# List Colouring Graphs of Bounded Treewidth

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# Vertex colouring

- Given a graph $G = (V, E)$, $\phi : V \to \{1, \ldots, k\}$ is a *proper c-colouring* of $G$ if, for all $uv \in E$, $\phi(u) \neq \phi(v)$.
- The *chromatic number* $\chi(G)$ of $G$ is the smallest $c$ such that there exists a proper $c$-colouring of $G$.

CHROMATIC NUMBER

Input: A graph $G = (V, E)$.

Question: What is $\chi(G)$?

- It is NP-complete to decide whether $\chi(G) \leq 3$.
- If $G$ has fixed treewidth at most $k$, $\chi(G)$ can be computed in linear time (Arnborg and Proskurowski, 1989).

# List Colouring

For graph $G(V, E)$ and a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper colouring $\phi$ of $G$ such that $c(v) \in L_v$ for all $v \in V$.

List Colouring

Input: A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$.

Question: Is there a proper list colouring $(G, \mathcal{L})$?

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Question: Is there a proper list colouring $(G, \mathcal{L})$?

**Theorem (Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider and Thomassen, 2011)**

LIST COLOURING *is W[1]-hard, parameterised by treewidth.*

# List Chromatic Number

The *list chromatic number* $\mathrm{ch}(G)$ of $G$ is the smallest integer $c$ such that, for any assignment of lists $(L_v)_{v \in V(G)}$ to the vertices of $G$ with $|L_v| \geq c$ for each $v$, there exists a proper list colouring of $(G, \mathcal{L})$.

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Question: What is $\mathrm{ch}(G)$?

**Theorem (Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider and Thomassen, 2011)**

*The* List Chromatic Number *problem, parameterised by the treewidth bound $k$, is fixed-parameter tractable, and solvable in linear time for any fixed $k$.*

# Edge Colouring

- Given a graph $G = (V, E)$, a *proper edge colouring* of $G$ is an assignment of colours to the edges of $G$ such that no two incident edges receive the same colour.
- The *edge chromatic number* $\chi'(G)$ of $G$ is the smallest integer $c$ such that there exists a proper edge colouring of $G$ using $c$ colours.
- It is NP-hard to determine whether $\chi'(G) \leq 3$ for cubic graphs (Holyer, 1981).
- $\chi'(G)$ can be computed in linear time on graphs of bounded treewidth (Zhou, Nakano and Nishizeki, 2005).

# List Edge Colouring

For graph $G(V,E)$ and a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper list colouring $\phi$ of $G$ such that $c(v) \in L_v$ for all $v \in V$.

List Edge Colouring

- Input: A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_e)_{e \in E(G)}$.
- Question: Is there a proper list edge colouring $(G, \mathcal{L})$?

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List Edge Colouring

**Input:** A graph $G = (V,E)$, together with a collection of colour lists $\mathcal{L} = (L_e)_{e \in E(G)}$.

**Question:** Is there a proper list edge colouring $(G, \mathcal{L})$?

**Theorem (Zhou, Matsuo, Nishizeki, 2005)**

List Edge Colouring *is NP-hard on series-parallel graphs.*

**Theorem (Marx, 2005)**

List Edge Colouring *is NP-hard on outerplanar graphs.*

# Total Colouring

- Given a graph $G = (V, E)$, a *proper total colouring* of $G$ is an assignment of colours to the vertices and edges of $G$ such that
  - no two adjacent vertices receive the same colour
  - no two incident edges receive the same colour
  - no edge receives the same colour as either of its endpoints.
- The *total chromatic number* $\chi_T(G)$ of $G$ is the smallest integer $c$ such that there exists a proper total colouring of $G$ using $c$ colours.
- It is NP-hard to determine $\chi_T(G)$ for regular bipartite graphs (McDiarmid and Sánchez-Arroyo, 1994).
- $\chi_T(G)$ can be computed in linear time on graphs of bounded treewidth (Isobe, Zhou and Nishizeki, 2007).

# List Total Colouring

For graph $G(V, E)$ and a collection of colour lists $\mathcal{L} = (L_x)_{x \in V \cup E}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper total colouring $\phi$ of $G$ such that $c(x) \in L_x$ for all $x \in V \cup E$.

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**Input:** A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_x)_{x \in V \cup E}$.

