University of Glasgow

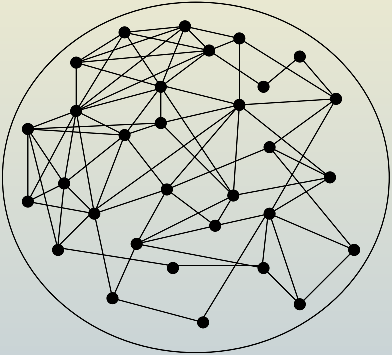
Subgraph counting problems

23rd March 2016

Kitty Meeks

The problem

Given a graph on n vertices, we are interested in subgraphs with k vertices that have particular properties.



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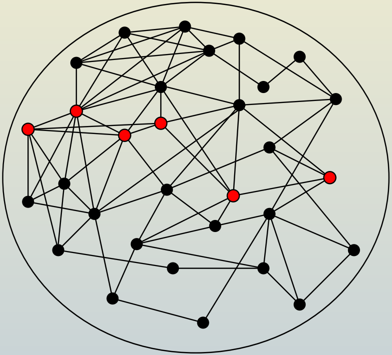

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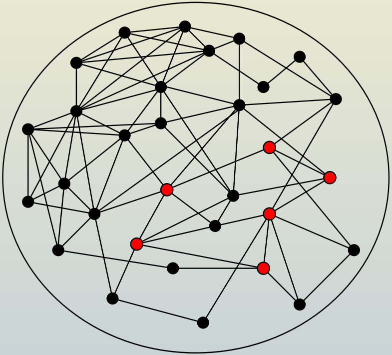

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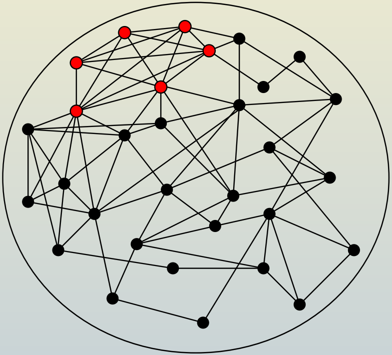

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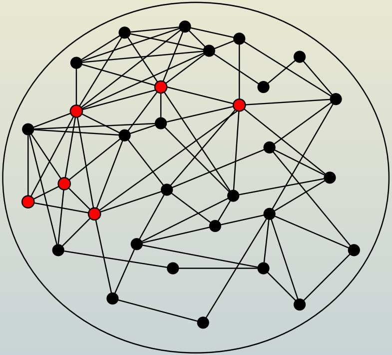

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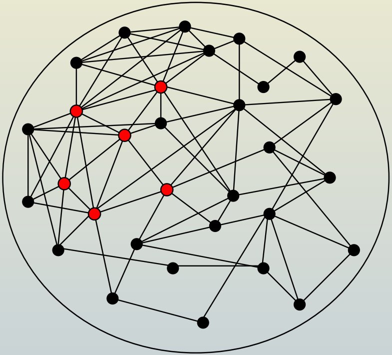

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- We can consider all possible k-vertex subgraphs in time $O(n^k)$.
- We would like to be able to answer questions about k-vertex subgraphs in time $f(k) \cdot n^{O(1)}$; in this case the problem belongs to the parameterised complexity class FPT.

Theorem (Based on Chen, Chor, Fellows, Huang, Juedes, Kanj and Xia, 2005)

Assuming the Exponential Time Hypothesis, there is no algorithm to determine whether an n-vertex graph contains a clique on k vertices in time $f(k) \cdot n^{o(1)}$, for any function f.

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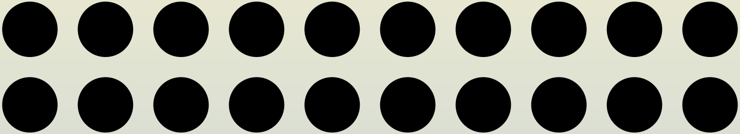
If we can decide, we can find a witness



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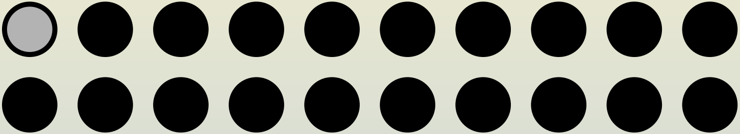

