

# Bigraphs with Sharing

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**Abstract.** Bigraphical reactive systems (BRSs) [1] are a fully graphical model for mobile computation in which both time and space are prominent. But the locality is a tree structure, space can not be shared. We extend the formalism to BRS with sharing, which models the system topology by a directed graph structure. We present a categorical characterisation of bigraphs with sharing and a complete axiomatisation for algebraic expressions of bigraphs with sharing.

## 1 Introduction

Bigraphical reactive systems (BRSs) is a recent formalism for modelling the temporal and spatial evolution of computation. They were initially introduced by Milner [1] to provide a fully graphical model capable of representing both connectivity and locality. Bigraphs and their operations have been shown to be expressible in well-known categories such as symmetric monoidal categories.

A BRS consists of a set of bigraphs and a set of reaction rules, which defines the dynamic evolution of the system by specifying how the set of bigraphs can be reconfigured. The two principal motivations that led to the development of BRSs are:

- to model directly ubiquitous systems by focusing on mobile connectivity and mobile locality [2,3];
- to provide a unification of existing theories by developing a general theory in which many existing calculi for concurrency and mobility may be represented, with a uniform behavioural theory [4,5,6,7].

However, BRSs assume an underlying model of space that is not overlapping, that is, spaces can be nested and their structure is defined by a forest. We have therefore extended the spatial model in BRS to *BRS with sharing*, where topology is defined by a directed acyclic graph (DAG). This extension allows natural modelling of systems in which space is overlapping such as signal ranges in wireless networks, offices belonging to more than one department, and overlapping biological zones and compartments.

An example bigraph with sharing is shown in Figure 1a. In the graphical notation, ovals are *nodes* that can be contained within others. Each node has a type called *control* and *ports* that may be connected by *links*. Each link may be identified by a unique (*outer*) name. Intuitively, nodes represent the spatial placement of agents while links stand for their communication capabilities.

Dashed rectangles denote *regions*. Their rôle is to describe parts of the system that are not necessarily adjacent. The grey squares are called *sites*. They can be regarded as holes in which the regions of another bigraph may be inserted via composition. Going more into detail, when a bigraph is inserted into another, its outer names are merged to the corresponding inner names of the host bigraph. It is worthwhile to remark that insertion is rendered by categorical composition.

Another peculiarity of bigraphs is the complete independence of the linking and the placing of nodes, as can be shown by the way links cross boundaries in the diagram in Figure 1a. This characteristic is formalized by defining bigraphs in terms of the constituent notions of *place graph* and *link graph*. A place graph is a DAG whose roots are the regions of the corresponding bigraph and leaves are its sites and atomic nodes. An example of place graph is drawn in Figure 1b. A node  $v$  is a parent of a node  $w$  only if  $v$  contains  $w$  in the original bigraph. A link graph consists of a hyper-graph whose vertices are the names and nodes of the corresponding bigraph and hyper-edges are its links. A formal definition of bigraphs with sharing is given in the next section.

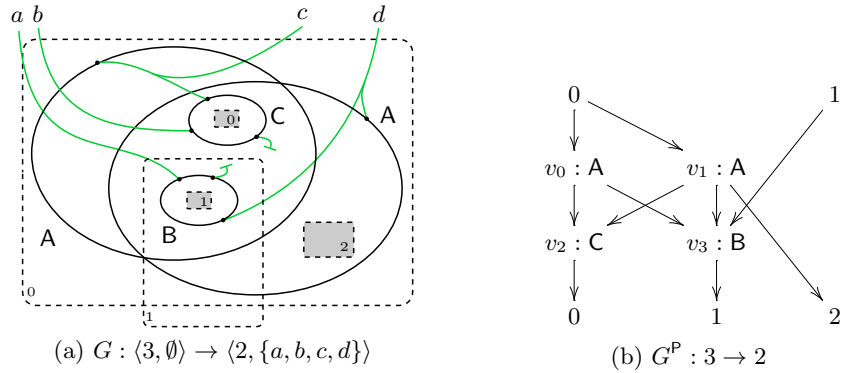


Fig. 1: An example of a bigraph (a) and the corresponding place graph (b) with node-identifiers.

## Outline

The paper is organised as follows. Bigraphs with sharing are formally defined in Section 2. In Section 3 we describe the categories used to characterise bigraphs with sharing. In Section 4 we define epimorphisms and monomorphisms in the category of place graphs with sharing. A complete axiomatisation for algebraic expressions of bigraphs with sharing is given in Section 5. Conclusions and directions for future work are in Sect. ??.

## Notation and Conventions

We treat a non-negative integer  $n$  as the finite ordinal  $n = \{0, 1, \dots, n-1\}$ . We write  $S \uplus T$  for the union of two sets  $S$  and  $T$  known or assumed to be disjoint.

In defining bigraphs we assume that names, node-identifiers and edge-identifiers are drawn from three infinite sets,  $\mathcal{X}$ ,  $\mathcal{V}$  and  $\mathcal{E}$ , disjoint from each other. An interface for bigraphs is a pair  $I = \langle m, X \rangle$  with  $m$  a finite ordinal and  $X \subset \mathcal{X}$  a finite set of names. We denote the interfaces of bigraphs by  $I, J, K$ . We call the trivial interface  $\epsilon \stackrel{\text{def}}{=} \langle 0, \emptyset \rangle$  the *origin*. If an interface  $I = \langle m, X \rangle$  has  $X = \emptyset$  we may write  $I$  as  $m$ ; if  $m = 0$  we may write it as  $X$ . When there is no ambiguity, we shall often write a name set  $\{x, y, z, \dots\}$  as  $\{xyz \dots\}$ . We write  $\mathbf{x}$  to indicate a sequence of distinct names.

We write  $\bigotimes_{i < n} G_i$  for the iterated tensor product  $G_0 \otimes \dots \otimes G_{n-1}$ . This equals  $\text{id}_\epsilon$  in case  $n = 0$ . We shall write  $G_0 G_1$  for composition, letting it bind tighter than tensor product.

## 2 Bigraphs with Sharing

We begin by extending the standard definition of place graphs (see [1, p. 15]) so that a node may have several parents in the place graph, that is, the place graph can be a DAG. Formally, place graphs with sharing are defined as follows.

**Definition 1 (concrete place graph with sharing).** A concrete place graph with sharing

$$F = (V_F, \text{ctrl}_F, \text{prnt}_F) : m \rightarrow n$$

is a triple having an inner face  $m$  and an outer face  $n$ , both finite ordinals. These index respectively the sites and roots of the place graph.  $F$  has a finite set  $V_F \subset \mathcal{V}$  of nodes, a control map  $\text{ctrl}_F : V_F \rightarrow \mathcal{K}$ , where  $\mathcal{K}$  is the signature of  $F$  (i.e. set of controls), and a parent relation

$$\text{prnt}_F \subseteq (m \uplus V_F) \times (V_F \uplus n)$$

which is acyclic i.e. if  $(v, v) \in \text{prnt}_F^i$  for some  $v \in V_F$  then  $i = 0$ .

