List-colouring problems on graphs of bounded treewidth

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Parameterised Complexity

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- Analyse complexity in terms of the problem size, n, and a *parameter*, k.
- A problem is *fixed parameter tractable* (FPT) if it can be solved in time $f(k) \cdot p(n)$, where p is a polynomial and f is any computable function.
- The standard method of showing that a problem is *not* FPT is to prove that it is W[1]-hard.

Vertex colouring

- Given a graph $G = (V, E)$, $\phi : V \to \{1, \ldots, k\}$ is a *proper c-colouring* of G if, for all $uv \in E$, $\phi(u) \neq \phi(v)$.
- The *chromatic number* $\chi(G)$ of G is the smallest c such that there exists a proper c-colouring of G.

CHROMATIC NUMBER

Input: A graph $G = (V, E)$.

Question: What is $\chi(G)$?

- It is NP-complete to decide whether $\chi(G) \leq 3$.
- If G has fixed treewidth at most k, $\chi(G)$ can be computed in linear time (Arnborg and Proskurowski, 1989).

List Colouring

For graph $G(V, E)$ and a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper colouring ϕ of G such that $c(v) \in L_v$ for all $v \in V$.

LIST COLOURING

Input: A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$.

Question: Is there a proper list colouring $(G, \mathcal{L})$?

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Theorem (Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider and Thomassen, 2011)

LIST COLOURING *is W[1]-hard, parameterised by treewidth.*

List Chromatic Number

The *list chromatic number* $\mathrm{ch}(G)$ of G is the smallest integer c such that, for any assignment of lists $(L_v)_{v \in V(G)}$ to the vertices of G with $|L_v| \geq c$ for each v, there exists a proper list colouring of $(G, \mathcal{L})$.

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Theorem (Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider and Thomassen, 2011)

The List Chromatic Number *problem, parameterised by the treewidth bound k, is fixed-parameter tractable, and solvable in linear time for any fixed k.*

Edge Colouring

- Given a graph $G = (V, E)$, a *proper edge colouring* of G is an assignment of colours to the edges of G such that no two incident edges receive the same colour.
- The *edge chromatic number* $\chi'(G)$ of G is the smallest integer c such that there exists a proper edge colouring of G using c colours.
- It is NP-hard to determine whether $\chi'(G) \leq 3$ for cubic graphs (Holyer, 1981).
- $\chi'(G)$ can be computed in linear time on graphs of bounded treewidth (Zhou, Nakano and Nishizeki, 2005).

Line Graphs

- Given a graph $G = (V, E)$, the line graph $L(G)$ of G is $(E, \{ef : e, f \in E \text{ and } e, f \text{ incident in } G\})$.
- A proper edge colouring of G corresponds to a proper vertex colouring of $L(G)$.
- If G has treewidth k and maximum degree at most Δ, then $L(G)$ has treewidth at most $(k+1)\Delta$.

List Edge Colouring

For graph $G(V, E)$ and a collection of colour lists $\mathcal{L} = (L_v)_{v \in V(G)}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper list colouring ϕ of G such that $c(v) \in L_v$ for all $v \in V$.

List Edge Colouring

Input: A graph $G = (V, E)$, together with a collection of colour lists $\mathcal{L} = (L_e)_{e \in E(G)}$.

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Theorem (Zhou, Matsuo, Nishizeki, 2005)

List Edge Colouring *is NP-hard on series-parallel graphs.*

Theorem (Marx, 2005)

List Edge Colouring *is NP-hard on outerplanar graphs.*

Total Colouring

- Given a graph $G = (V, E)$, a *proper total colouring* of G is an assignment of colours to the vertices and edges of G such that
  - no two adjacent vertices receive the same colour
  - no two incident edges receive the same colour
  - no edge receives the same colour as either of its endpoints.
- The *total chromatic number* $\chi_T(G)$ of G is the smallest integer c such that there exists a proper total colouring of G using c colours.
- It is NP-hard to determine $\chi_T(G)$ for regular bipartite graphs (McDiarmid and Sánchez-Arroyo, 1994).
- $\chi_T(G)$ can be computed in linear time on graphs of bounded treewidth (Isobe, Zhou and Nishizeki, 2007).

