Parameterised Subgraph Counting Problems

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Joint work with Mark Jerrum (QMUL)

What is a counting problem?

Decision problems

Given a graph G, does G contain a Hamilton cycle?

Given a bipartite graph G, does G contain a perfect matching?

What is a counting problem?

Decision problems	**Counting problems**
Given a graph G, does G contain a Hamilton cycle?	How many Hamilton cycles are there in the graph G?
Given a bipartite graph G, does G contain a perfect matching?	How many perfect matchings are there in the bipartite graph G?

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- Measure running time in terms of a parameter as well as the total input size
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  - How many vertex-covers of size k are there in G?
  - How many k-cliques are there in G?
  - Given a graph G of treewidth at most k, how many Hamilton cycles are there in G?

The theory of parameterised counting

Efficient algorithms: Fixed parameter tractable (FPT)
Running time $f(k) \cdot n^{O(1)}$

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Intractable problems: #W[1]-hard
A #W[1]-complete problem: **p-#**$\textsc{Clique}$.

#W[1]-completeness

- To show the problem Π' (with parameter κ') is #W[1]-hard, we give a reduction from a problem Π (with parameter κ) to Π'.

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- An fpt Turing reduction from (Π, κ) to (Π', κ') is an algorithm A with an oracle to Π' such that
  1. A computes Π,
  2. A is an fpt-algorithm with respect to κ, and
  3. there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all oracle queries "$\Pi'(y) =?$" posed by A on input x we have $\kappa'(y) \leq g(\kappa(x))$.

  In this case we write $(\Pi, \kappa) \ \leq_T^{fpt} \ (\Pi', \kappa')$.

Subgraph Counting Model

Let Φ be a family $(\phi_1, \phi_2, \ldots)$ of functions, such that ϕ_k is a mapping from labelled graphs on k-vertices to $\{0, 1\}$.

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p-#INDUCED SUBGRAPH WITH PROPERTY(Φ) (ISWP(Φ))
Input: A graph $G = (V, E)$ and an integer k.
Parameter: k.
Question: What is the cardinality of the set

$$\{(v_1, \ldots, v_k) \in V^k : v_1, \ldots, v_k \text{ all distinct, } \\ \text{and } \phi_k(G[v_1, \ldots, v_k]) = 1\}?$$

Examples

- **p**-#$\textsc{Sub}(\mathcal{H})$

  e.g. **p**-#$\textsc{Clique}$, **p**-#$\textsc{Path}$, **p**-#$\textsc{Cycle}$, **p**-#$\textsc{Matching}$

Examples

- **p-**$\#$SUB($\mathcal{H}$)

  e.g. **p-**$\#$CLIQUE, **p-**$\#$PATH, **p-**$\#$CYCLE, **p-**$\#$MATCHING

- **p-**$\#$CONNECTED INDUCED SUBGRAPH

Examples

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- **p-#**$\textsc{Connected Induced Subgraph}$

- **p-#**$\textsc{Planar Induced Subgraph}$

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- **p-#**$\textsc{Planar Induced Subgraph}$

- **p-#**$\textsc{Even Induced Subgraph}$
  p-#$\textsc{Odd Induced Subgraph}$

Complexity Questions

- Is the corresponding decision problem in FPT?
- Is there a fixed parameter algorithm for **p-**#Induced Subgraph With Property(Φ)?
- Can we approximate **p-**#Induced Subgraph With Property(Φ) efficiently?

