# Randomised enumeration of small witnesses using a decision oracle

IPEC, Aarhus, 25th August 2016

Kitty Meeks

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**Examples**
$k$-CLIQUE
$k$-PATH

**Non-examples**
$k$-VERTEX COVER
$k$-DOMINATING SET

**DECISION**

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University of Glasgow

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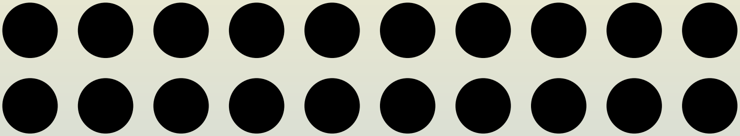
| **EXACT COUNTING**<br>Exactly how many witnesses? | **ENUMERATION**<br>List all witnesses |

## If we can decide, we can find a witness



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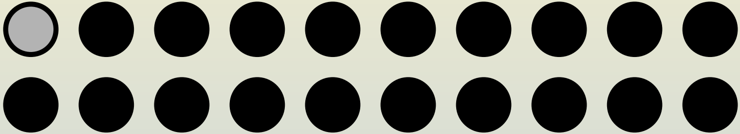

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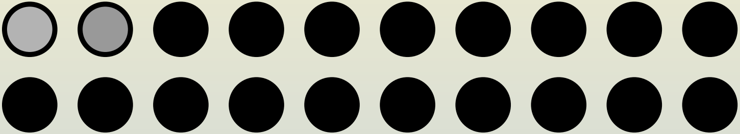

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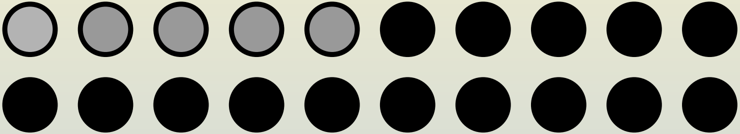

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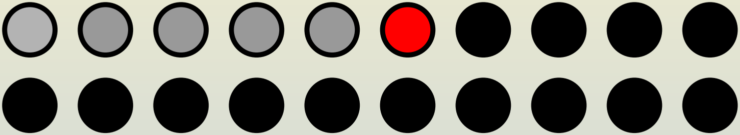

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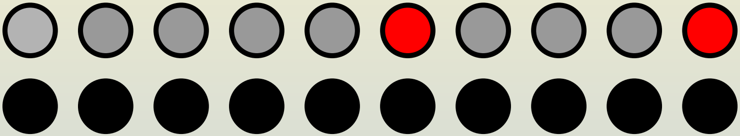

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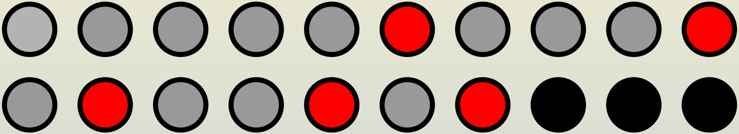

## If we can decide, we can find a witness

**Theorem (Björklund, Kaski and Kowalik, ESA 2014)**

*There exists an algorithm that extracts a witness using at most*

$$2k\left(\log_2 \frac{n}{k} + 2\right)$$

*queries to a deterministic decision algorithm.*

# With an **extension** oracle, we can find all witnesses

**EXT-ORA($X, Y$)**
*Input:* $X \subseteq U$ and $Y \subseteq X$
*Output:* 1 if there exists a witness $W$ with $Y \subseteq W \subseteq X$; 0 otherwise.