Then, according to the new definition, it is possible to have  $(v, \_) \notin \text{prnt}_F$  and  $(v, w_i) \in \text{prnt}_F$  for for some  $v \in m \uplus V_F$  and some  $w_i \in n \uplus V_F$ , with  $i \geq 0$ . Any place (i.e. node, root or site) having more than one parent is said to be shared. A place with no children is called *idle*. Two places with a common parent are called *siblings*.

*Example 1.* Consider place graph with sharing  $G^P : 3 \rightarrow 2$  drawn in Figure 1b. The node set is  $V_G = \{v_i \mid i < 4\}$ , the control map is  $\text{ctrl}_G = \{(v_0, \mathbf{A}), (v_1, \mathbf{A}), (v_2, \mathbf{C}), (v_3, \mathbf{B})\}$  and the parent relation is

$$\begin{aligned} \text{prnt}_G = \{ & (0, v_2), (1, v_3), (2, v_1), (v_2, v_0), (v_2, v_1), \\ & (v_3, v_0), (v_3, v_1), (v_3, 1), (v_0, 0), (v_1, 0) \} . \end{aligned}$$

Composition for place graphs with sharing is extended in order to allow DAGs. It is now based on composition of binary relations.

**Definition 2 (composition).** *If  $F : k \rightarrow m$  and  $G : m \rightarrow n$  are two place graphs with sharing with  $V_F \cap V_G = \emptyset$ , their composite*

$$GF = (V, ctrl, prnt) : k \rightarrow n$$

has nodes  $V = V_F \uplus V_G$  and control map

$$ctrl(v) \stackrel{def}{=} \begin{cases} ctrl_F(v) & \text{if } v \in V_F, \\ ctrl_G(v) & \text{if } v \in V_G. \end{cases}$$

Its parent relation  $prnt \subseteq (k \uplus V_F \uplus V_G) \times (V_F \uplus V_G \uplus n)$  is defined as follows:

$$prnt_G^{\triangleleft} \cup prnt_F^{\triangleright} \cup R_{G \circ F}$$

where

$$\begin{aligned} prnt_G^{\triangleleft} &\stackrel{def}{=} \{(v, w) \mid (v, w) \in prnt_G \text{ and } v \in V_G\} \\ prnt_F^{\triangleright} &\stackrel{def}{=} \{(v, w) \mid (v, w) \in prnt_F \text{ and } w \in V_F\} \\ R_{G \circ F} &\stackrel{def}{=} (prnt_G \setminus prnt_G^{\triangleleft})(prnt_F \setminus prnt_F^{\triangleright}) . \end{aligned}$$

*Example 2.* Let place graphs with sharing  $G : 2 \rightarrow 1$ ,  $F : 2 \rightarrow 2$  and their composition  $GF$  as in Figure 2. The parent relation for place graphs  $G$  and  $F$  is

$$\begin{aligned} prnt_G &= \{(0, v_0), (1, v_1), (v_2, v_0), (v_0, 0), (v_1, 0)\} \\ prnt_F &= \{(0, 0), (0, w_0), (1, w_2), (w_0, 0), (w_0, 1), (w_2, 0), (w_2, w_1), (w_1, 1)\} . \end{aligned}$$

To construct the parent relation for the composed place graph  $GF$  we first define the following relations:

$$\begin{aligned} prnt_G^{\triangleleft} &\subseteq V_G \times (V_G \uplus 1) = \{(v_2, v_0), (v_0, 0), (v_1, 0)\} \\ prnt_F^{\triangleright} &\subseteq (2 \uplus V_F) \times V_F = \{(0, w_0), (1, w_2), (w_2, w_1)\} \\ R_{G \circ F} &\subseteq (2 \uplus V_F) \times (V_G \uplus 1) = \{(0, v_0), (w_0, v_0), (w_0, v_1), (w_2, v_0), (w_1, v_1)\} . \end{aligned}$$

Their union gives rise to the parent relation

$$prnt = prnt_G^{\triangleleft} \cup prnt_F^{\triangleright} \cup R_{G \circ F} .$$

We now prove associativity of composition for concrete place graphs with sharing. We are going to use this result in the next section.

**Proposition 1 (associativity of composition).** *If  $A : m \rightarrow n$ ,  $B : k \rightarrow m$ ,  $C : h \rightarrow k$  are three concrete place graphs with sharing with disjoint node sets, then*

$$A(BC) = (AB)C .$$

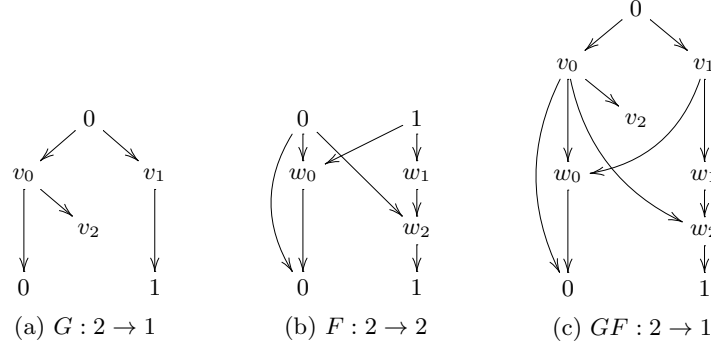


Fig. 2: Example of composition.

*Proof.* Let us define  $A(BC) = G_0$  and  $(AB)C = G_1$ . Since the node sets  $V_A, V_B, V_C$  are all disjoint and

$$\text{dom}(A) = \text{cod}(B) = \text{cod}(BC) \quad \text{dom}(AB) = \text{dom}(B) = \text{cod}(C) ,$$

then by Definition 2 all the composition are defined. We have to prove that  $G_0 = G_1$ . Again by Definition 2,  $G_0, G_1 : h \rightarrow n$ ,  $V_{G_0} = V_{G_1} = V_A \uplus V_B \uplus V_C$  and for every node  $v$

$$\text{ctrl}_{G_0}(v) = \text{ctrl}_{G_1}(v) = \begin{cases} \text{ctrl}_A(v) & \text{if } v \in V_A , \\ \text{ctrl}_B(v) & \text{if } v \in V_B , \\ \text{ctrl}_C(v) & \text{otherwise .} \end{cases}$$