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Approximation Algorithms

An FPTRAS for a parameterised counting problem Π with parameter k is a randomised approximation scheme that takes an instance I of Π (with $|I| = n$), and numbers $\epsilon > 0$ and $0 < \delta < 1$, and in time $f(k) \cdot g(n, 1/\epsilon, \log(1/\delta))$ (where f is any function, and g is a polynomial in n, $1/\epsilon$ and $\log(1/\delta)$) outputs a rational number z such that

$$\mathbb{P}[(1-\epsilon)\Pi(I) \leq z \leq (1+\epsilon)\Pi(I)] \geq 1-\delta.$$

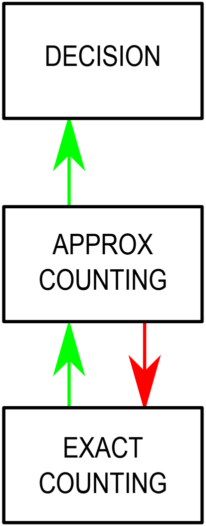
The Diagram of Everything

DECISION

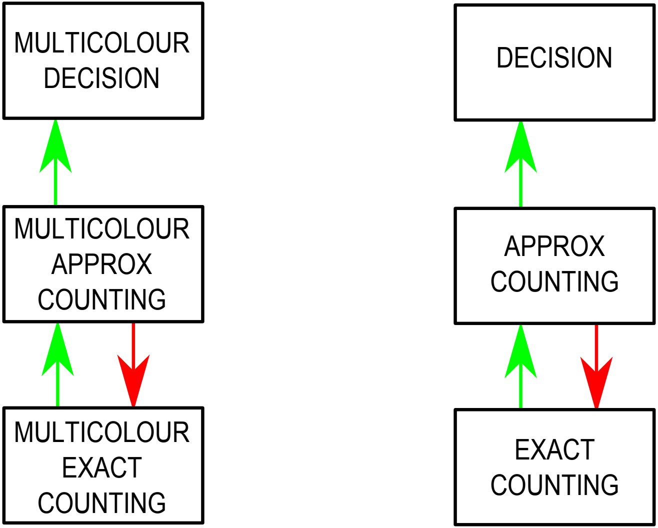
APPROX COUNTING

EXACT COUNTING

The Diagram of Everything



The Diagram of Everything



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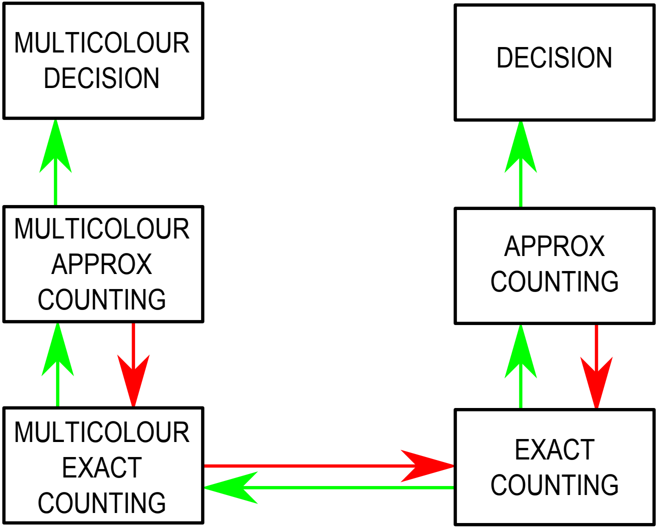

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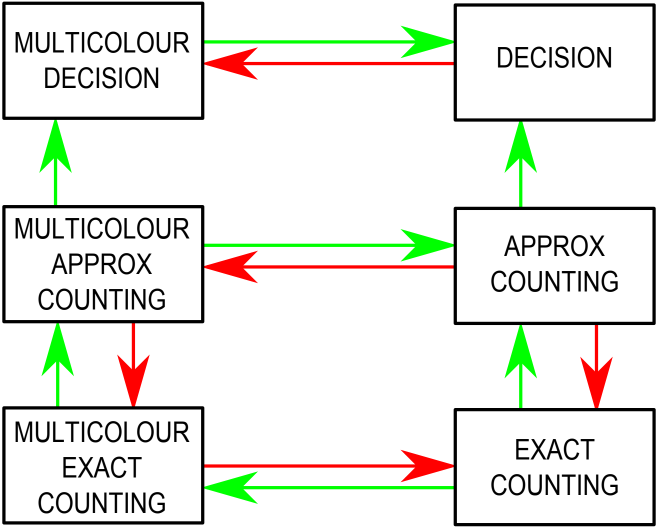

