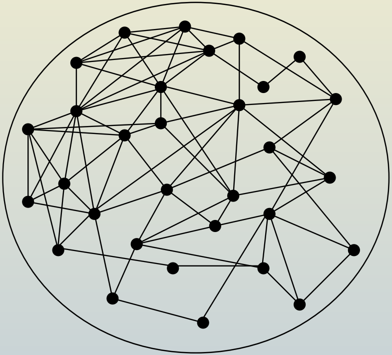
The complexity of finding and counting small subgraphs

OU Winter Combinatorics Meeting, 20th January 2016

Kitty Meeks

The problem

Given a graph on n vertices, we are interested in subgraphs with k vertices that have particular properties.



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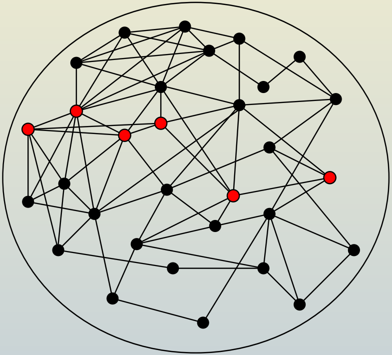

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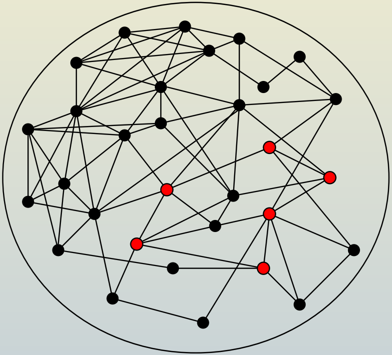

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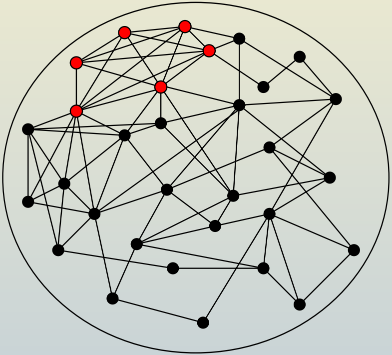

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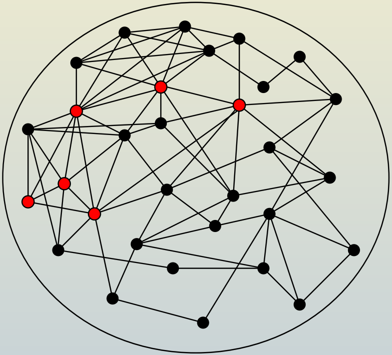

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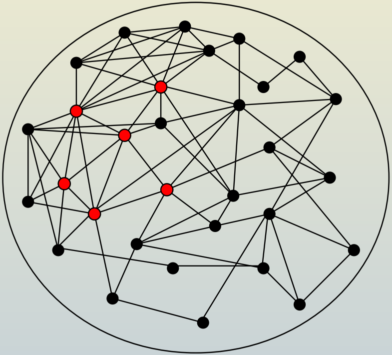

For example:

- Paths on k vertices
- Cycles on k vertices
- Cliques on k vertices
- Connected k-vertex induced subgraphs
- k-vertex induced subgraphs with an even number of edges

DECISION

Is there a witness?

Deciding, counting and enumerating

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APPROX COUNTING
Approximately how many witnesses?

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We are interested in what happens as n and k both tend to infinity, independently, with $k << n$.

- We can consider all possible k-vertex subgraphs in time $O(n^k)$.
- We would like to be able to answer questions about k-vertex subgraphs in time $f(k) \cdot n^{O(1)}$.