It remains to prove that  $\text{prnt}_{G_0} = \text{prnt}_{G_1}$ . Since both relations are subsets of  $(h \uplus V_{G_0}) \times (V_{G_0} \uplus n)$ , we have to show that  $(v, w) \in \text{prnt}_{G_0}$  if and only if  $(v, w) \in \text{prnt}_{G_1}$  for every element  $(v, w)$ . The parent relations are defined as

$$\text{prnt}_{G_0} = \text{prnt}_A^{\triangleleft} \cup \text{prnt}_{BC}^{\triangleright} \cup R_{A \circ BC} \quad (1)$$

$$\text{prnt}_{G_1} = \text{prnt}_{AB}^{\triangleleft} \cup \text{prnt}_C^{\triangleright} \cup R_{AB \circ C} . \quad (2)$$

To analyse the single components we compute the parent relations for the compositions  $BC$  and  $AB$

$$\text{prnt}_{BC} = \text{prnt}_B^{\triangleleft} \cup \text{prnt}_C^{\triangleright} \cup R_{B \circ C}$$

$$\text{prnt}_{AB} = \text{prnt}_A^{\triangleleft} \cup \text{prnt}_B^{\triangleright} \cup R_{A \circ B} .$$

Therefore,

$$\begin{aligned}
prnt_{BC}^{\triangleright} &= \{(v, w) \mid (v, w) \in prnt_{BC}, w \in V_B \uplus V_C\} \\
&= \{(v, w) \mid (v, w) \in prnt_{BC}, w \in V_B\} \\
&\quad \cup \{(v, w) \mid (v, w) \in prnt_{BC}, w \in V_C\} \\
&= \{(v, w) \mid (v, w) \in prnt_{BC}, v \in h \uplus V_C, w \in V_B\} \\
&\quad \cup \{(v, w) \mid (v, w) \in prnt_B, v \in V_B, w \in V_B\} \\
&\quad \cup \{(v, w) \mid (v, w) \in prnt_C, w \in V_C\} \\
&= R_{B \circ C}^{\triangleright} \cup prnt_B^{\boxplus} \cup prnt_C^{\triangleright}
\end{aligned}$$

and similarly

$$prnt_{AB}^{\triangleleft} = prnt_A^{\triangleleft} \cup prnt_B^{\boxplus} \cup R_{A \circ B}^{\triangleleft} .$$

Then (1) and (2) can be rewritten as

$$\begin{aligned}
prnt_{G_0} &= prnt_A^{\triangleleft} \cup prnt_B^{\boxplus} \cup prnt_C^{\triangleright} \cup R_{B \circ C}^{\triangleright} \cup R_{A \circ BC} \\
prnt_{G_1} &= prnt_A^{\triangleleft} \cup prnt_B^{\boxplus} \cup prnt_C^{\triangleright} \cup R_{A \circ B}^{\triangleleft} \cup R_{AB \circ C} .
\end{aligned}$$

Hence, to prove that  $prnt_{G_0} = prnt_{G_1}$  we have to show that

$$R_{A \circ BC} \cup R_{B \circ C}^{\triangleright} = R_{A \circ B}^{\triangleleft} \cup R_{AB \circ C}$$

holds. We start by proving  $\Rightarrow$ . We have the following cases:

1. If  $(v, w) \in R_{A \circ BC}$  then there exists a  $w' \in m$  such that  $(v, w') \in prnt_{BC}$  and  $(w', w) \in prnt_A$ , where  $v \in h \uplus V_C \uplus V_B$  and  $w \in V_A \uplus n$ . There are two sub-cases:
  - (a) If  $v \in V_B$  then  $(v, w') \in prnt_B$ . Therefore,  $(v, w) \in R_{A \circ B}^{\triangleleft}$ .
  - (b) If  $v \in h \uplus V_C$  then there exists a  $w'' \in k$  such that  $(v, w'') \in prnt_C$  and  $(w'', w') \in prnt_B$ . But then  $(w'', w) \in prnt_{AB}$ . It follows that  $(v, w) \in R_{AB \circ C}$ .
2. If  $(v, w) \in R_{B \circ C}^{\triangleright}$  then there exists a  $w' \in k$  such that  $(v, w') \in prnt_C$  and  $(w', w) \in prnt_B$ , where  $v \in h \uplus V_C$  and  $w \in V_B$ . But we also have that  $(w', w) \in prnt_{AB}$ . Hence,  $(v, w) \in R_{AB \circ C}$ .

The proof for  $\Leftarrow$  is symmetric. This concludes the proof.

It remains to define identities, tensor product and symmetries for concrete place graphs with sharing. As in [1], identities and symmetries are special classes of bijective relations from an ordinal to itself, while tensor product  $G_0 \otimes G_1$  is performed by placing  $G_0$  and  $G_1$  side-by-side.

**Definition 3 (identities).** *The identity place graph at  $m$  is*

$$id_m \stackrel{def}{=} (\emptyset, \emptyset, id_m) : m \rightarrow m$$

where  $id_m = \{(i, i) \mid i < m\}$  is the identity relation on  $m$ .

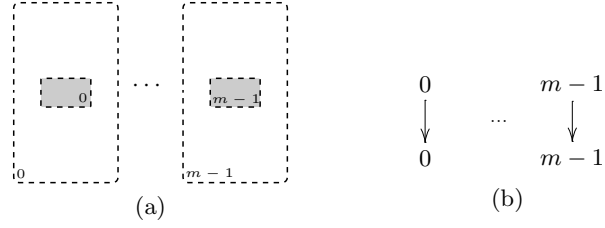


Fig. 3: An example of identity  $\text{id}_m$  (a) and the corresponding place graph (b).

An example of identity is drawn in Figure 3. We now prove that identities as defined above, are the neutral elements for composition of concrete place graphs with sharing.

**Proposition 2 (neutral elements for composition).** *For any concrete place graph with sharing  $G : m \rightarrow n$  the following holds*

$$G \text{id}_m = G = \text{id}_n G .$$

*Proof.* All the compositions are defined since

$$\text{dom}(G) = \text{cod}(\text{id}_m) \qquad \text{dom}(\text{id}_n) = \text{cod}(G)$$

and  $V_{\text{id}_m} = V_{\text{id}_n} = \emptyset$ . The composite  $G \text{id}_m = (V, \text{ctrl}, \text{prnt})$  is defined according to 2. In particular, we have  $V = V_G \uplus \emptyset$ ,  $\text{ctrl} = \text{ctrl}_G$  and

$$\text{prnt} = \text{prnt}_G^{\triangleleft} \cup \text{prnt}_{\text{id}_m}^{\triangleright} \cup R_{G \circ \text{id}_m} .$$

But,  $\text{prnt}_{\text{id}_m}^{\triangleright} = \emptyset$  and

$$R_{G \circ \text{id}_m} = (\text{prnt}_G \setminus \text{prnt}_G^{\triangleleft}) \text{id}_m = (\text{prnt}_G \setminus \text{prnt}_G^{\triangleleft}) .$$

It follows that  $\text{prnt} = \text{prnt}_G$  and then  $G \text{id}_m = G$ . The proof for  $\text{id}_n G = G$  is similar.