Monotone properties I: **p-#**$\textsc{Sub}(\mathcal{H})$

Theorem (Arvind & Raman, 2002)

There is an FPTRAS for **p-#**$\textsc{Sub}(\mathcal{H})$ *whenever all graphs in* $\mathcal{H}$ *have bounded treewidth.*

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Theorem (Curticapean & Marx, 2014)

p-#$\textsc{Sub}(\mathcal{H})$ *is in FPT if all graphs in* $\mathcal{H}$ *have bounded vertex-cover number; otherwise* **p-#**$\textsc{Sub}(\mathcal{H})$ *is* $\#$*W[1]-complete.*

Monotone properties II: properties with more than one minimal element

Theorem (Jerrum & M.)

Let Φ be a monotone property, and suppose that there exists a constant t such that, for every $k \in \mathbb{N}$, all minimal graphs satisfying ϕ_k have treewidth at most t. Then there is an FPTRAS for **p-#**INDUCED SUBGRAPH WITH PROPERTY(Φ).

Monotone properties II: properties with more than one minimal element

Theorem (M.)

Suppose that there is no constant t such that, for every $k \in \mathbb{N}$, all minimal graphs satisfying ϕ_k have treewidth at most t. Then **p-#**$\textsc{Induced Subgraph With Property}(\Phi)$ *is #W[1]-complete.*

Monotone properties II: properties with more than one minimal element

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Theorem (Jerrum & M.)

p-#$\textsc{Connected Induced Subgraph}$ *is #W[1]-complete.*

Non-monotone properties

Theorem

Let Φ be a family $(\phi_1, \phi_2, \ldots)$ of functions ϕ_k from labelled k-vertex graphs to $\{0, 1\}$ that are not identically zero, such that the function mapping $k \mapsto \phi_k$ is computable. Suppose that

$$|\{|E(H)| : |V(H)| = k \text{ and } \Phi \text{ is true for } H\}| = o(k^2).$$

Then **p-#**Induced Subgraph With Property*(Φ) is #W[1]-complete.*

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Then **p-#**$\textsc{Induced Subgraph With Property}$*($\Phi$) is #W[1]-complete.*

E.g. **p-#**$\textsc{Planar Induced Subgraph}$, **p-#**$\textsc{Regular Induced Subgraph}$

Even induced subgraphs: FPT???

Theorem (Goldberg, Grohe, Jerrum & Thurley (2010); Lidl & Niederreiter (1983))

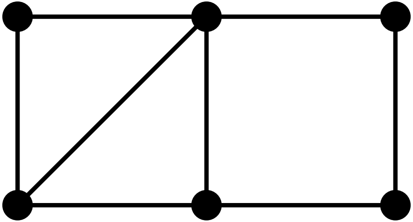
Given a graph G, there is a polynomial-time algorithm which computes the number of induced subgraphs of G having an even number of edges.

Even induced subgraphs: decision

Let G be a graph on $n \geq 2^{2k}$ vertices. Then:

- If $k \equiv 0 \mod 4$ or $k \equiv 1 \mod 4$ then G contains a k-vertex subgraph with an even number of edges.
- If $k \equiv 2 \mod 4$ then G contains a k-vertex subgraph with an even number of edges *unless* G is a clique.
- If $k \equiv 3 \mod 4$ then G contains a k-vertex subgraph with an even number of edges *unless* G is either a clique or the disjoint union of two cliques.