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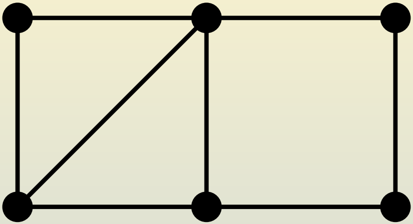
Hardness, using poset lattices



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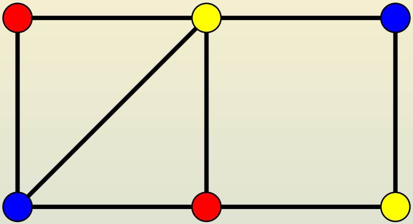

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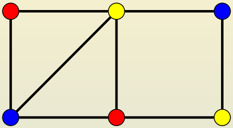

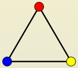

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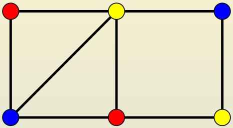

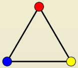

$$\text{mask}\left\{\overbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}}^{\text{underlying structure}}\right. \cdot \begin{pmatrix} \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

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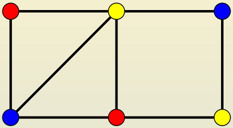
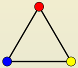
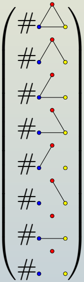




underlying structure

mask

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot$$



$$= \begin{pmatrix} 4 \\ 4 \\ 2 \\ 6 \\ 4 \\ 2 \\ 2 \end{pmatrix}$$

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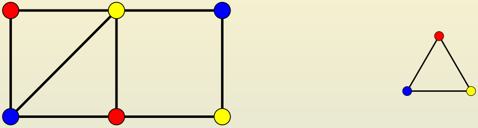

$$
\text{mask}\left\{
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\#\\ \#\\ \#\\ \#\\ \#\\ \#\\ \#\\ \#
\end{pmatrix}
=
\begin{pmatrix}
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\end{pmatrix}
$$

Method used by: Jerrum & M. ('14, '16); matrix inversion more generally used by Flum & Grohe ('04), Bläser & Curticapean ('12), Curticapean ('13).

Tractability, using Ramsey

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Proposition

Let $G = (V, E)$ be an n-vertex graph, where $n \geq 2^{2k}$. Then the number of k-vertex subsets $U \subset V$ such that U induces either a clique or independent set in G is at least

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Method used by: Arvind & Raman ('02), Khot & Raman ('02), Jerrum & M. ('16).

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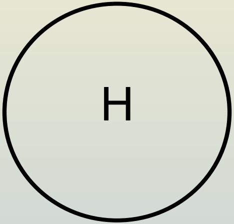

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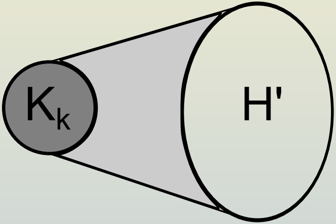

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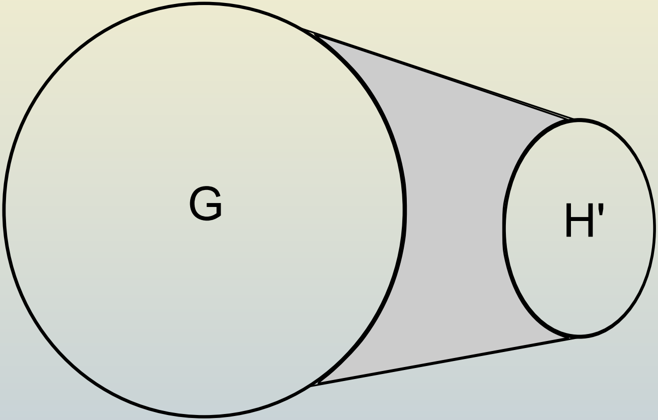

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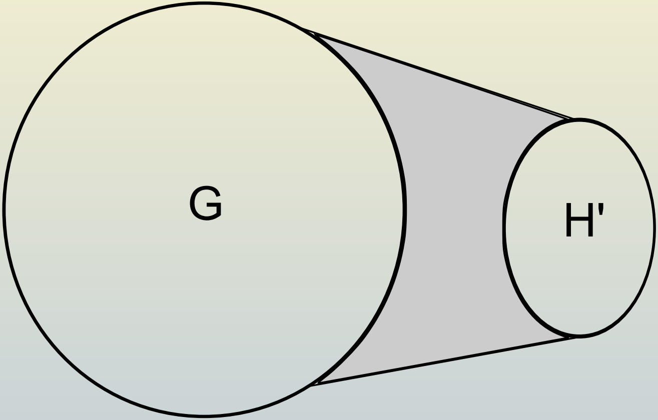

Method used by: Chen, Thurley & Weyer ('08), Khot & Raman ('02), Curticapean & Marx ('14), Jerrum & M. ('15).