**Question:** Is there a proper list total colouring $(G, \mathcal{L})$?

**Theorem (Zhou, Matsuo, Nishizeki, 2005)**

List Total Colouring *is NP-hard on series-parallel graphs.*

# List Edge and Total Chromatic numbers

- The *list edge chromatic number* $\text{ch}'(G)$ of $G$ is the smallest integer $c$ such that, for any assignment of lists $(L_e)_{e \in E(G)}$ to the edges of $G$ with $|L_e| \geq c$ for each $e$, there exists a proper list edge colouring of $(G, \mathcal{L})$.

$$\Delta(G) \leq \chi'(G) \leq \text{ch}'(G) \leq 2\Delta(G) - 1$$

- The *list total chromatic number* $\text{ch}_T$ of $G$ is the smallest integer $c$ such that, for any assignment of lists $(L_e)_{e \in E(G)}$ to the edges of $G$ with $|L_e| \geq c$ for each $e$, there exists a proper list total colouring of $(G, \mathcal{L})$.

$$\Delta(G) + 1 \leq \chi_T(G) \leq \text{ch}_T(G) \leq 2\Delta(G) + 1$$

# Parameterised complexity of colouring problems

| | General problem | Parameter treewidth | List version, parameter treewidth | List Chromatic number, parameter treewidth |
|---|---|---|---|---|
| Vertex colouring | NP-c | FPT | W[1]-hard | FPT |
| Edge colouring | | | | |
| Total colouring | | | | |

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# The Combinatorial Results

**Theorem**

*Let $G$ be a graph with treewidth at most $k$ and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}'(G) = \Delta(G)$.*

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# Edge colouring: background

**Theorem (Vizing, 1964)**

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**Conjecture (Vizing)**

$ch'(G) \leq \Delta(G)+1$.

# The List (Edge) Colouring Conjecture

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*For any graph $G$,*

$$\mathrm{ch}'(G) = \chi'(G).$$

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**Theorem (Kahn, 1996)**

*For any $\epsilon > 0$, if $\Delta(G)$ is sufficiently large,*

$$\mathrm{ch}'(G) \leq (1 + \epsilon)\Delta(G).$$

# The List (Edge) Colouring Conjecture

**Conjecture (List (Edge) Colouring Conjecture)**

*For any graph $G$,*

$$\mathrm{ch}'(G) = \chi'(G).$$

- Our result proves a special case: if $\Delta(G)$ is sufficiently large compared with the treewidth of $G$,

$$\chi'(G) \leq ch'(G) = \Delta(G) \leq \chi'(G).$$

# The Total Colouring Conjecture

**Conjecture (Total Colouring Conjecture)**

*For any graph $G$,*

$$\mathrm{ch}_T(G) \leq \Delta(G) + 2.$$

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Again, we prove a special case of this conjecture: if $\Delta(G)$ is sufficiently large compared with the treewidth of $G$, we have the stronger bound

$$\mathrm{ch}_T(G) = \Delta(G) + 1.$$

# List Edge Chromatic Number: The Proof

**Theorem**

*Let $G$ be a graph with treewidth at most $k$ and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}'(G) = \Delta(G)$.*

- Sufficient to prove that, if $G$ has treewidth at most $k$, then $\mathrm{ch}'(G) \leq \max\{\Delta(G), (k+2)2^{k+2}\}$.

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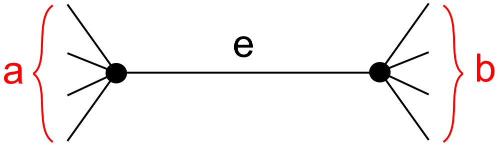
- We may assume any proper subgraph $G'$ of $G$ has $\mathrm{ch}'(G') \leq \Delta_0$.

# List Edge Chromatic Number: The Proof




$$a + b < \Delta_0$$

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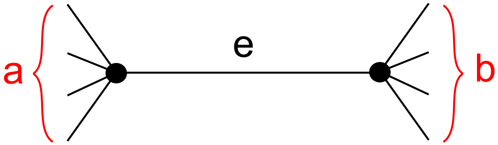


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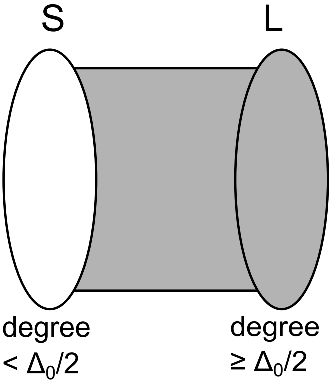
- We may assume every edge is incident with at least $\Delta_0$ others.