Total Graphs

- Given a graph $G = (V, E)$, the total graph $T(G)$ of G has vertex set $V \cup E$ and edge set

$$E \cup \{ef : e, f \in E \text{ and } e, f \text{ incident in } G\} \\ \cup \{ve : v \in V, e \in E, e \text{ incident with } v\}).$$

- A proper total colouring of G corresponds to a proper vertex colouring of $T(G)$.
- If G has treewidth k and maximum degree at most Δ, then $T(G)$ has treewidth at most $(k+1)(\Delta+1)$.

List Total Colouring

For graph $G(V, E)$ and a collection of colour lists $\mathcal{L} = (L_x)_{x \in V \cup E}$, there is a proper list colouring of $(G, \mathcal{L})$ if there is a proper total colouring ϕ of G such that $c(x) \in L_x$ for all $x \in V \cup E$.

LIST TOTAL COLOURING

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Theorem (Zhou, Matsuo, Nishizeki, 2005)

List Total Colouring *is NP-hard on series-parallel graphs.*

List Edge and Total Chromatic numbers

- The *list edge chromatic number* $\mathrm{ch}'(G)$ of G is the smallest integer c such that, for any assignment of lists $(L_e)_{e\in E(G)}$ to the edges of G with $|L_e| \geq c$ for each e, there exists a proper list edge colouring of $(G, \mathcal{L})$.

$$\Delta(G) \leq \chi'(G) \leq \mathrm{ch}'(G) \leq 2\Delta(G) - 1$$

- The *list total chromatic number* ch_T of G is the smallest integer c such that, for any assignment of lists $(L_e)_{e\in E(G)}$ to the edges of G with $|L_e| \geq c$ for each e, there exists a proper list edge colouring of $(G, \mathcal{L})$.

$$\Delta(G) + 1 \leq \chi_T(G) \leq \mathrm{ch}_T(G) \leq 2\Delta(G) + 1$$

Parameterised complexity of colouring problems

	General problem	Parameter treewidth	List version, parameter treewidth	List Chromatic number, parameter treewidth
Vertex colouring	NP-c	FPT	W[1]-hard	FPT
Edge colouring				
Total colouring				

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Theorem

LIST EDGE CHROMATIC NUMBER *and* LIST TOTAL CHROMATIC NUMBER *are fixed parameter tractable, parameterised by the treewidth bound k, and are solvable in linear time for any fixed k.*

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- If G has treewidth k and bounded maximum degree, then $L(G)$ and $T(G)$ both have bounded treewidth.
- So it is possible in this case to compute the $\mathrm{ch}(L(G))$ or $\mathrm{ch}(T(G))$ in linear time.

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- If G has treewidth k and bounded maximum degree, then $L(G)$ and $T(G)$ both have bounded treewidth.
- So it is possible in this case to compute the $\mathrm{ch}(L(G))$ or $\mathrm{ch}(T(G))$ in linear time.
- It remains to consider the case that $\Delta(G)$ is very large compared with the treewidth.

Bounded treewidth and large maximum degree

Theorem

Let G be a graph with treewidth at most k and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}'(G) = \Delta(G)$.

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Let G be a graph with treewidth at most k and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}'(G) = \Delta(G)$.

- So we have

$$\Delta(G) = \mathrm{ch}'(G) \geq \chi'(G) \geq \Delta(G),$$

  and in particular $\mathrm{ch}'(G) = \chi'(G)$.
- This is a special case of the List (Edge) Colouring Conjecture, which asserts that

$$\mathrm{ch}'(G) = \chi'(G)$$

  for every graph G.

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- Sufficient to prove that, if G has treewidth at most k, then $\mathrm{ch}'(G) \leq \max\{\Delta(G), (k+2)2^{k+2}\}$.

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- Let $(G, \mathcal{L} = \{L_e : e \in E\})$ be an edge-minimal counterexample. Assume $|L_e| = \Delta_0 = \max\{\Delta(G), (k+2)2^{k+2}\}$ for each e.