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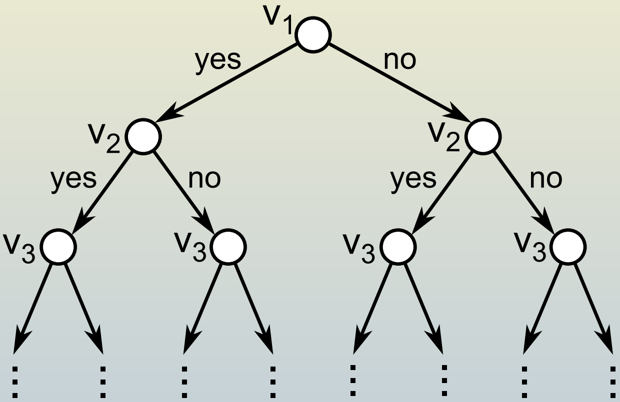

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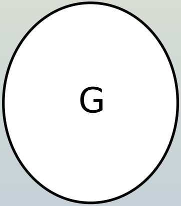

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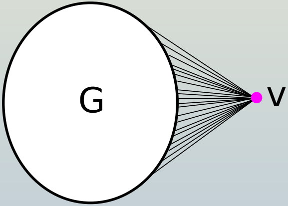

# Enumerating without using extension

## Theorem

*There is a randomised algorithm to enumerate all witnesses of size k in a self-contained k-witness problem exactly once, whose expected number of calls to a deterministic decision oracle is at most $2^{O(k)} \log^2 n \cdot N$, where $N$ is the total number of witnesses.*

*Moreover, if an oracle call can be executed in time $g(k) \cdot n^{O(1)}$, then the expected total running time of the algorithm is*

$$2^{O(k)} \cdot g(k) \cdot n^{O(1)} \cdot N.$$

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## Definition

*A family $\mathcal{F}$ of hash functions from $[n]$ to $[k]$ is said to be k-perfect if, for every subset $A \subset [n]$ of size k, there exists $f \in \mathcal{F}$ such that the restriction of $f$ to $A$ is injective.*

## Theorem (Alon, Yuster, Zwick, 1995)

*For all $n, k \in \mathbb{N}$ there is a k-perfect family $\mathcal{F}_{n,k}$ of hash functions from $[n]$ to $[k]$ of cardinality $2^{O(k)} \cdot \log n$. Furthermore, given n and k, a representation of the family $\mathcal{F}_{n,k}$ can be computed in time $2^{O(k)} \cdot n \log n$.*

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- PROBLEM: although we're now looking for colourful witnesses, we still only have a decision algorithm for the uncoloured version...

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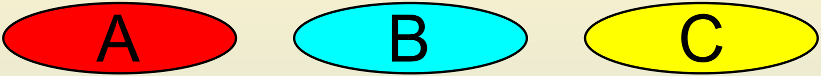

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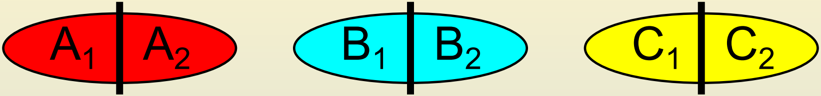

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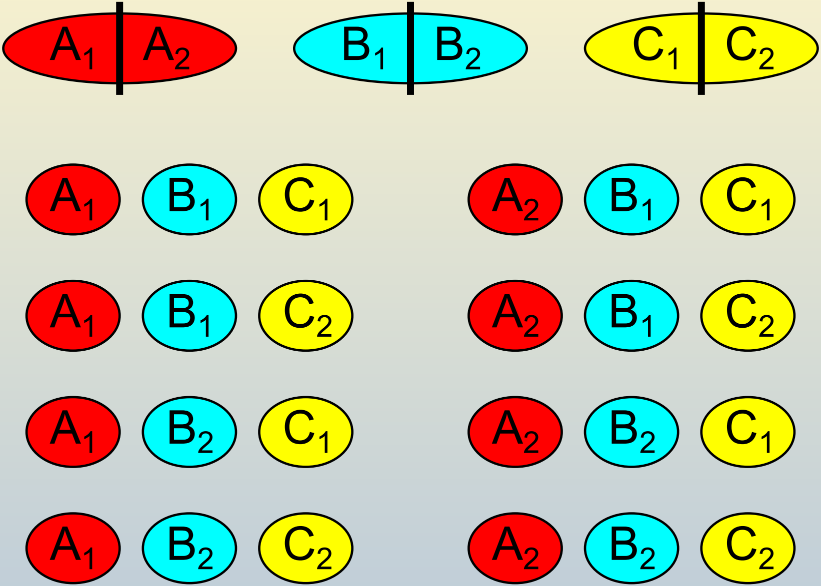