# List Edge Chromatic Number: The Proof

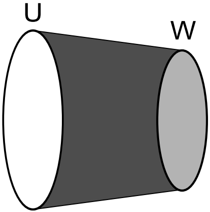
- Every edge is incident with at least one vertex in $L$.



# List Edge Chromatic Number: The Proof

We want

- $\Gamma(u) = W \ \forall u \in U$
- $|U| \geq |W|$
- $U$ independent



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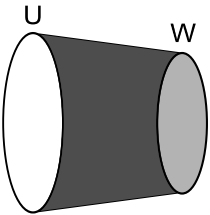


**Theorem (Galvin,1995)**

*If $G$ is a bipartite graph then* $\mathrm{ch}'(G) = \Delta(G)$.

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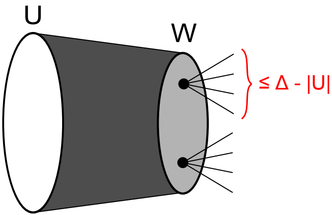


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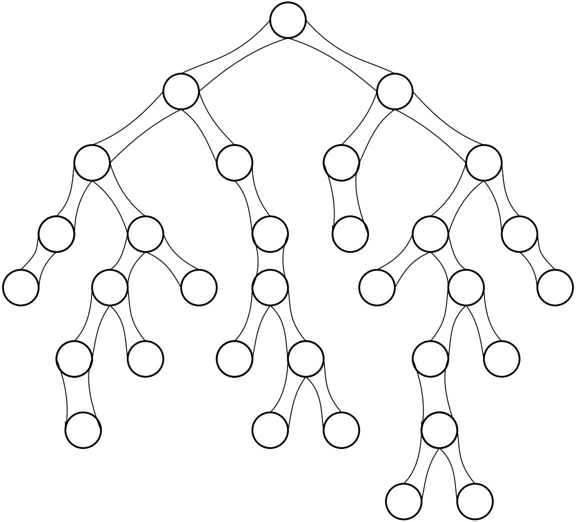

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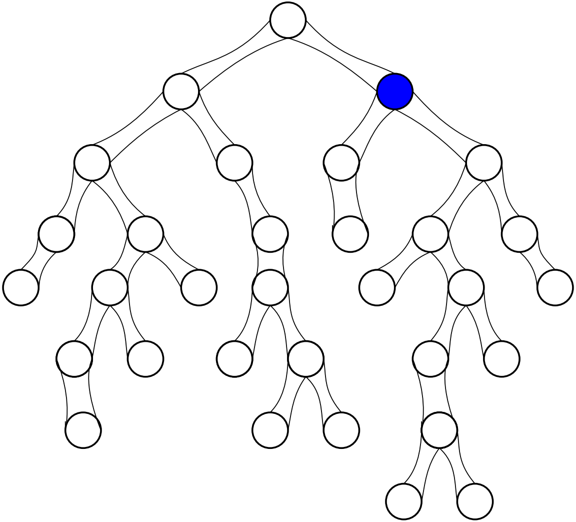

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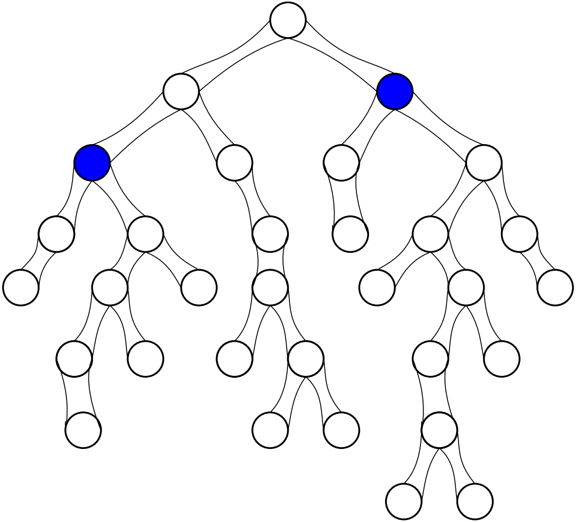