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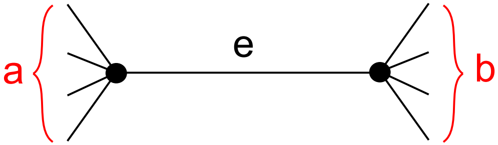
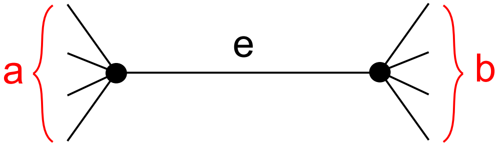
- We may assume any proper subgraph G' of G has $\mathrm{ch}'(G') \leq \Delta_0$.

Bounded treewidth and large maximum degree



$$a + b < \Delta_0$$

Bounded treewidth and large maximum degree



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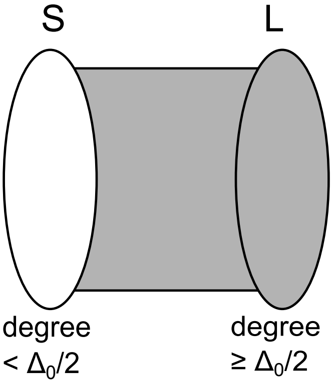
- We may assume every edge is incident with at least Δ_0 others.

Bounded treewidth and large maximum degree

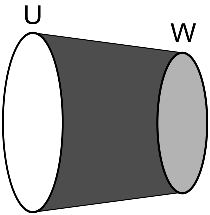
- Every edge is incident with at least one vertex in L.



Bounded treewidth and large maximum degree

We want

- $\Gamma(u) = W \;\forall u \in U$
- $|U| \geq |W|$
- U independent



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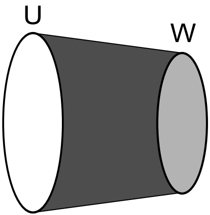

Theorem (Galvin,1995)

If G is a bipartite graph then $\mathrm{ch}'(G) = \Delta(G)$.

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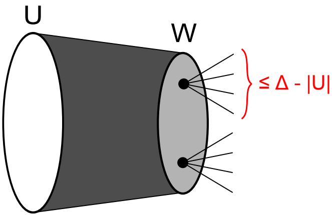

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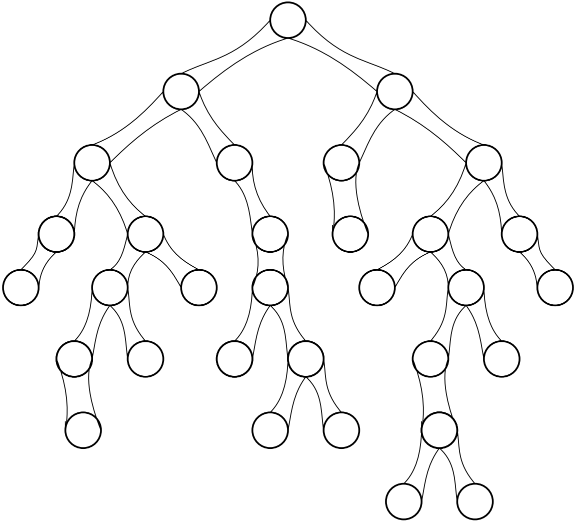

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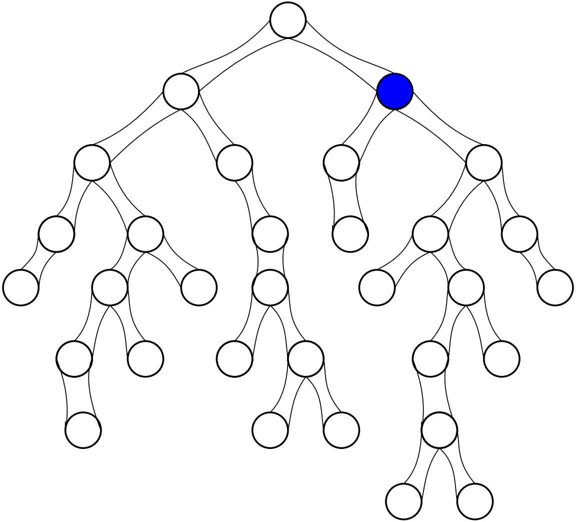