We now define tensor product of concrete place graphs with sharing.

**Definition 4 (tensor product).** *If  $G_0 : m_0 \rightarrow n_0$  and  $G_1 : m_1 \rightarrow n_1$  are two concrete place graphs with sharing with  $V_{G_0} \cap V_{G_1} = \emptyset$ , their tensor product*

$$G_0 \otimes G_1 = (V, \text{ctrl}, \text{prnt}) : m_0 + m_1 \rightarrow n_0 + n_1$$

*has nodes  $V = V_{G_0} \uplus V_{G_1}$  and control map*

$$\text{ctrl}(v) \stackrel{\text{def}}{=} \text{ctrl}_{G_i}(v) \qquad \text{if } v \in V_{G_i} \text{ with } i = 0, 1.$$

*Its parent relation  $\text{prnt} \subseteq ((m_0 + m_1) \uplus V_{G_0} \uplus V_{G_1}) \times (V_{G_0} \uplus V_{G_1} \uplus (n_0 + n_1))$  is defined as follows:*

$$\text{prnt}_{G_0} \cup \text{prnt}_{G_1}^{(m_0, n_0)}$$

where,

$$\begin{aligned} \text{prnt}_{G_1}^{(m_0, n_0)} &= \{(v, w) \mid (v, w) \in \text{prnt}_{G_1} \text{ and } v, w \in V_{G_1}\} \\ &\cup \{(m_0 + i, w) \mid (i, w) \in \text{prnt}_{G_1}, w \in V_{G_1} \text{ and } i \in m_1\} \\ &\cup \{(v, n_0 + i) \mid (v, i) \in \text{prnt}_{G_1}, v \in V_{G_1} \text{ and } i \in n_1\} \\ &\cup \{(m_0 + i, n_0 + j) \mid (i, j) \in \text{prnt}_{G_1}, i \in m_1 \text{ and } j \in n_1\} . \end{aligned}$$

Therefore, according to the definition above, tensor product is not commutative. Moreover, tensor product over interfaces  $m$  and  $n$  is given by  $m + n$ . As for composition, we prove that tensor product enjoys associative property and has neutral elements.

**Proposition 3 (associativity of tensor product).** *If  $A : m_0 \rightarrow n_0$ ,  $B : m_1 \rightarrow n_1$ ,  $C : m_2 \rightarrow n_2$  are three concrete place graphs with sharing with disjoint node sets, then*

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C .$$

*Proof.* Let us define  $A \otimes (B \otimes C) = G_0$  and  $(A \otimes B) \otimes C = G_1$ . Since the node sets  $V_A, V_B, V_C$  are all disjoint by Definition 4 all the products are defined. We have to prove that  $G_0 = G_1$ . Associativity of  $\uplus$  and  $+$  assures that  $V_{G_0} = V_{G_1}$ ,  $m_0 + (m_1 + m_2) = (m_0 + m_1) + m_2$  and  $n_0 + (n_1 + n_2) = (n_0 + n_1) + n_2$ . Moreover, by construction  $\text{ctrl}_{G_0} = \text{ctrl}_{G_1}$ . It remains to prove that  $\text{prnt}_{G_0} = \text{prnt}_{G_1}$ . By construction the following equalities hold:

$$\begin{aligned} \text{prnt}_{G_0} &= \text{prnt}_A \cup \text{prnt}_{B \otimes C}^{(m_0, n_0)} \\ \text{prnt}_{G_1} &= \text{prnt}_{A \otimes B} \cup \text{prnt}_C^{(m_0 + m_1, n_0 + n_1)} \end{aligned}$$

with

$$\begin{aligned} \text{prnt}_{B \otimes C} &= \text{prnt}_B \cup \text{prnt}_C^{(m_1, n_1)} \\ \text{prnt}_{A \otimes B} &= \text{prnt}_A \cup \text{prnt}_B^{(m_0, n_0)} . \end{aligned}$$

Hence, we have

$$\text{prnt}_{G_0} = \text{prnt}_A \cup \text{prnt}_B^{(m_0, n_0)} \cup \text{prnt}_C^{(m_0 + m_1, n_0 + n_1)} = \text{prnt}_{G_1} .$$

**Proposition 4 (neutral element for tensor product).** *For any concrete place graph with sharing  $G : m \rightarrow n$  the following holds*

$$G \otimes \text{id}_0 = G = \text{id}_0 \otimes G .$$

*Proof.* Immediate from

$$\text{prnt}_{G \otimes \text{id}_0} = \text{prnt}_G \cup \text{prnt}_{\text{id}_0}^{(m, n)} = \text{prnt}_G \cup \emptyset = \text{prnt}_G = \text{prnt}_{\text{id}_0 \otimes G} .$$



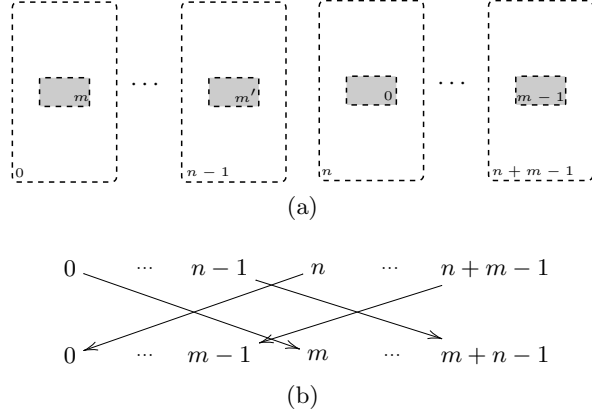


Fig. 4: An example of symmetry  $\gamma_{m,n}$  (with  $m' = m + n - 1$ ) (a) and the corresponding place graph (b).

Finally, symmetries for place graphs with sharing are defined as in [1]. An example of symmetry is drawn in Figure 4.

**Definition 5 (symmetries).** *The symmetry  $\gamma_{m,n} : m + n \rightarrow n + m$  is given by*

$$\gamma_{m,n} \stackrel{\text{def}}{=} (\emptyset, \emptyset, \text{prnt})$$

where  $\text{prnt} = \{(i, i + n) \mid i \in m\} \cup \{(i, i - m) \mid i \in n\}$ .

