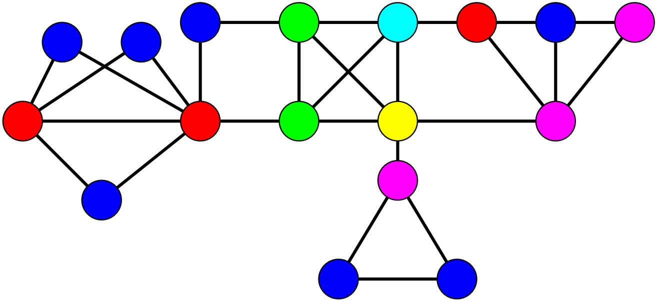

Sources: brighton-hove.dbprimary.com, englishclub.com, picgifs.com

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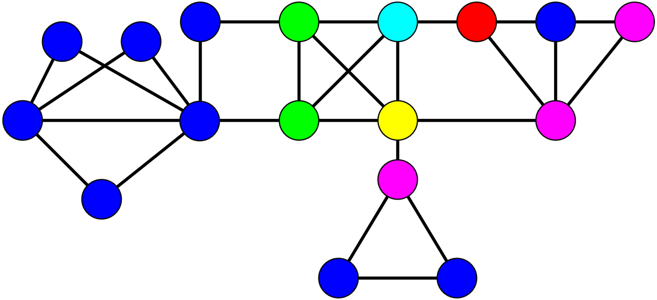

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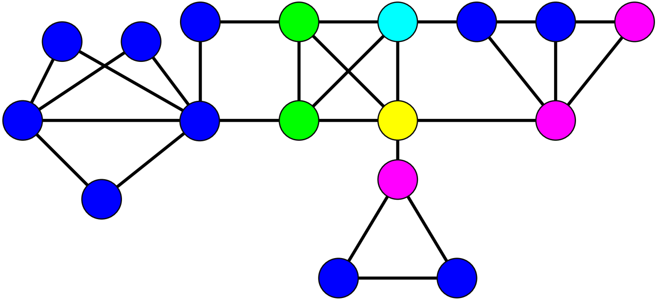

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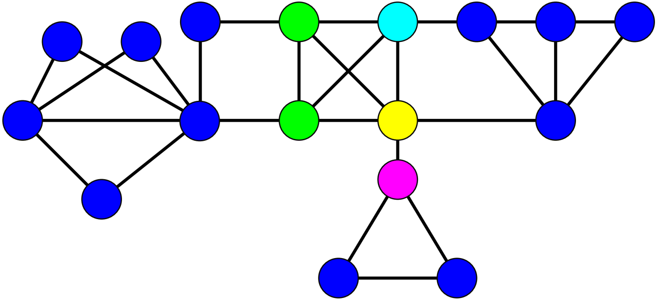

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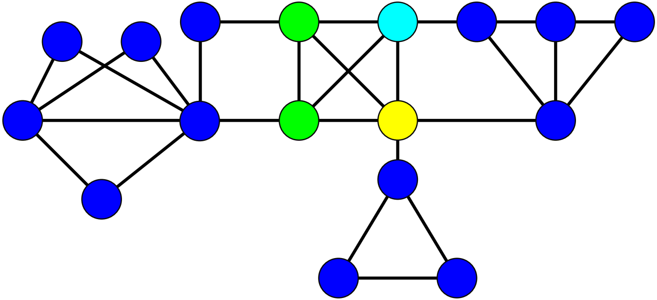

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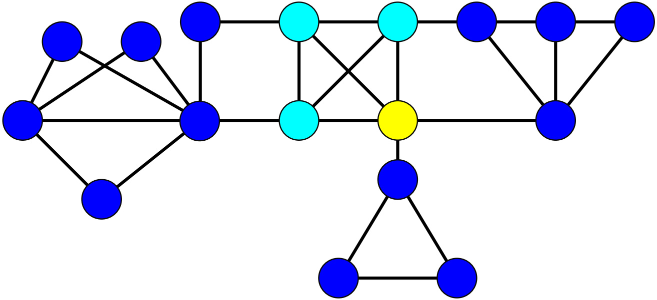