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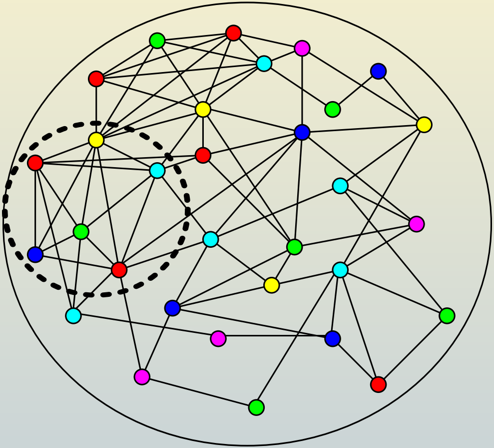

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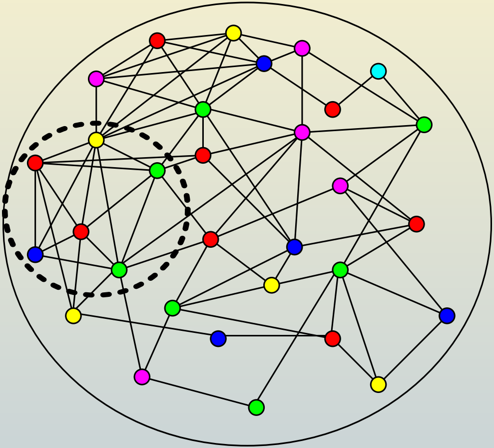

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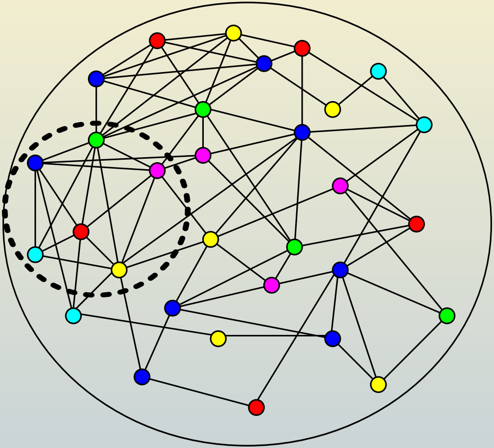

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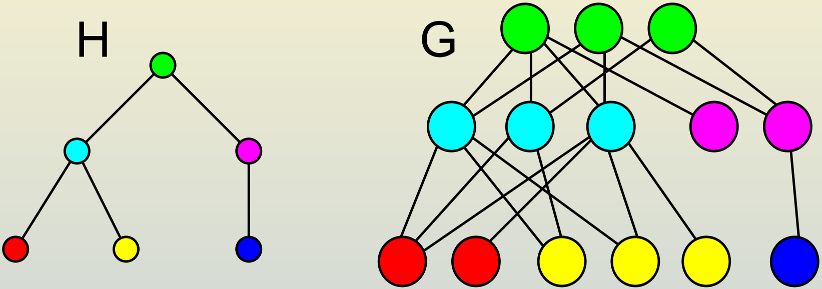
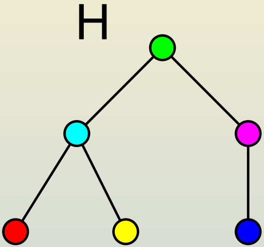
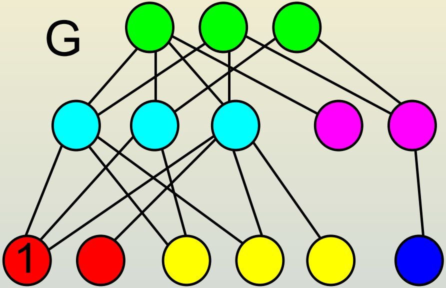
Theorem (Alon, Yuster and Zwick 1995)