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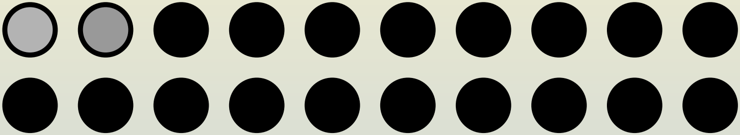

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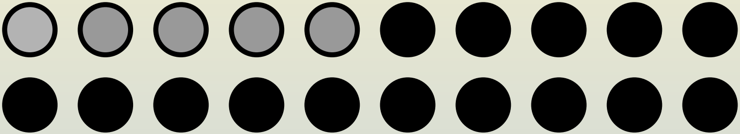

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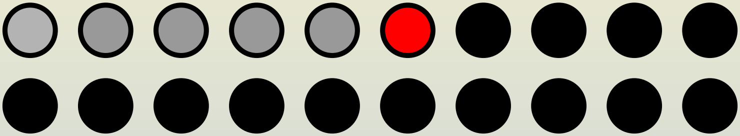

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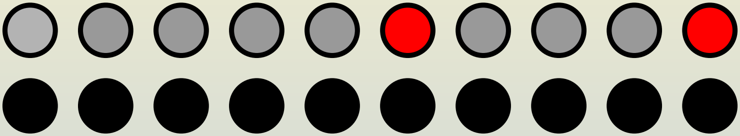

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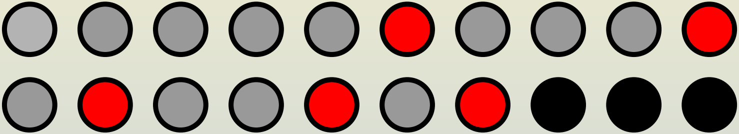

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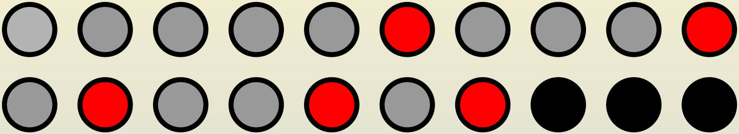

Theorem (Björklund, Kaski and Kowalik, 2014)

There exists an algorithm that extracts a witness using at most

$$2k\left(\log_2 \frac{n}{k} + 2\right)$$

queries to a deterministic decision algorithm.

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Exact counting is harder than decision

Question: Does G contain a connected induced subgraph on k vertices?

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Many other examples: k-vertex paths, k-vertex cycles, k-edge matchings, k-vertex regular induced subgraphs...

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Suppose that, for each k and any graph G on n vertices, the number of k-vertex subgraphs of G that have our property is either

1. *zero, or*
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$$\frac{1}{g(k)p(n)}\binom{n}{k}.$$

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These problems are still hard for exact counting.

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*Suppose that we have an $f(k) \cdot n^{O(1)}$ decision algorithm for the **multicolour** version of the problem. Then we can enumerate (and hence count) all witnesses in time $g(k) \cdot n^{O(1)} \cdot N$, where N is the total number of witnesses.*

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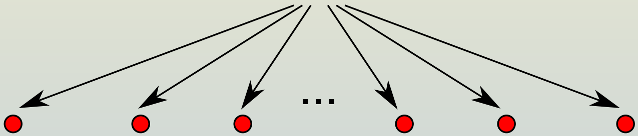

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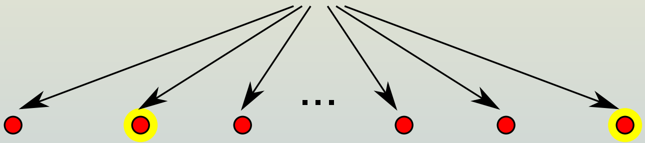

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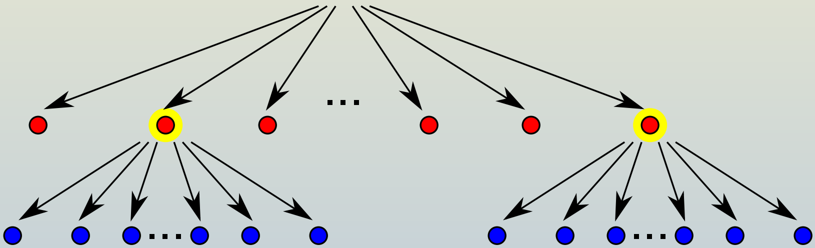