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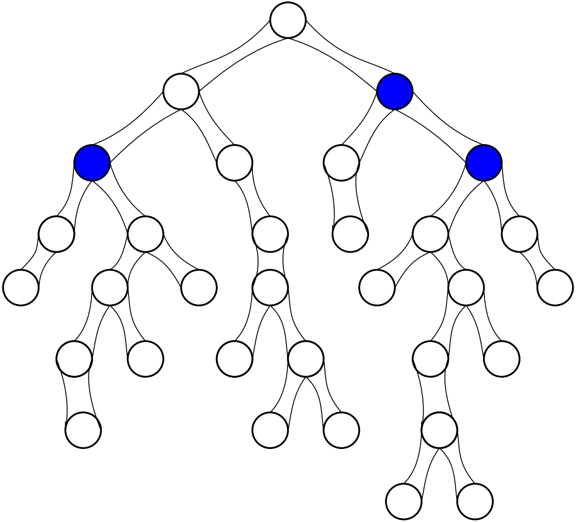

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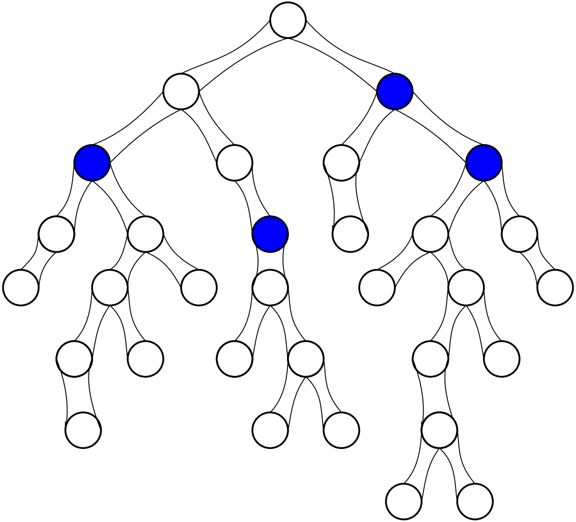

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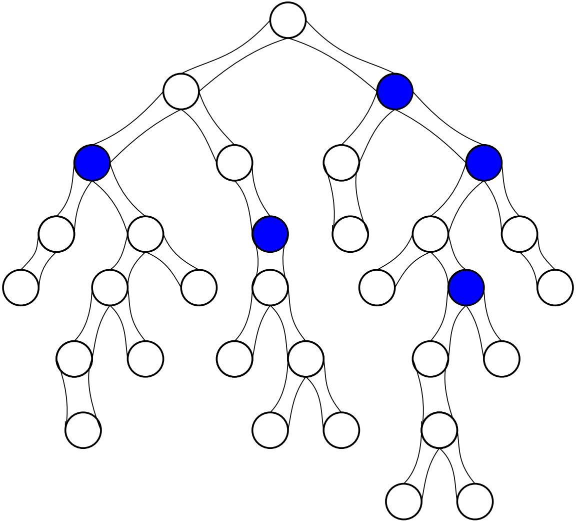

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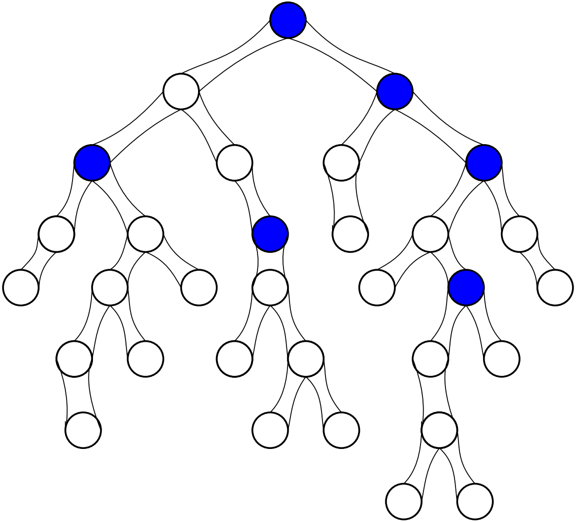