Even induced subgraphs: exact counting is #W[1]-complete



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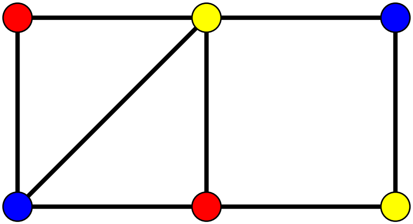

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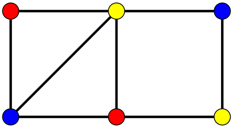

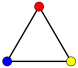

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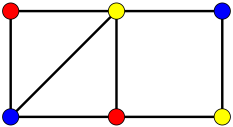

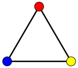

underlying structure

mask

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

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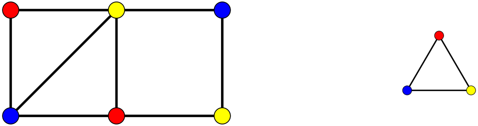

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\end{pmatrix}}^{\text{underlying structure}}\right. \cdot \begin{pmatrix}\#\\ \#\\ \#\\ \#\\ \#\\ \#\\ \#\\ \#\end{pmatrix} = \begin{pmatrix}4\\ .\\ .\\ .\\ .\\ .\\ .\end{pmatrix}
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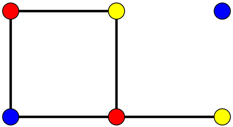

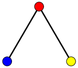

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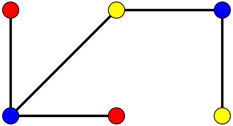
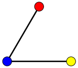
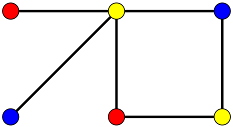
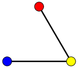




$$\text{mask}\left\{\overbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}}^{\text{underlying structure}}\right. \cdot \begin{pmatrix} \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Even induced subgraphs: exact counting is #W[1]-complete





underlying structure

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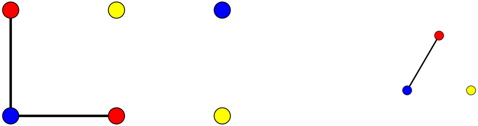
Even induced subgraphs: exact counting is #W[1]-complete



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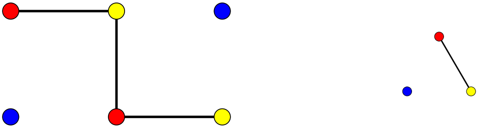
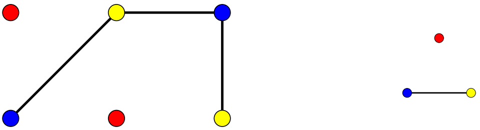


$$\text{mask}\ \overbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}}^{\text{underlying structure}} \cdot \begin{pmatrix} \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \\ \# \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 6 \\ 4 \\ 2 \\ . \end{pmatrix}$$

Even induced subgraphs: exact counting is #W[1]-complete



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\end{pmatrix}
=
\begin{pmatrix}
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\end{pmatrix}
$$

Inverting the matrix

Theorem

Let $(P, \leq)$ be a finite lattice and $f : P \to \mathbb{R}$ a function. Set S to be the upward closure of the support of f, that is,

$$S = \{x \in P : \exists y \in P \text{ with } y \leq x \text{ and } f(y) \neq 0\},$$

and suppose that $S = \{x_1, \dots, x_n\}$. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix given by $a_{ij} = f(x_i \wedge x_j)$. Then

$$\det(A) = \prod_{i=1}^{n} \sum_{\substack{x_j \leq x_i \\ x_j \in S}} f(x_j)\mu(x_j, x_i).$$

Based on results of Rajarama Bhat (1991) and Haukkanen (1996).

Even induced subgraphs: an FPTRAS

Lemma

Suppose that, for each k and any graph G on n vertices, the number of k-vertex (labelled) subgraphs of G that satisfy ϕ_k is either

1. *zero, or*
2. *at least*
$$\frac{1}{g(k)p(n)}\binom{n}{k},$$
where p is a polynomial and g is a computable function.