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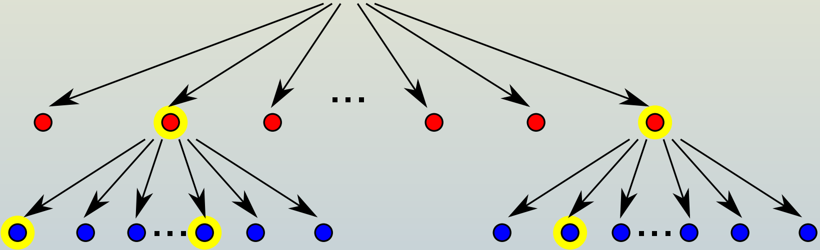

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- IDEA: create many coloured instances, and enumerate the colourful copies in each (omitting duplicates)

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- PROBLEM: although we're now looking for colourful witnesses, we still only have a decision algorithm for the uncoloured version...

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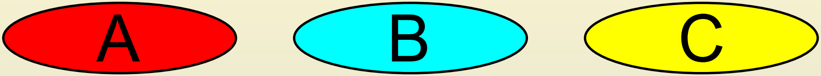

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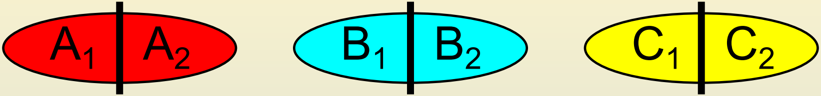

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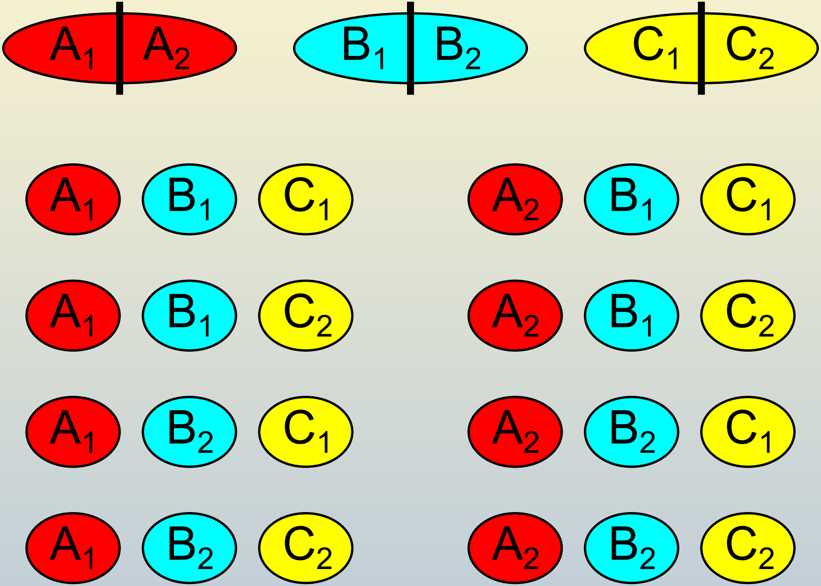

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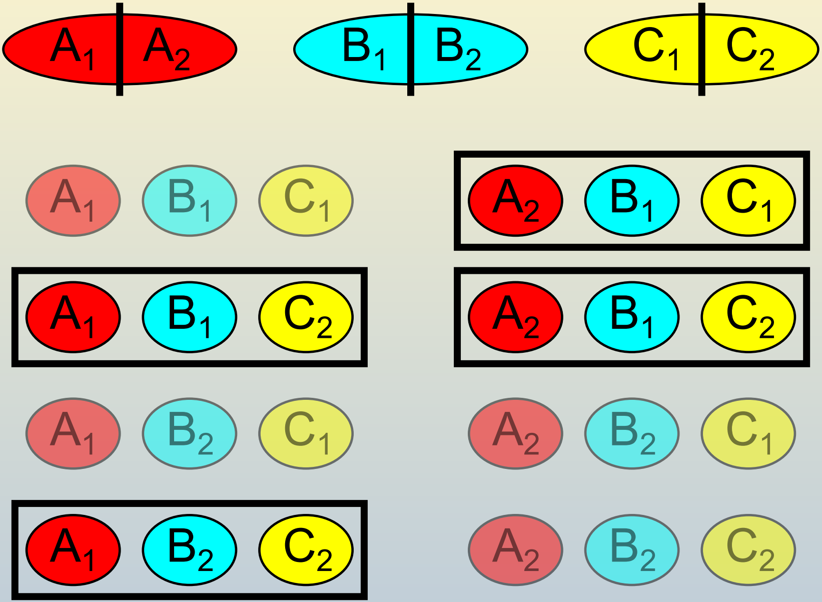