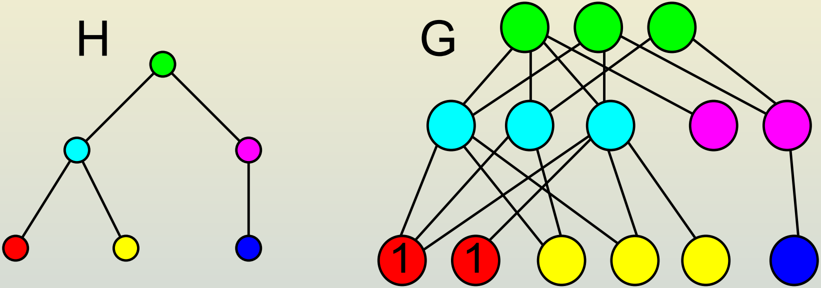
For all $n, k \in \mathbb{N}$, there is a k-perfect family $\mathcal{F}_{n,k}$ of hash functions from $[n]$ to $[k]$ of cardinality $2^{O(k)} \cdot \log n$, and (given n and k) the family $\mathcal{F}_{n,k}$ can be computed in time $2^{O(k)} \cdot n \cdot \log^2 n$.

Tractability, using bounded treewidth




H


G
1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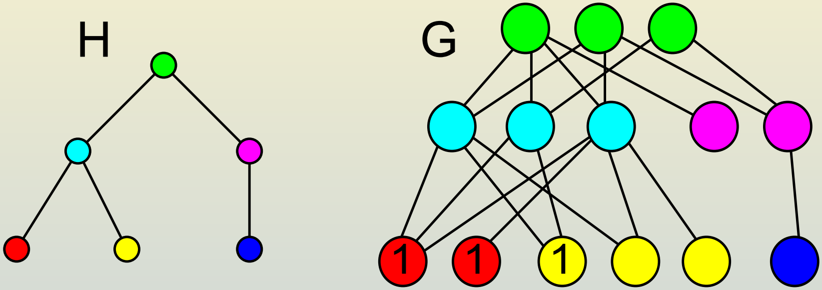

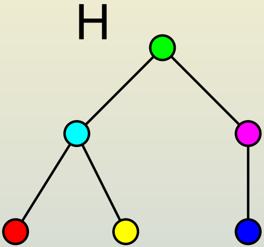
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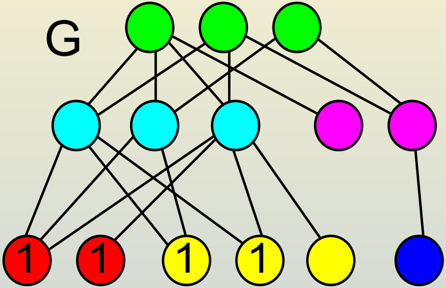
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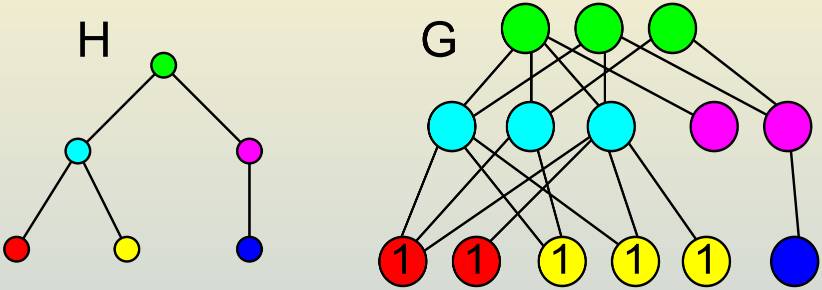

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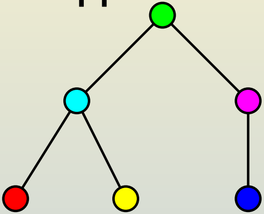

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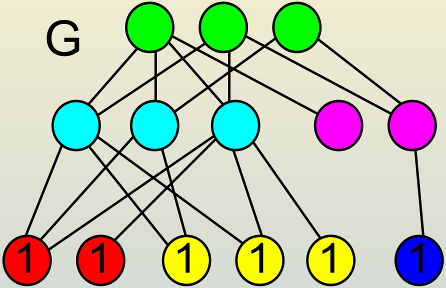

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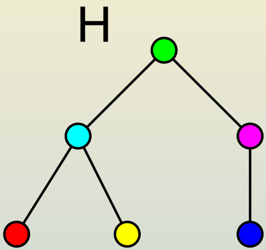