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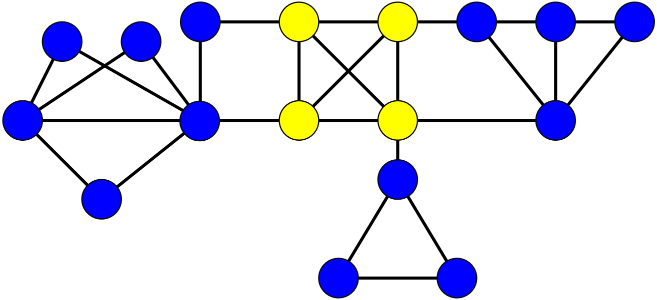

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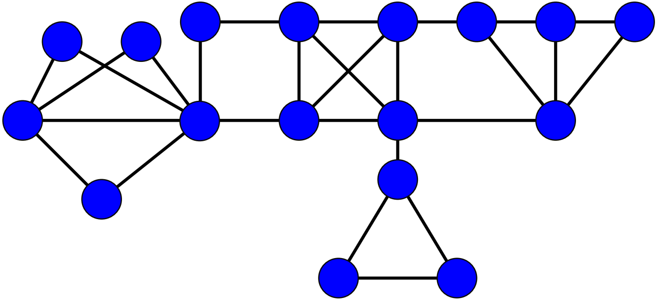

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# Existing results concerning Free Flood It

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FREE FLOOD IT can be solved in polynomial time when restricted to

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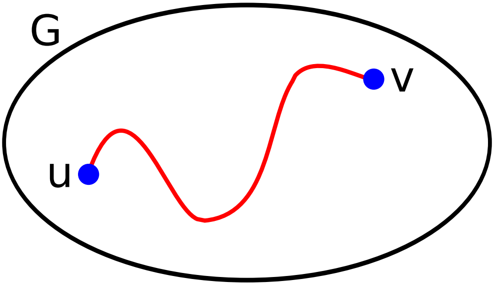
or

- if only two colours are used

# Connecting pairs of vertices



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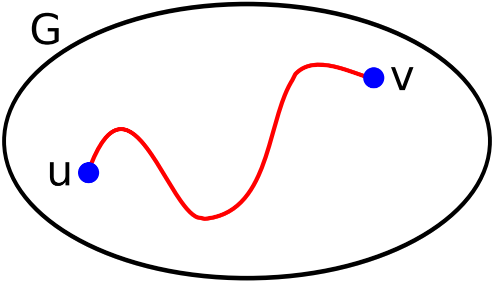


Using this fact, we can compute in time $O(|V|^3|E|c^2)$ the number of moves required to connect any given pair of vertices in a graph $G = (V, E)$ coloured with $c$ colours.

# Spanning trees

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*The number of moves required to flood a coloured graph $G$ is equal to the minimum, taken over all spanning trees $T$ of $G$, of the number of moves required to flood $T$.*

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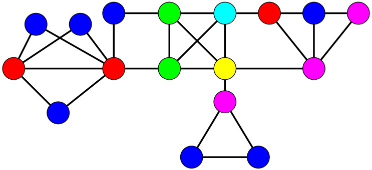

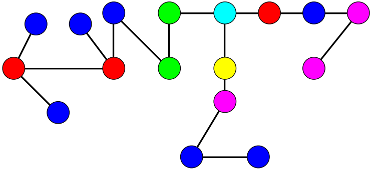

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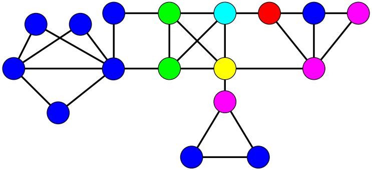

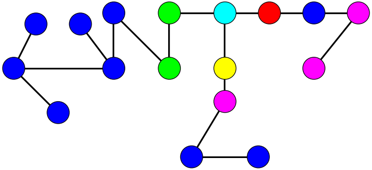

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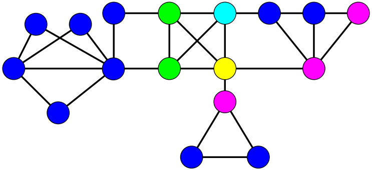

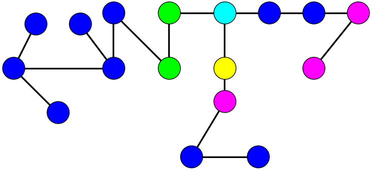

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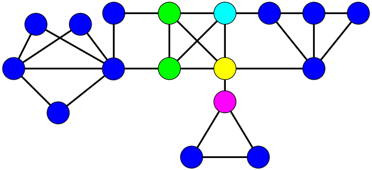