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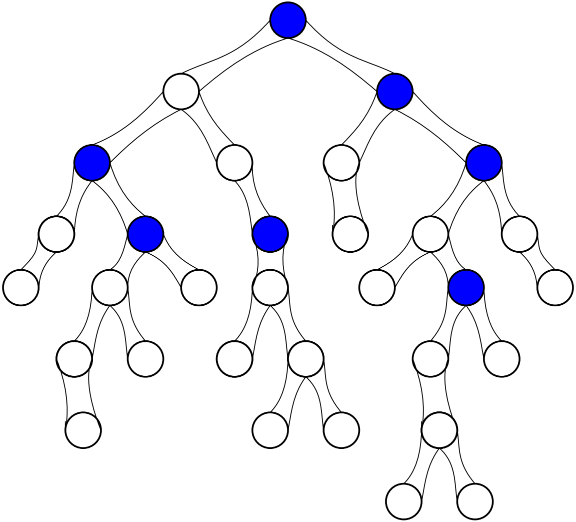

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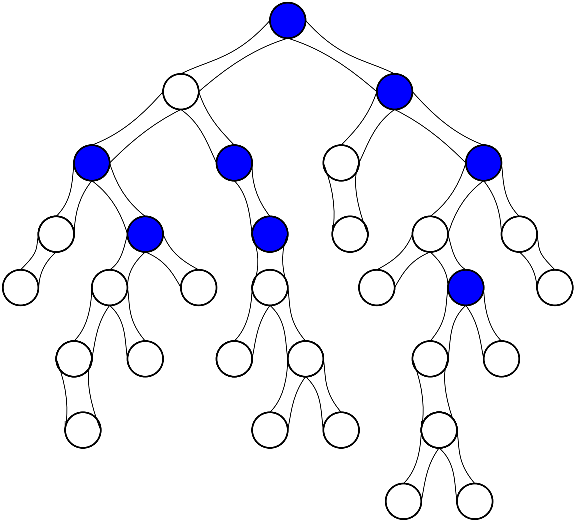

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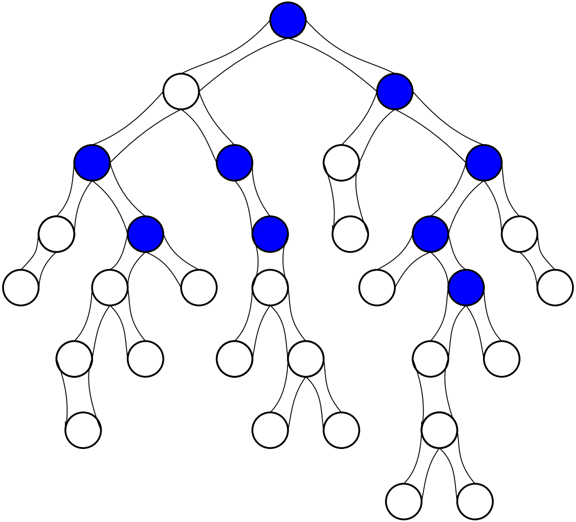

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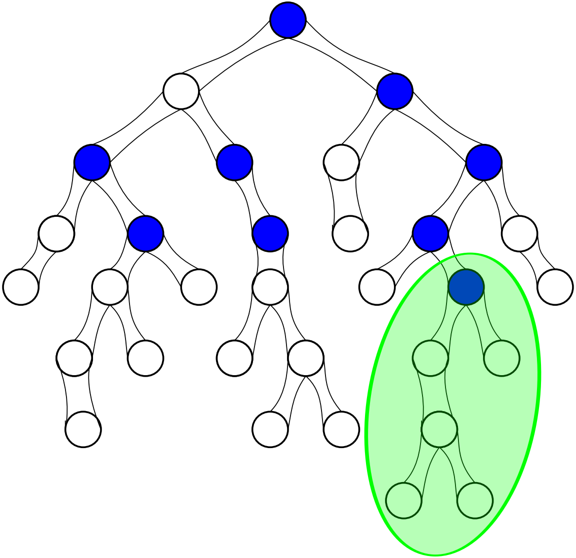

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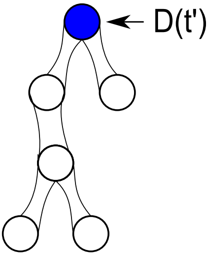

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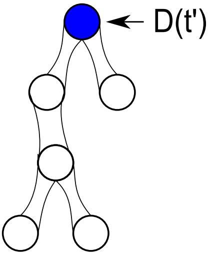