Observe that links and names are unaffected by the introduction of overlapping places. Hence, the definition of link graphs remains unchanged. A concrete bigraph with sharing  $G : \langle m, X \rangle \rightarrow \langle n, Y \rangle$  is the pair of its constituents, a place graph with sharing and a link graph. It is written  $G = \langle G^P, G^L \rangle$ . Also the definition of support for bigraphs with sharing is analogous to the one presented in [1]. In particular, the support of concrete place graph with sharing  $G_0 = (V, \text{ctrl}, \text{prnt}) : m \rightarrow n$  is its node set  $V$ . We write  $G_0 \simeq G_1$  to indicate that  $G_0$  and  $G_1$  are support equivalent.

## Discussion

Before presenting a categorical semantics of bigraphs with sharing, we explain why we choose to extend the original definition of bigraphs, rather than encode sharing within the formalism. There are two possible encodings. The first is to introduce dummy controls to represent intersections of nodes. For instance, if nodes A and B share a region, their intersection is represented as a separate node of control  $A \cap B$ . A graphical representation is given in Figure 5b. The immediate consequence of this approach is that place graphs are still representable by forests. However, a major disadvantage is that the number of dummy nodes to be added grows exponentially with the number of intersecting nodes. Moreover,

this encoding is not complete because it cannot represent sharing when no nodes are involved, e.g. a node shared by two regions. This can be a limiting factor especially in the definition of reaction rules. Another shortcoming is that a node shared between A and B is placed inside the dummy node  $A \cap B$ , thus both A and B appear as if they do not have a child.

The second is to keep a copy of a shared node inside each of its parents and connect the copies with a special link. For example, when a node C is shared between A and B, both A and B contain a node of control  $\bar{C}$  and the two  $\bar{C}$ s are linked together. This is drawn in Figure 5c. Note that control  $\bar{C}$  is defined exactly as C but with an extra port to handle the special link. However, this approach does not allow one to express sharing without nodes, e.g. two nodes sharing a site. Another problem arises when occurrences have to be counted, for computing a reaction rate (this is relevant in a stochastic setting). In this case, a shared node has to be copied (and counted)  $n$  times, where  $n$  is the number of sharing nodes.

Our extension yields several advantages. First, its completeness allows the representation of any place graph with sharing. Second, the modelling phase is more natural and immediate, because no additional links, copies of nodes and controls have to be introduced. Third, the structure of place graphs with sharing appears to have many similarities with standard categorical notions as we will show in the next section.

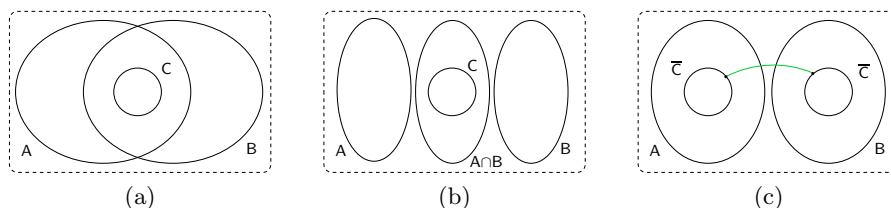


Fig. 5: (a) An example of a bigraph with sharing and two possible encodings (b) and (c)

### 3 Categories of Bigraphs with Sharing

We start by recalling basic notions on monoidal categories and bialgebras. Refer to [8] for a more detailed account.

#### 3.1 Monoidal Categories

A *monoidal category*  $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$  consists of a category  $\mathcal{C}$ , a bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , an object  $I \in \mathcal{C}$  and three natural isomorphisms  $\alpha, \lambda, \rho$ . Explicitly,

$$\alpha = \alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \quad (3)$$

is natural for all  $A, B, C \in \mathcal{C}$ , and the pentagonal diagram

$$\begin{array}{ccc}
A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} & ((A \otimes B) \otimes C) \otimes D \\
\text{id} \otimes \alpha \downarrow & & & \uparrow \alpha \otimes \text{id} \\
A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha} & & (A \otimes (B \otimes C)) \otimes D
\end{array} \quad (4)$$

commutes for all  $A, B, C, D \in \mathcal{C}$ . Again,  $\lambda$  and  $\varrho$  are natural

$$\lambda_A : I \otimes A \cong A, \quad \varrho_A : A \otimes I \cong A \quad (5)$$

for all objects  $A \in \mathcal{C}$ , the triangular diagram

$$\begin{array}{ccc}
A \otimes (I \otimes B) & \xrightarrow{\alpha} & (A \otimes I) \otimes B \\
\text{id} \otimes \lambda \searrow & & \swarrow \varrho \otimes \text{id} \\
& A \otimes B &
\end{array} \quad (6)$$

commutes for all  $A, B \in \mathcal{C}$ , and also

$$\lambda_I = \varrho_I : I \otimes I \rightarrow I . \quad (7)$$

The bifunctor  $\otimes$  assigns to each pair of objects  $A, B \in \mathcal{C}$  an object  $A \otimes B \in \mathcal{C}$  and to each pair of arrows  $f : A \rightarrow A', g : B \rightarrow B'$  an arrow  $f \otimes g : A \otimes B \rightarrow A' \otimes B'$ . Thus  $\otimes$  a bifunctor means that the interchange law

$$\text{id}_A \otimes \text{id}_B = \text{id}_{A \otimes B} \quad (f' \otimes g')(f \otimes g) = (f'f) \otimes (g'g) \quad (8)$$

holds whenever the composites  $f'f$  and  $g'g$  are defined. A monoidal category is said to be *strict* if  $\alpha, \lambda, \varrho$  are the identity morphisms. It is said to be *partial* if  $\otimes$  is partial. Any monoidal category is monoidally equivalent to a strict monoidal category.

A monoidal category  $\mathcal{C}$  is said to be *symmetric* when it is equipped with isomorphisms

$$\gamma_{A,B} : A \otimes B \cong B \otimes A , \quad (9)$$

natural in  $A, B \in \mathcal{C}$ , such that the diagrams

$$\gamma_{A,B} \gamma_{B,A} = \text{id}_{B \otimes A} \quad \varrho_B = \lambda_B \gamma_{B,I} : B \otimes I \cong B , \quad (10)$$

$$\begin{array}{ccc}
A \otimes (B \otimes C) & \xrightarrow{\alpha} & (A \otimes B) \otimes C \xrightarrow{\gamma} & C \otimes (A \otimes B) \\
\text{id} \otimes \gamma \downarrow & & & \downarrow \alpha \\
A \otimes (C \otimes B) & \xrightarrow{\alpha} & (A \otimes C) \otimes B \xrightarrow{\gamma \otimes \text{id}} & (C \otimes A) \otimes B
\end{array} \quad (11)$$

all commute.