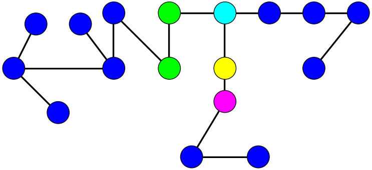

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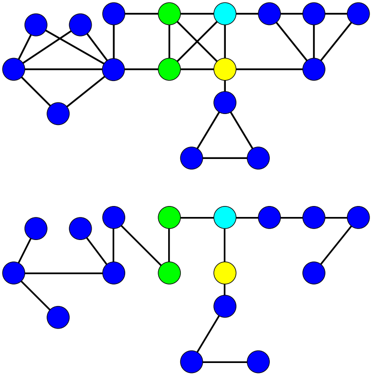

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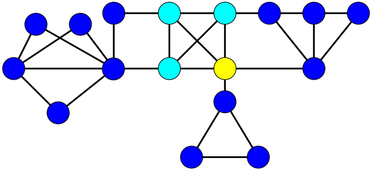

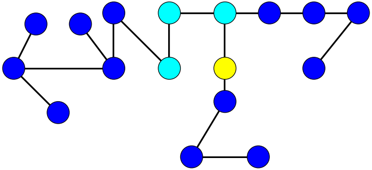

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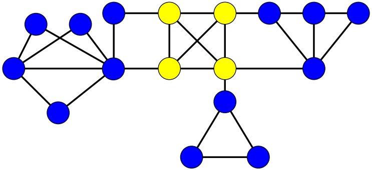

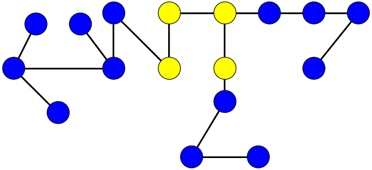

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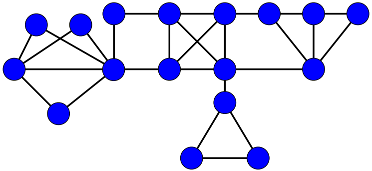
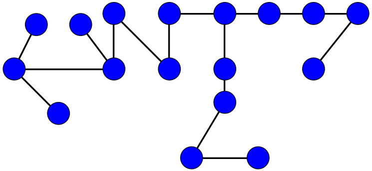
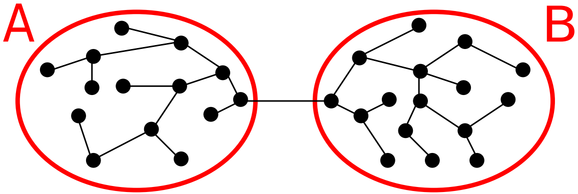




# Proof of spanning trees result: key step




The number of moves required to flood $T$ with colour $d$ is at most the sum of the numbers of moves required to flood $A$ and $B$ respectively with colour $d$.

# This is useless!

- In general, a graph has an exponential number of spanning trees.

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- Besides, FREE FLOOD IT is still NP-hard even on trees.

# ... or is it?

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$$P \neq NP$$

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$$\cancel{P \neq NP}$$

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Source: finditinscotland.com

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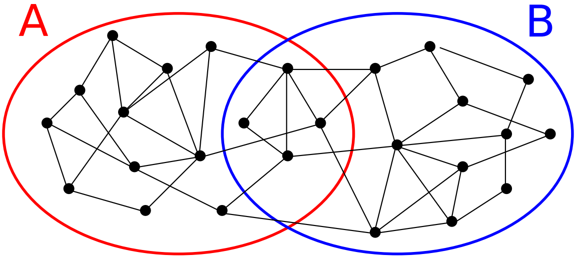
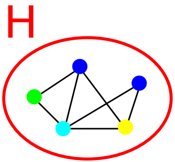


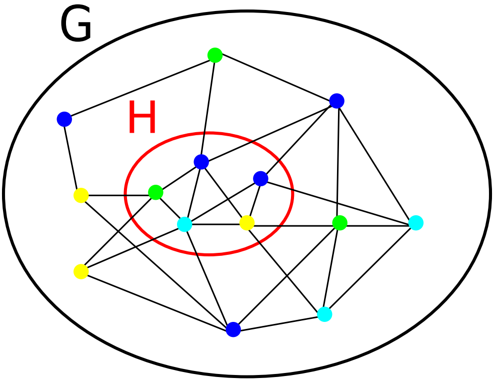

The number of moves required to flood $G$ with colour $d$ is at most the sum of the numbers of moves required to flood $A$ and $B$ respectively with colour $d$.