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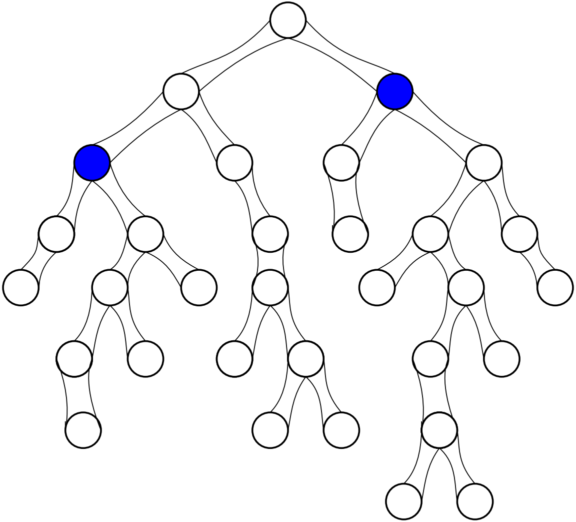

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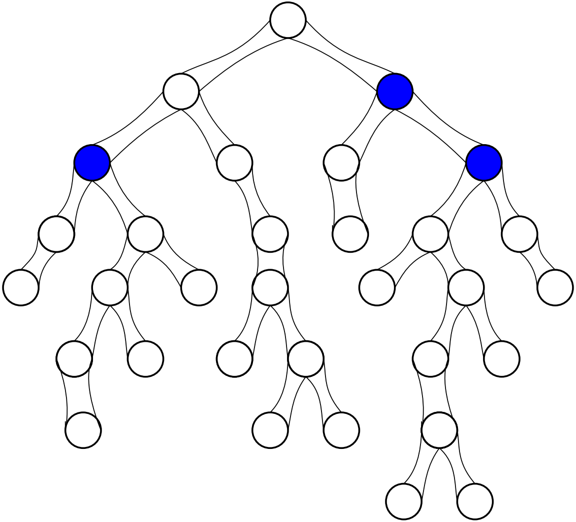

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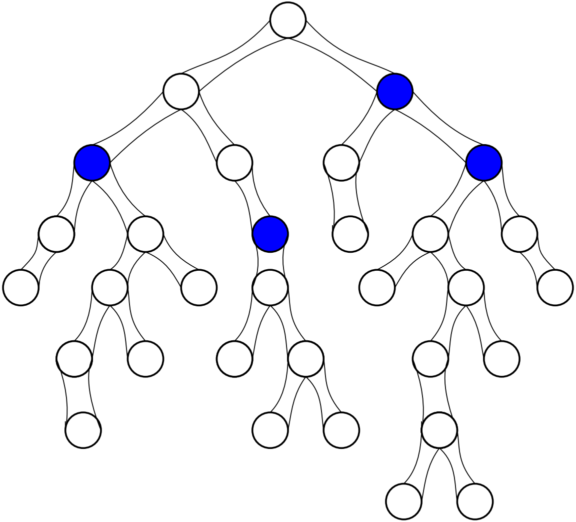

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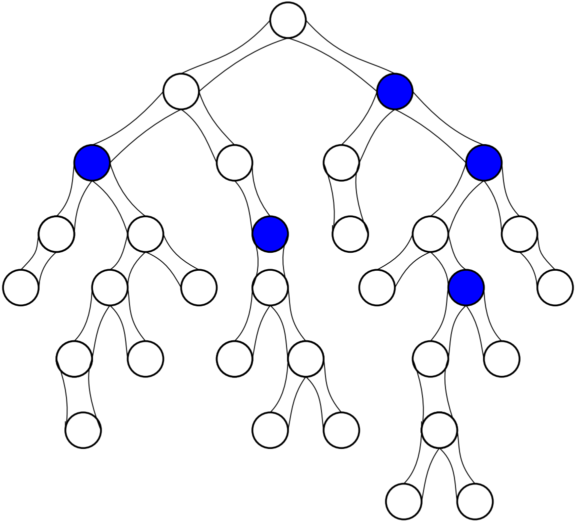