Then there exists an FPTRAS for **p-#**$\mathrm{ISWP}(\Phi)$.

Even induced subgraphs: an FPTRAS

Theorem

Let $k \geq 3$ and let G be a graph on $n \geq 2^{2k}$ vertices. Then either G contains no even k-vertex subgraph or else G contains at least

$$\frac{1}{2^{2k^2} k^2 n^2} \binom{n}{k}$$

even k-vertex subgraphs.

Even induced subgraphs: an FPTRAS

Theorem (Erdős and Szekeres)

Let $k \in \mathbb{N}$. Then there exists $R(k) < 2^{2k}$ such that any graph on $n \geq R(k)$ vertices contains either a clique or independent set on k vertices.

Corollary

Let $G = (V, E)$ be an n-vertex graph, where $n \geq 2^{2k}$. Then the number of k-vertex subsets $U \subset V$ such that U induces either a clique or independent set in G is at least

$$\frac{(2^{2k} - k)!}{(2^{2k})!} \frac{n!}{(n-k)!}.$$

Corollary

Let $G = (V, E)$ be an n-vertex graph, where $n \geq 2^{2k}$. Then the number of k-vertex subsets $U \subset V$ such that U induces either a clique or independent set in G is at least

$$\frac{(2^{2k}-k)!}{(2^{2k})!}\frac{n!}{(n-k)!}.$$

- If at least half of these "interesting" subsets are independent sets, we are done.
- Thus we may assume from now on that G contains at least $\frac{(2^{2k}-k)!}{2(2^{2k})!}\frac{n!}{(n-k)!}$ k-cliques.

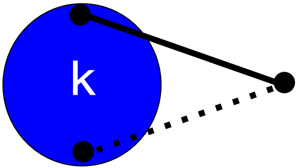
Even induced subgraphs: an FPTRAS

Definition

Let $A \subset \{1, \ldots, k\}$. We say that a k-clique H in G is A-replaceable if there are subsets $U \subset V(H)$ and $W \subset V(G) \setminus V(H)$, with $|U| = |W| \in A$, such that $G[(H \setminus U) \cup W]$ has an even number of edges.

- If every k-clique in G is $\{1, 2\}$-replaceable, we are done.
- Thus we may assume from now on that there is at least one k-clique H in G that is not $\{1, 2\}$-replaceable.
- We also assume G contains at least one even k-vertex subgraph.

Even induced subgraphs: an FPTRAS



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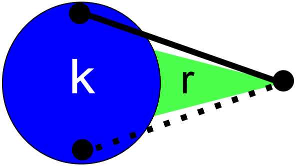

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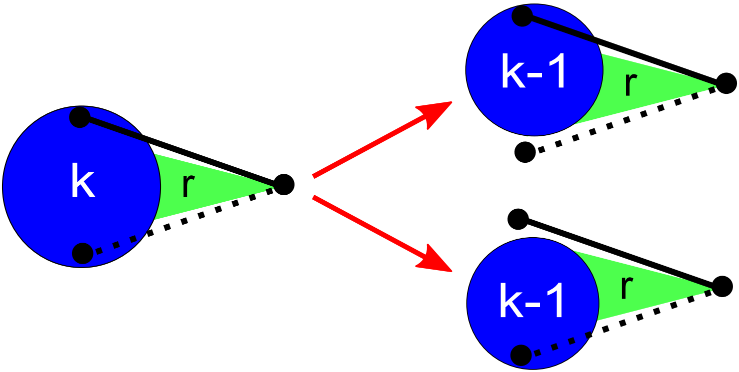
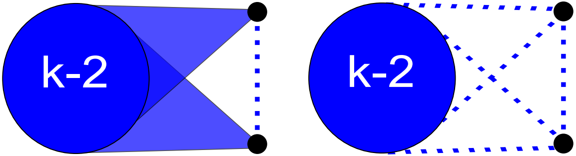