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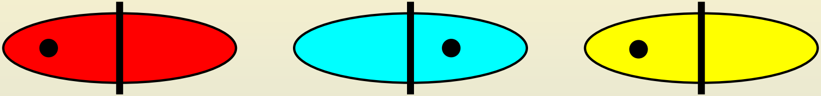

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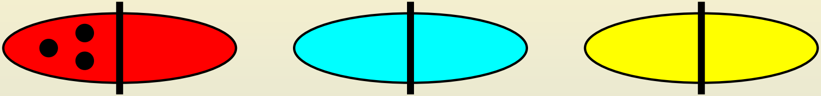

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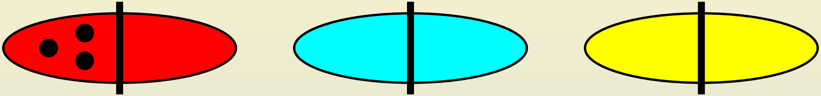

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It can then be shown that, for **any** witness, the **expected** number of combinations in which it survives at each level is at most one.

Theorem

Suppose that we can decide if there is at least one witness in time $f(k) \cdot n^{O(1)}$. Then there is a randomised algorithm which enumerates all witnesses in expected time $f(k) \cdot n^{O(1)} \cdot N$, where N is the total number of witnesses in the instance.

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Corollary

Suppose that we can decide if there is at least one witness in time $f(k) \cdot n^{O(1)}$ and that, for each k and any graph G on n vertices, the total number of witnesses is at most $f(k)n^{O(1)}$. Then there is an efficient algorithm to count witnesses approximately.

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0 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\binom{n}{k}$

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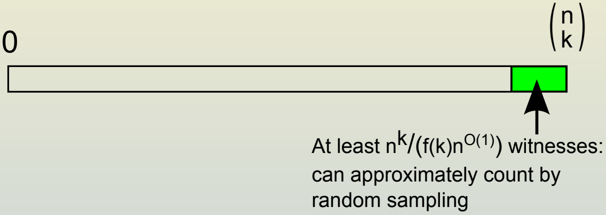

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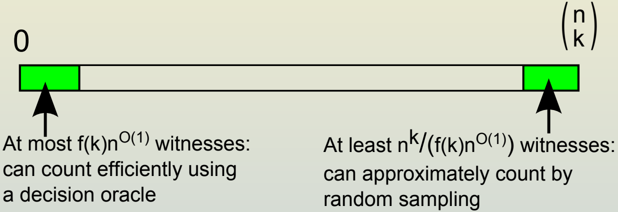

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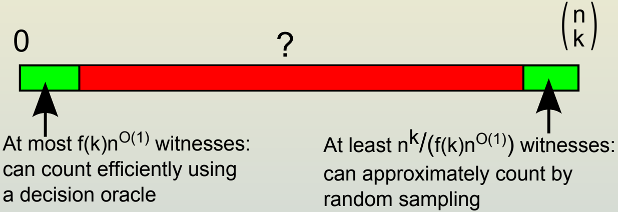

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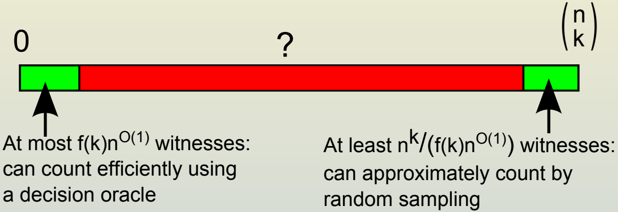

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