# ... or is it?




The number of moves required to flood a subgraph doesn't increase when we play in a larger graph.

# ... or is it?

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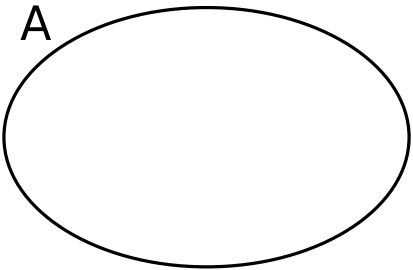
# Application I: Graphs with polynomially many connected subgraphs

**Theorem**

Free Flood It *can be solved in polynomial time on graphs that have only a polynomial number of connected subgraphs.*



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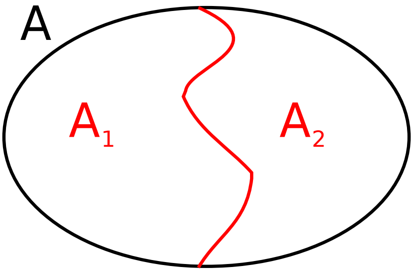

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Classes of graphs with only a polynomial number of connected subgraphs include:

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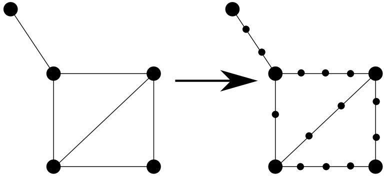
Classes of graphs with only a polynomial number of connected subgraphs include:

- paths
- cycles
- subdivisions of any fixed graph $H$



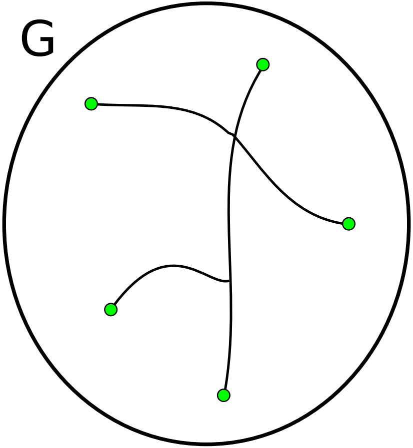
# Application II: Connecting $k$ points

Given a coloured graph $G$ and a subset $U$ of at most $k$ vertices, $k$-LINKING FLOOD IT is the problem of determining the number of moves required to create a single monochromatic component containing $U$.

**Theorem**

*$k$-LINKING FLOOD IT can be solved in time $O(|V|^{k+3}|E|c^2 2^k)$ on a graph $G = (V, E)$ coloured with $c$ colours.*

# Application II: Connecting $k$ points




The number of moves required to connect $U$ is equal to the minimum, taken over all subtrees $T$ of $G$ that contain $U$, of the number of moves required to flood $T$.

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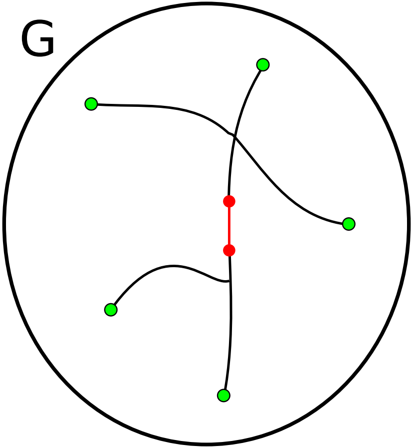

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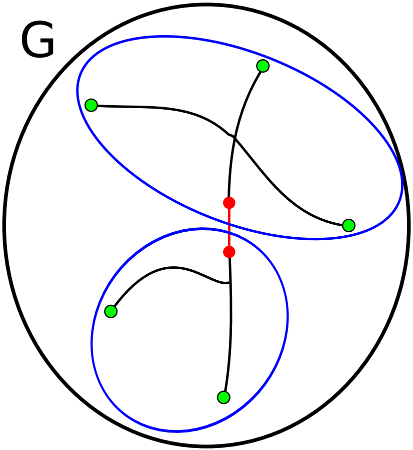

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- This allows us to prove nice complexity results:
  - FREE FLOOD IT is solvable in polynomial time on graphs with polynomially many connected subgraphs.
  - $k$-LINKING FLOOD IT is solvable in polynomial time on arbitrary graphs (for fixed $k$).

# Open Problems

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- On what other minor-closed classes of trees is Free Flood It solvable in polynomial time?
- Extremal problems...
- Does the Loch Ness Monster exist?

Thank you