Even induced subgraphs: an FPTRAS

If $\binom{k}{2}$ is odd, the following have an even number of edges:



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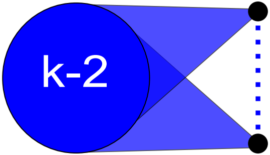
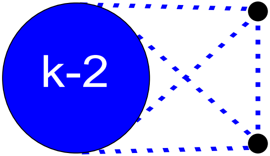
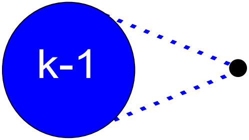
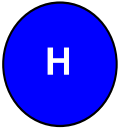




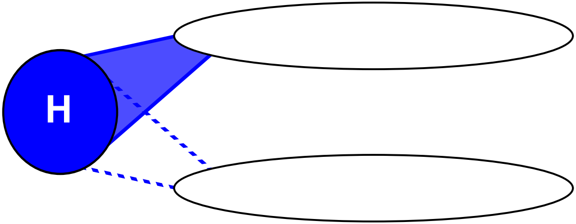
If $k \equiv 2 \mod 4$, this also has an even number of edges:



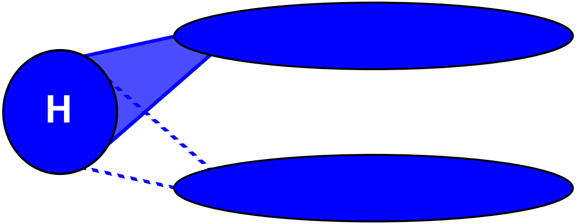
Even induced subgraphs: an FPTRAS



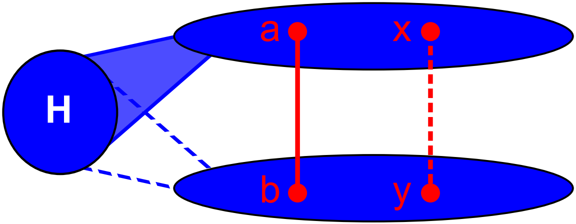
Even induced subgraphs: an FPTRAS



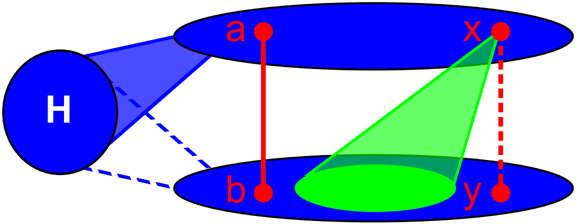
Even induced subgraphs: an FPTRAS



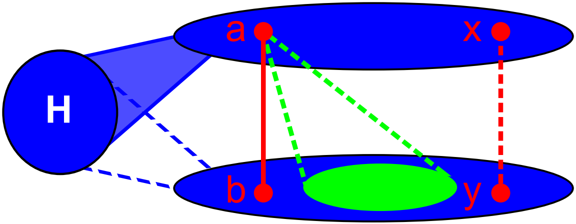
Even induced subgraphs: an FPTRAS



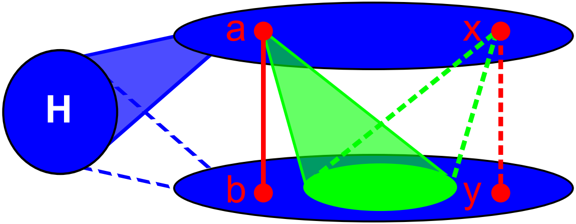
Even induced subgraphs: an FPTRAS



Even induced subgraphs: an FPTRAS



Even induced subgraphs: an FPTRAS



Open problems

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- Can similar results be obtained for properties that only hold for graphs H where

$$e(H) \equiv r \mod p,$$

for $p > 2$?

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- What if we consider an arbitrary property that depends only on the number of edges?

THANK YOU