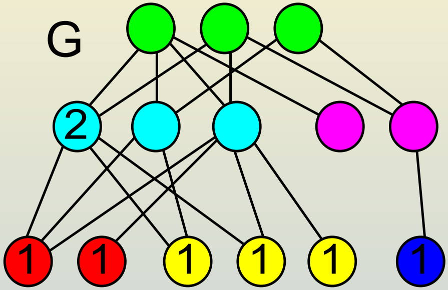

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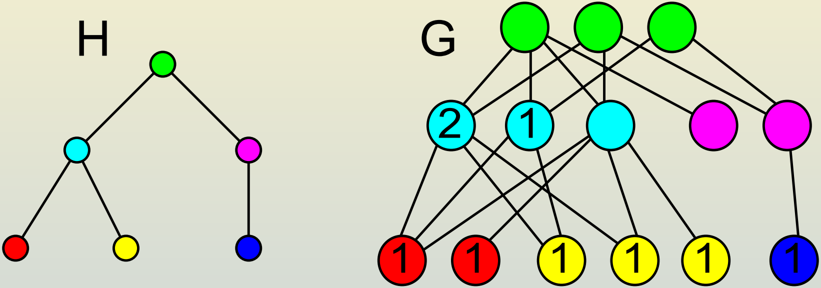

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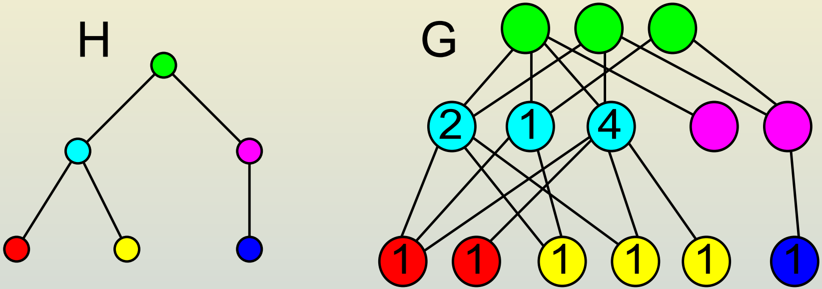

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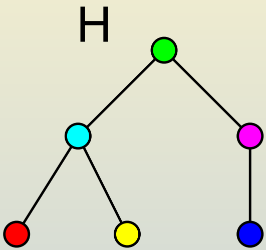

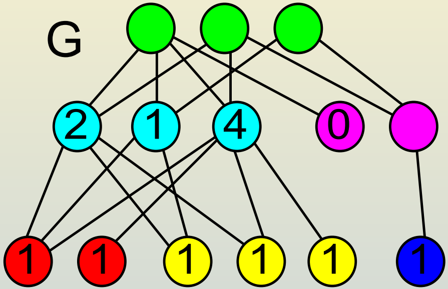

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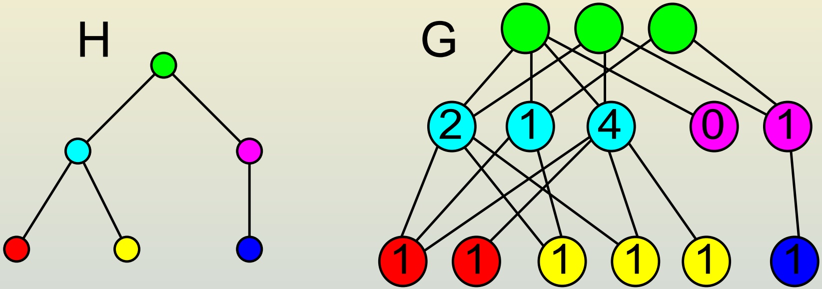