- At most $k+1$ vertices from $L$

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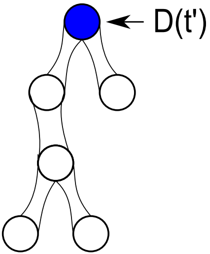


- At most $k+1$ vertices from $L$
- At least $\Delta_0/2-k$ vertices not in $D(t')$, all from $S$

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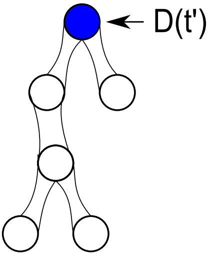


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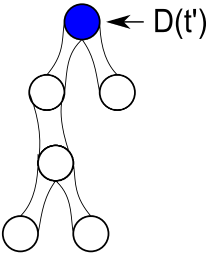


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- At most $2^{k+1}$ different neighbourhoods for these vertices
- So there exists a subset $U$ with $|U| \geq k+1$ and every vertex in $U$ having the same neighbourhood $W$ ($|W| \leq k+1$)

# List Total Chromatic Number: The Proof

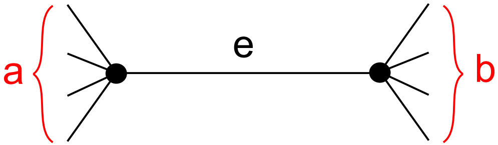
**Theorem**

*Let $G$ be a graph with treewidth at most $k$ and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}_T(G) = \Delta(G) + 1$.*

- Sufficient to prove that, if $G$ has treewidth at most $k$, then $\mathrm{ch}_T(G) \leq \max\{\Delta(G), (k+2)2^{k+2}\} + 1 = \Delta_0 + 1$.
- As before, fix an edge-minimal counterexample $(G, \mathcal{L} = \{L_e : e \in E\})$.

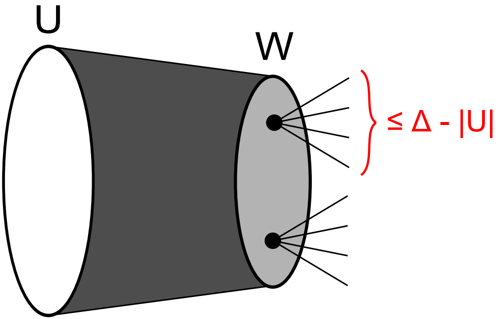
# List Total Chromatic Number: The Proof

- Here we may assume every edge is incident with at least $\Delta_0 - 1$ others.




$$a + b < \Delta_0 - 1$$

# Total Colouring



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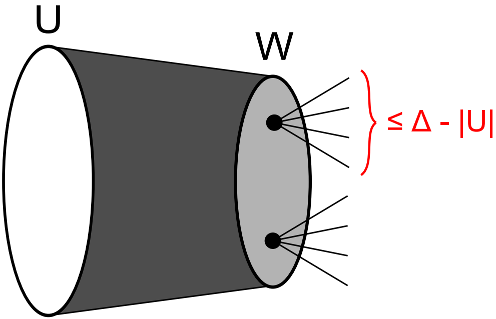


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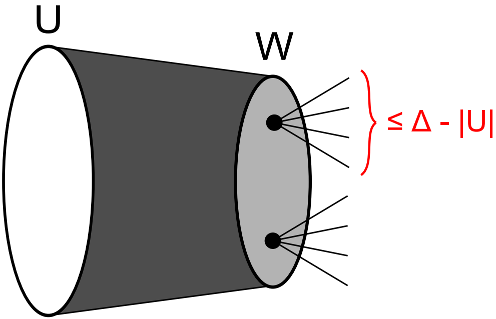


- Extend colouring to edges as before
- Each vertex $u \in U$ has a list of $\Delta_0 > 2(k+1) \geq 2d(u)$ colours

# Complexity Results

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| Edge colouring | NP-c | FPT | W[1]-hard | |
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# Complexity Results

**Theorem**

LIST EDGE CHROMATIC NUMBER *and* LIST TOTAL CHROMATIC NUMBER *are fixed parameter tractable, parameterised by the treewidth bound* $k$*, and are solvable in linear time for any fixed* $k$*.*

# Complexity Results

**Theorem**

List Edge Chromatic Number *and* List Total Chromatic Number *are fixed parameter tractable, parameterised by the treewidth bound $k$, and are solvable in linear time for any fixed $k$.*

- If $\Delta(G) \geq (k+2)2^{k+2}$, we know the value of $\mathrm{ch}'(G)$ and $\mathrm{ch}_T(G)$.
- It remains to deal with the case in which the maximum degree is bounded by a function of the treewidth $k$.