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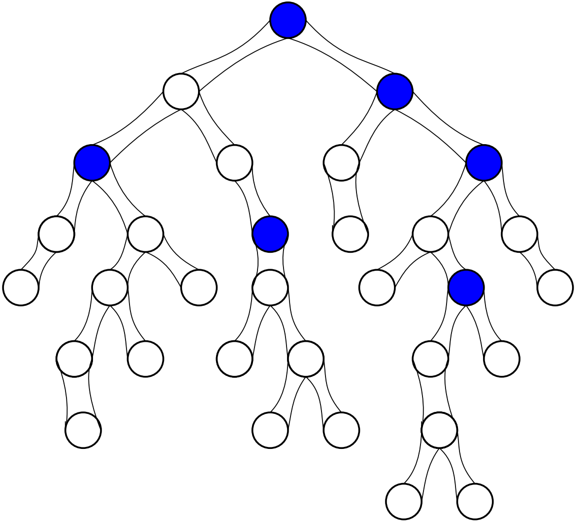

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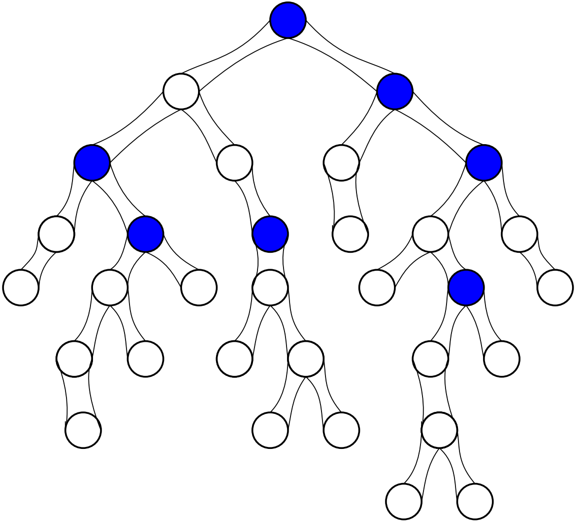

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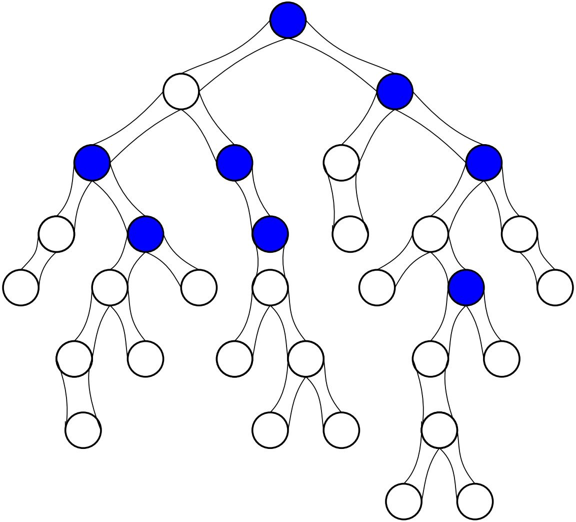

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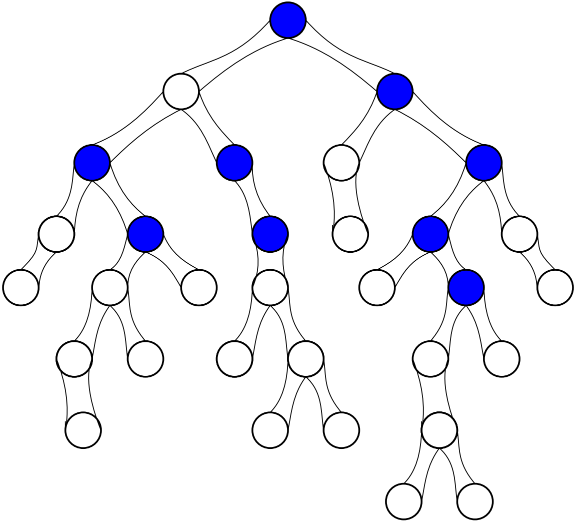