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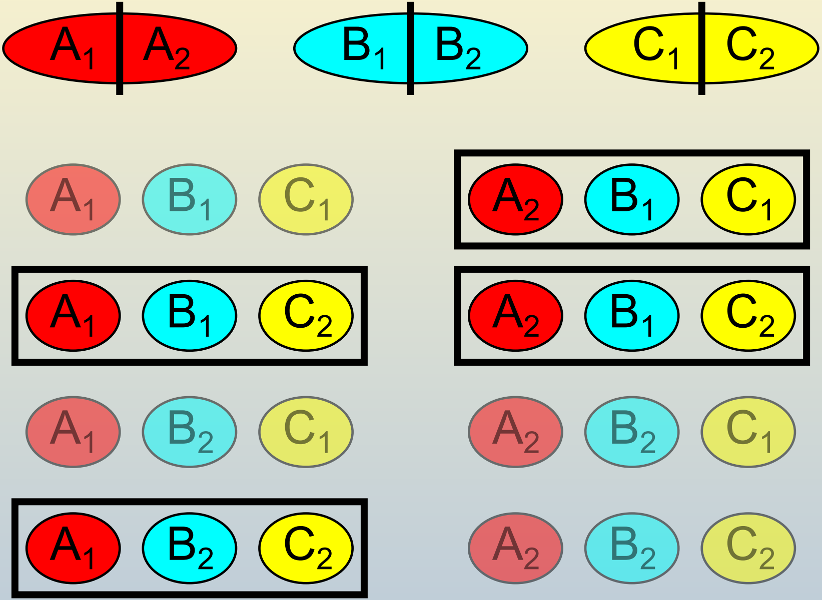

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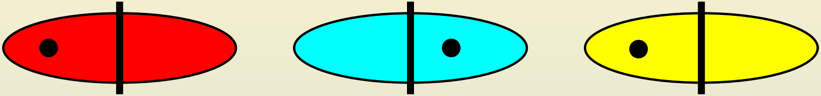

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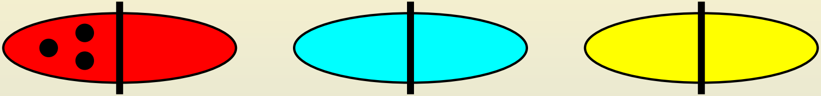

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If a witness contains vertices of only $\ell < k$ colours:

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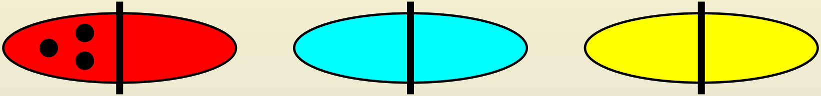

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It can then be shown that, for **any** witness, the **expected** number of combinations in which it survives at each level is at most one.

# Application to counting

## Theorem

*Let $\Pi$ be a self-contained k-witness problem, and suppose that $0 < \delta \leq \frac{1}{2}$ and $M \in \mathbb{N}$. Then there exists a randomised algorithm which makes at most $2^{O(k)} \log^2 n \, M \log(\delta^{-1})$ calls to a deterministic decision oracle for $\Pi$, and*

1. *if the number of witnesses in the instance of $\Pi$ is at most $M$, outputs with probability at least $1 - \delta$ the exact number of witnesses in the instance;*
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*Moreover, if there is an algorithm solving the decision version of $\Pi$ in time $g(k) \cdot n^{O(1)}$, then the expected running time of the randomised algorithm is bounded by $2^{O(k)} \cdot g(k) \cdot n^{O(1)} \cdot M \cdot \log(\delta^{-1})$.*

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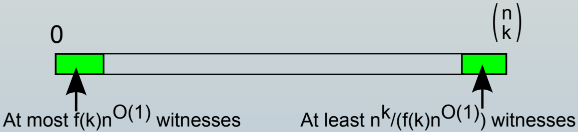

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