# Line Graphs and Total Graphs

- Given a graph $G = (V, E)$, the line graph $L(G)$ of $G$ is $(E, \{ef : e, f \in E \text{ and } e, f \text{ incident in } G\})$.
- A proper edge colouring of $G$ corresponds to a proper vertex colouring of $L(G)$.

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- A proper edge colouring of $G$ corresponds to a proper vertex colouring of $L(G)$.
- Given a graph $G = (V, E)$, the total graph $T(G)$ of $G$ has vertex set $V \cup E$ and edge set

$$
\begin{aligned}
&E \cup \{ef : e, f \in E \text{ and } e, f \text{ incident in } G\} \\
&\quad \cup \{ve : v \in V, e \in E, e \text{ incident with } v\}).
\end{aligned}
$$

- A proper total colouring of $G$ corresponds to a proper vertex colouring of $T(G)$.

# Line Graphs and Total Graphs

**Proposition**

*If $G$ has treewidth $k$ and maximum degree at most $\Delta$, then $L(G)$ has treewidth at most $(k+1)\Delta$.*

- Suppose $(T, \mathcal{D} = \{D(t) : t \in T\})$ is a width $k$ tree decomposition for $G$.
- Set $D'(t) = \{e \in E : e$ has an endpoint in $D(t)\}$, and $\mathcal{D}' = \{D'(t) : t \in T\}$.
- $(T, \mathcal{D}')$ is a tree decomposition for $L(G)$.
- $\max_{t \in T} |D'(t)| \leq \max_{t \in T} |D(t)| \cdot \Delta \leq (k+1)\Delta$.

# Line Graphs and Total Graphs

**Proposition**

*If $G$ has treewidth $k$ and maximum degree at most $\Delta$, then $T(G)$ has treewidth at most $(k+1)(\Delta+1)$.*

- Suppose $(T, \mathcal{D} = \{D(t) : t \in T\})$ is a width $k$ tree decomposition for $G$.
- Set $D'(t) = D(t) \cup \{e \in E : e \text{ has an endpoint in } D(t)\}$, and $\mathcal{D}' = \{D'(t) : t \in T\}$.
- $(T, \mathcal{D}')$ is a tree decomposition for $T(G)$.
- $\max_{t \in T} |D'(t)| \leq \max_{t \in T}\{|D(t)| + |D(t)| \cdot \Delta \leq (k+1)(\Delta+1)$.

# Bounded Maximum Degree

- If both the treewidth and maximum degree of $G$ are bounded, computing $\mathrm{ch}'(G)$ or $\mathrm{ch}_T(G)$ is equivalent to computing $ch(H)$ for a graph $H$ of bounded treewidth ($H = L(G)$ or $T(G)$).
- This can be done in linear time:

**Theorem (Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider and Thomassen, 2011)**

*The* LIST CHROMATIC NUMBER *problem, parameterised by the treewidth bound $k$, is fixed-parameter tractable, and solvable in linear time for any fixed $k$.*

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Suppose we are given $G$ together with a tree decomposition $(T, \mathcal{D})$ of width $k$.

1. Determine whether $\Delta(G) \geq (k+2)2^{k+2}$.
2. If $\Delta(G) \geq (k+2)2^{k+2}$ we know $\mathrm{ch}'(G) = \Delta(G)$ and $\mathrm{ch}_T(G) = \Delta(G) + 1$.
3. Otherwise, $L(G)$ and $T(G)$ have bounded treewidth.
   - Compute a bounded width tree decomposition for $L(G)$ or $T(G)$.
   - Solve List Chromatic Number for $L(G)$ or $T(G)$ in linear time.

# Parameterised complexity of colouring problems - again!

| | General problem | Parameter treewidth | List version, parameter treewidth | List Chromatic number, parameter treewidth |
|---|---|---|---|---|
| Vertex colouring | NP-c | FPT | W[1]-hard | FPT |
| Edge colouring | NP-c | FPT | W[1]-hard | FPT |
| Total colouring | NP-c | FPT | W[1]-hard | FPT |

THANK YOU