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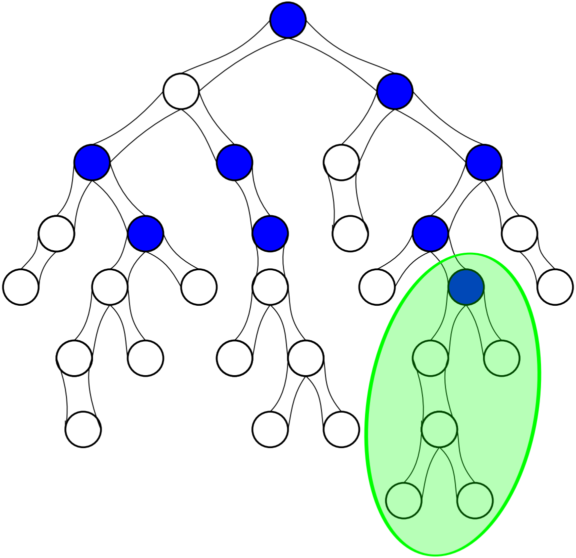

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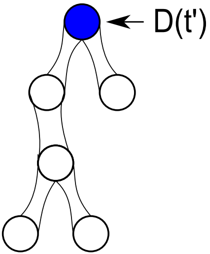

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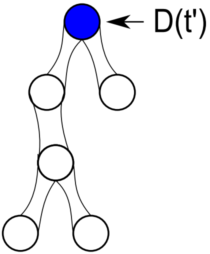
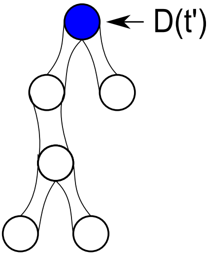


- At most $k + 1$ vertices from L

Bounded treewidth and large maximum degree



- At most $k + 1$ vertices from L
- At least $\Delta_0/2 - k$ vertices not in $D(t')$, all from S

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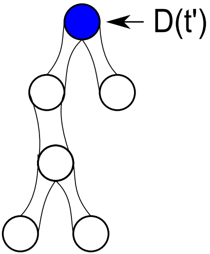

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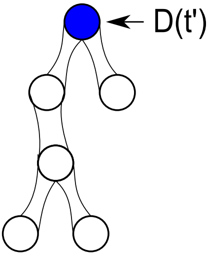

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- At least $\Delta_0/2 - k$ vertices not in $D(t')$, all from S
- At most 2^{k+1} different neighbourhoods for these vertices
- So there exists a subset U with $|U| \geq k+1$ and every vertex in U having the same neighbourhood W ($|W| \leq k+1$)

Total Colouring

Theorem

Let G be a graph with treewidth at most k and $\Delta(G) \geq (k+2)2^{k+2}$. Then $\mathrm{ch}_T(G) = \Delta(G) + 1$.

- This is a special case of the Total Colouring Conjecture, which asserts that

$$\mathrm{ch}_T(G) \leq \Delta(G) + 2$$

for every graph G.

Summary of Algorithms

Suppose we are given G together with a tree decomposition $(T, \mathcal{D})$ of width k.

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1. Determine whether $\Delta(G) \geq (k+2)2^{k+2}$.
2. If $\Delta(G) \geq (k+2)2^{k+2}$ we know $\mathrm{ch}'(G) = \Delta(G)$ and $\mathrm{ch}_T(G) = \Delta(G) + 1$.

Summary of Algorithms

Suppose we are given G together with a tree decomposition $(T, \mathcal{D})$ of width k.

1. Determine whether $\Delta(G) \geq (k+2)2^{k+2}$.
2. If $\Delta(G) \geq (k+2)2^{k+2}$ we know $\mathrm{ch}'(G) = \Delta(G)$ and $\mathrm{ch}_T(G) = \Delta(G) + 1$.
3. Otherwise, $L(G)$ and $T(G)$ have bounded treewidth.
   - Compute a bounded width tree decomposition for $L(G)$ or $T(G)$.
   - Solve List Chromatic Number for $L(G)$ or $T(G)$ in linear time.

Parameterised complexity of colouring problems - again!

	General problem	Parameter treewidth	List version, parameter treewidth	List Chromatic number, parameter treewidth
Vertex colouring	NP-c	FPT	W[1]-hard	FPT
Edge colouring	NP-c	FPT	W[1]-hard	FPT
Total colouring	NP-c	FPT	W[1]-hard	FPT

THANK YOU

http://arxiv.org/abs/1110.4077