### 3.2 Monoids and Co-monoids

A *monoid*  $(A, \mu, \eta)$  in a monoidal category  $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho)$  is an object  $A$  equipped with arrows  $\mu : A \otimes A \rightarrow A$ , called the *multiplication*, and  $\eta : I \rightarrow A$ , called the *unit*, such that the following diagrams

$$\begin{array}{ccccc} A \otimes (A \otimes A) & \xrightarrow{\alpha} & (A \otimes A) \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\ \text{id} \otimes \mu \downarrow & & & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & & & A \end{array} \quad (12)$$

$$\begin{array}{ccccc} I \otimes A & \xrightarrow{\eta \otimes \text{id}} & A \otimes A & \xleftarrow{\text{id} \otimes \eta} & A \otimes I \\ & \searrow \lambda & \downarrow \mu & \swarrow \varrho & \\ & & A & & \end{array} \quad (13)$$

are commutative. When  $\mathcal{C}$  is symmetric and

$$\mu \gamma = \mu \quad (14)$$

holds, we say  $A$  is *commutative*.

Dually, a *co-monoid*  $(A, \Delta, \epsilon)$  in a monoidal category  $\mathcal{C}$  is an object  $A$  equipped with morphisms  $\Delta : A \rightarrow A \otimes A$ , called the *co-multiplication*, and  $\epsilon : A \rightarrow I$ , called the *co-unit*, satisfying

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A \otimes A & & \\ \Delta \downarrow & & \downarrow \Delta \otimes \text{id} & & \\ A \otimes A & \xrightarrow{\text{id} \otimes \Delta} & A \otimes (A \otimes A) & \xrightarrow{\alpha} & (A \otimes A) \otimes A \end{array} \quad (15)$$

$$\begin{array}{ccccc} A \otimes I & \xleftarrow{\text{id} \otimes \epsilon} & A \otimes A & \xrightarrow{\epsilon \otimes \text{id}} & I \otimes A \\ & \searrow \varrho & \uparrow \Delta & \swarrow \lambda & \\ & & A & & \end{array} \quad (16)$$

When  $\mathcal{C}$  is symmetric and

$$\gamma \Delta = \Delta \quad (17)$$

holds, we say  $A$  is *co-commutative*.

### 3.3 Bialgebras

A *bialgebra* in a strict symmetric monoidal category  $\mathcal{C}$  is given by a tuple  $A = (A, \mu, \eta, \Delta, \epsilon, \gamma)$  where  $A$  is an object of  $\mathcal{C}$ ,  $\gamma$  is a symmetry,  $(A, \mu, \eta)$  is a monoid and  $(A, \Delta, \epsilon)$  is a co-monoid, satisfying

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{\mu} & A & & \\ \Delta \otimes \Delta \downarrow & & \downarrow \Delta & & \\ A \otimes A \otimes A \otimes A & \xrightarrow{\text{id} \otimes \gamma \otimes \text{id}} & A \otimes A \otimes A \otimes A & \xrightarrow{\mu \otimes \mu} & A \otimes A \end{array} \quad (18)$$

$$\begin{array}{ccc}
\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ & \searrow^{\epsilon \otimes \epsilon} & \swarrow_{\epsilon} \\ & I \otimes I \cong I & \end{array} & 
\begin{array}{ccc} I \otimes I \cong I & & \\ \eta \swarrow & & \searrow_{\eta \otimes \eta} \\ A & \xrightarrow{\Delta} & A \otimes A \end{array} & 
\begin{array}{ccc} I & \xrightarrow{\text{id}} & I \\ \eta \searrow & & \swarrow_{\epsilon} \\ & A & \end{array}
\end{array} \quad (19)$$

We say  $A$  is *commutative* (resp. *co-commutative*) when it is commutative (resp. co-commutative) as a monoid. It is *bicommutative* when it is both commutative and co-commutative. A bialgebra is *qualitative* when the following equality holds:

$$\mu\Delta = \text{id} . \quad (19)$$

Now we list some standard categories we are going to use in the remainder of this paper.

**Rel** is the category with objects all finite ordinals  $n$  and arrows  $R : m \rightarrow n$  all binary relations from  $m$  to  $n$ . The monomorphisms are injective relations while epimorphisms are surjective relations. They form two subcategory indicated with  $\mathbf{Rel}_m$  and  $\mathbf{Rel}_e$ , respectively.

$\mathbf{Finord} = \mathbf{Set}_\omega$  is the category with objects all finite ordinals  $n$  and arrows  $f : m \rightarrow n$  all functions from  $m$  to  $n$ . It is a subcategory of **Rel** and  $\mathcal{I} : \mathbf{Finord} \rightarrow \mathbf{Rel}$  is the inclusion functor. It is also possible to define a functor  $\mathcal{F} : \mathbf{Rel} \rightarrow \mathbf{Finord}$  as follows:

- every object  $n \in \mathbf{Rel}$  is mapped to an object  $2^n \in \mathbf{Finord}$ .
- every arrow  $R : m \rightarrow n$  in **Rel** is associated to an arrow  $f : 2^m \rightarrow 2^n$  in **Finord** such that

$$f(x) = y \text{ iff } (i, j) \in R \text{ and for every } i \in x \text{ there is a } j \in y .$$

Refer to [1, pag. 18] for the definitions of *precategories*, *s-categories* and *spm categories*.

### 3.4 Categories of Bigraphs with Sharing

We now introduce the categories in which bigraphs with sharing and their operations can be expressed. In the following we presume a basic signature  $\mathcal{K}$ . We start off by defining the precategory of concrete place graphs with sharing:

**Definition 6.**  $\mathbf{SPg}(\mathcal{K})$  is the precategory whose arrows are concrete place graphs with sharing and objects are finite ordinals. Composition and identities are as in Definitions 2 and 3.

By Definition 2 composition is a partial operation. Moreover, when  $G_0G_1$  is defined then  $\text{dom}(G_0) = \text{cod}(G_1)$ . Additionally, Proposition 1 states associativity of composition and identities are shown to be neutral elements for composition in Proposition 2. This three properties assure that  $\mathbf{SPg}(\mathcal{K})$  is indeed a precategory.

We defined tensor product  $\otimes$  for concrete place graphs with sharing in Definition 4. Therefore, it is possible to refine the definition of  $\mathbf{SPg}(\mathcal{K})$ . However, we first have to prove that  $\otimes$  is a bifunctor. We have two propositions:

**Proposition 5 (bifactoriality 1).** *If  $A_0 : n_0 \rightarrow n_1$ ,  $A_1 : n_1 \rightarrow n_2$ ,  $B_0 : m_0 \rightarrow m_1$  and  $B_1 : m_1 \rightarrow m_2$  are four concrete place graphs with sharing with disjoint node sets, then*

$$(A_1 \otimes B_1)(A_0 \otimes B_0) = (A_1 A_0) \otimes (B_1 B_0) .$$

*Proof.* Let us define  $(A_1 \otimes B_1)(A_0 \otimes B_0) = G_0$  and  $(A_1 A_0) \otimes (B_1 B_0) = G_1$ . By Definitions 2 and 4,

$$V_{G_0} = V_{G_1} = \bigsqcup_{i=0,1} V_{A_i} \uplus V_{B_i}$$

and

$$ctrl_{G_0}(v) = ctrl_{G_1}(v) = \begin{cases} ctrl_{A_i}(v) & \text{if } v \in V_{A_i} \\ ctrl_{B_i}(v) & \text{if } v \in V_{B_i} \end{cases}$$

with  $i = 0, 1$ .