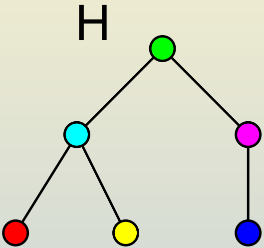
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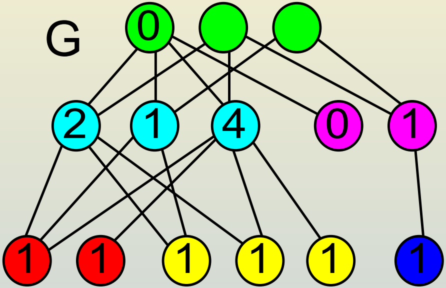
G
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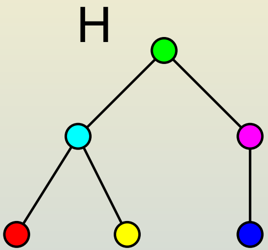

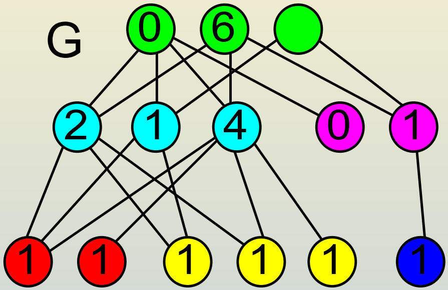

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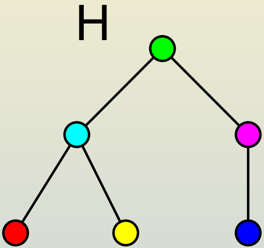

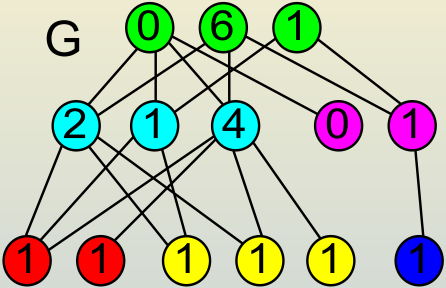

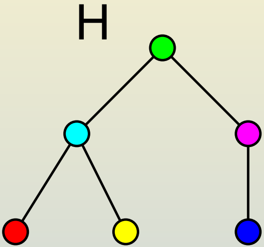

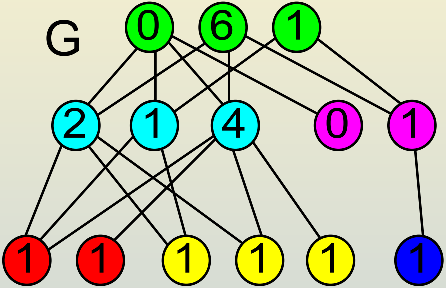

Method used by: Alon, Yuster & Zwick (1995), Arvind & Raman (2002), Jerrum & M. (2015).

Hardness, using unbounded treewidth

Theorem (Excluded Grid Theorem, Robertson and Seymour 1986)

There is a computable function $w : N \to N$ *such that the* $(k \times k)$ *grid is a minor of every graph of treewidth at least* $w(k)$.

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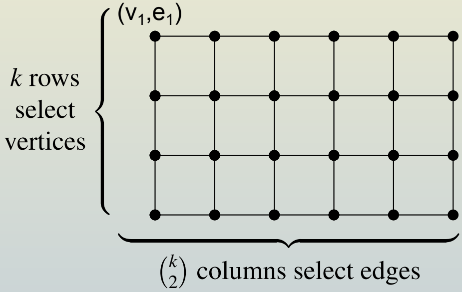

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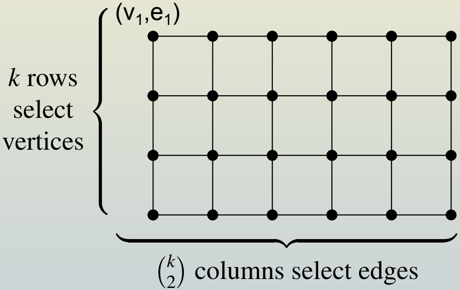
Theorem (Excluded Grid Theorem, Robertson and Seymour 1986)

There is a computable function $w : N \to N$ such that the $(k \times k)$ grid is a minor of every graph of treewidth at least $w(k)$.



Method used by: Grohe, Schwentick & Segoufin (2001), Dalmau & Jonsson (2004), Grohe (2007), Färnqvist & Jonnson (2007), Chen, Thurley & Weyer (2008), Chen & Müller (2008), M. (2016).

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- for which exact counting is tractable? (Other than the “trivial” examples...)

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Thank you