It remains to prove that  $prnt_{G_0} = prnt_{G_1}$ . We have

$$prnt_{G_0} = prnt_{A_1 \otimes B_1}^{\triangleleft} \cup prnt_{A_0 \otimes B_0}^{\triangleright} \cup R_{(A_1 \otimes B_1) \circ (A_0 \otimes B_0)} \quad (20)$$

$$prnt_{G_1} = prnt_{A_1 A_0} \cup prnt_{B_1 B_0}^{(n_0, n_2)} , \quad (21)$$

with

$$\begin{aligned} prnt_{A_1 \otimes B_1}^{\triangleleft} &= prnt_{A_1}^{\triangleleft} \cup prnt_{B_1}^{(-, n_2)^{\triangleleft}} \\ prnt_{A_0 \otimes B_0}^{\triangleright} &= prnt_{A_0}^{\triangleright} \cup prnt_{B_0}^{(n_0, -)^{\triangleright}} \end{aligned}$$

and

$$\begin{aligned} prnt_{A_1 A_0} &= prnt_{A_1}^{\triangleleft} \cup prnt_{A_0}^{\triangleright} \cup R_{A_1 \circ A_0} \\ prnt_{B_1 B_0}^{(n_0, n_2)} &= prnt_{B_1}^{(-, n_2)^{\triangleleft}} \cup prnt_{B_0}^{(n_0, -)^{\triangleright}} \cup R_{B_1 \circ B_0}^{(n_0, n_2)} . \end{aligned}$$

We write  $prnt_{B_1}^{(-, n_2)^{\triangleleft}}$  and  $prnt_{B_0}^{(n_0, -)^{\triangleright}}$  because the sets do not contain elements with sites and roots to be incremented. Hence, it remains to prove that

$$R_{(A_1 \otimes B_1) \circ (A_0 \otimes B_0)} = R_{A_1 \circ A_0} \cup R_{B_1 \circ B_0}^{(n_0, n_2)} . \quad (22)$$

The left-hand-side of Equation (22) can be rewritten as follows

$$\begin{aligned}
R_{(A_1 \otimes B_1) \circ (A_0 \otimes B_0)} &= (\text{prnt}_{A_1 \otimes B_1} \setminus \text{prnt}_{A_1 \otimes B_1}^{\triangleleft}) (\text{prnt}_{A_0 \otimes B_0} \setminus \text{prnt}_{A_0 \otimes B_0}^{\triangleright}) \\
&= ((\text{prnt}_{A_1} \cup \text{prnt}_{B_1}^{(n_1, n_2)}) \setminus (\text{prnt}_{A_1}^{\triangleleft} \cup \text{prnt}_{B_1}^{(-, n_2)^{\triangleleft}})) \\
&\quad ((\text{prnt}_{A_0} \cup \text{prnt}_{B_0}^{(n_0, n_1)}) \setminus (\text{prnt}_{A_0}^{\triangleright} \cup \text{prnt}_{B_0}^{(n_0, -)^{\triangleright}})) \\
&= ((\text{prnt}_{A_1} \setminus \text{prnt}_{A_1}^{\triangleleft}) \cup (\text{prnt}_{B_1}^{(n_1, n_2)} \setminus \text{prnt}_{B_1}^{(-, n_2)^{\triangleleft}})) \\
&\quad ((\text{prnt}_{A_0} \setminus \text{prnt}_{A_0}^{\triangleright}) \cup (\text{prnt}_{B_0}^{(n_0, n_1)} \setminus \text{prnt}_{B_0}^{(n_0, -)^{\triangleright}})) \\
&= R_{A_1 \circ A_0} \cup R_{B_1 \circ B_0}^{(n_0, n_2)} .
\end{aligned}$$

This concludes the proof.

**Proposition 6 (bifunctoriality 2).** *If  $\text{id}_m$  and  $\text{id}_n$  are two place graph identities then*

$$\text{id}_m \otimes \text{id}_n = \text{id}_{m \otimes n} .$$

*Proof.* The parent relation of the left-hand-side is  $\text{id}_{m+n}$ . Since  $m \otimes n = m + n$  we have equality.

It is easy to see that Propositions 3, 4, 5, and 6 satisfy Equations (4), (6), and (8). Symmetries as in Definition 5 also satisfy Equations (10) and (11). Hence, we can state the following:

**Proposition 7.** *( $\mathbf{SPg}(\mathcal{K}), \otimes, 0$ ) is a symmetric strict monoidal precategory.*

This category can be further enriched by assigning support to every concrete place with sharing.

**Proposition 8 (concrete place graphs with sharing).**  *$\mathbf{SPg}(\mathcal{K})$  is an  $s$ -category in which  $V_G$  is the support assigned to every arrow  $G$ .*

As in [1, pag. 24] composition becomes a total operation when supports are hidden.

**Proposition 9 (abstract place graphs with sharing).** *The support quotient*

$$\mathbf{SPg}(\mathcal{K}) \stackrel{\text{def}}{=} \mathbf{SPg}(\mathcal{K}) / \simeq$$

*is the spm category whose objects are finite ordinals and whose arrows  $\llbracket G \rrbracket : m \rightarrow n$ , called abstract place graphs with sharing, are support equivalence classes of  $\text{hom}(m, n)$  in  $\mathbf{SPg}(\mathcal{K})$ . We write*

$$\llbracket \cdot \rrbracket : \mathbf{SPg}(\mathcal{K}) \rightarrow \mathbf{SPg}(\mathcal{K})$$

*to indicate the support quotient functor.*

We now analyse the relationship between  $\mathbf{SPg}(\mathcal{K})$  and other categories. This is summarised in Figure 6.

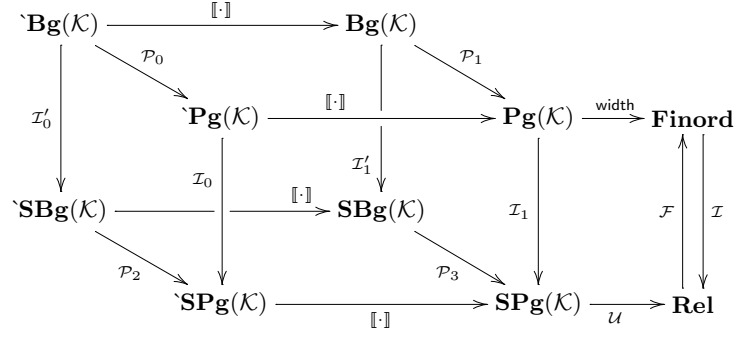


Fig. 6: Bigraphical categories.

Recall that  $\mathbf{Pg}(\mathcal{K})$  is the spm category whose objects are finite ordinals and whose arrows are abstract place graphs *without* sharing. Their spatial structure is given by a function instead of a relation. Since functions are total right-unique relations,  $\mathbf{Pg}(\mathcal{K})$  is a subcategory of  $\mathbf{SPg}(\mathcal{K})$  where  $\mathcal{I}_1 : \mathbf{Pg}(\mathcal{K}) \rightarrow \mathbf{SPg}(\mathcal{K})$  is the inclusion functor. Following the same argument,  $\mathbf{P}g(\mathcal{K})$  is a subcategory of  $\mathbf{S}Pg(\mathcal{K})$  with  $\mathcal{I}_0$  as inclusion functor.

In [1, pag. 25], the author proves that  $\mathbf{Pg}(\mathcal{K})$  is a wide category by constructing a functor  $\text{width} : \mathbf{Pg}(\mathcal{K}) \rightarrow \mathbf{Finord}$ . In order to prove the same result for  $\mathbf{SPg}(\mathcal{K})$ , we first define a functor  $\mathcal{U} : \mathbf{SPg}(\mathcal{K}) \rightarrow \mathbf{Rel}$  as follows:

- identity on objects,
- every arrow  $G : m \rightarrow n$  in  $\mathbf{SPg}(\mathcal{K})$  is associated to an arrow  $R : m \rightarrow n$  in  $\mathbf{Rel}$  such that for every  $i \in m$  and  $j \in n$

$$(i, j) \in R \text{ iff there is a path from } i \text{ to } j \text{ in } G .$$

Therefore, functor  $\mathcal{FU} : \mathbf{SPg}(\mathcal{K}) \rightarrow \mathbf{Finord}$  implies  $\mathbf{SPg}(\mathcal{K})$  is a wide category.

We are now ready to define category  $\mathbf{SBg}(\mathcal{K})$  of abstract bigraphs with sharing. We know that  $\mathbf{B}g(\mathcal{K})$  is the category whose objects are interfaces  $I = \langle m, X \rangle$  and arrows are concrete bigraphs  $B = \langle B^P, B^L \rangle$  with  $B^P$  an arrow in  $\mathbf{P}g(\mathcal{K})$  and  $B^L$  an arrow in  $\mathbf{L}g(\mathcal{K})$ . Additionally,  $B^P$  and  $B^L$  are required to share the same set of nodes and the same control map. Functor  $[\cdot] : \mathbf{B}g(\mathcal{K}) \rightarrow \mathbf{B}g(\mathcal{K})$  maps *lean-support* equivalent bigraphs to an abstract bigraph. See [1, pag. 26] for a formal definition. We follow the same approach to define categories of bigraphs with sharing.  $\mathbf{S}B`g(\mathcal{K})$  is the s-category whose objects are interfaces and arrows are concrete bigraphs with sharing  $B = \langle B^P, B^L \rangle$  with  $B^P$  an arrow in  $\mathbf{S}P`g(\mathcal{K})$  and  $B^L$  an arrow in  $\mathbf{L}g(\mathcal{K})$ . Hence,  $\mathbf{S}B`g(\mathcal{K})$  has interfaces as objects and lean-support equivalences classes of concrete bigraphs with sharing as arrows. The relation between concrete and abstract bigraphs with sharing is encoded by functor  $[\cdot] : \mathbf{S}B`g(\mathcal{K}) \rightarrow \mathbf{S}B`g(\mathcal{K})$ .  $\mathbf{B}g(\mathcal{K})$  and  $\mathbf{B}g(\mathcal{K})$  are subcategories of  $\mathbf{S}B`g(\mathcal{K})$  and  $\mathbf{S}B`g(\mathcal{K})$ , respectively.

Finally projection functors  $\mathcal{P}_i$  forget link graphs to move from categories of bigraphs to categories of place graphs.



## 4 Properties

In this section we characterise epimorphisms (epis) and monomorphisms (monos) in bigraphs with sharing. We then investigate which subcategory of  $\mathbf{SPg}(\mathcal{K})$  enjoys relative pushouts (RPOs).

**Proposition 10 (epis).** *A concrete place graph with sharing is epi iff no root is idle.*

*Proof.* Recall that  $B : m \rightarrow n$  is epi if  $B_0B = B_1B$  implies  $B_0 = B_1$  for any  $B_0, B_1 : n \rightarrow h$ . By Definition 2 we have

$$\text{prnt}_{B_iB} = \text{prnt}_{B_i}^{\triangleleft} \cup \text{prnt}_B^{\triangleright} \cup R_{B_i \circ B} \quad (23)$$

with  $i = 0, 1$ . By hypothesis we have

$$\text{prnt}_{B_0B} = \text{prnt}_{B_1B} \quad . \quad (24)$$

It is immediate to see that node sets and control maps of  $B_0$  and  $B_1$  are equal. Therefore, we only have to prove that  $\text{prnt}_{B_0} = \text{prnt}_{B_1}$ . By (23) and (24), we obtain

$$\text{prnt}_{B_0}^{\triangleleft} \cup R_{B_0 \circ B} = \text{prnt}_{B_1}^{\triangleleft} \cup R_{B_1 \circ B} \quad . \quad (25)$$

Now assume  $\text{prnt}_{B_0}^{\triangleleft} \neq \text{prnt}_{B_1}^{\triangleleft}$ . Then we have two cases:

1. There exists an element  $(v, w) \in \text{prnt}_{B_0}^{\triangleleft}$  such that  $(v, w) \notin \text{prnt}_{B_1}^{\triangleleft}$ . Then by (25)  $(v, w) \in R_{B_1 \circ B}$ . But by construction,  $v \in V_{B_0}$  and  $w \in V_{B_0} \uplus h$ , while for any element  $(u, t) \in R_{B_1 \circ B}$  we have  $u \in V_B \uplus m$  and  $t \in V_{B_1} \uplus h$ . Therefore  $(v, w) \in \text{prnt}_{B_1}^{\triangleleft}$ . This is a contradiction.
2. There is an element  $(v, w) \in \text{prnt}_{B_1}^{\triangleleft}$  such that  $(v, w) \notin \text{prnt}_{B_0}^{\triangleleft}$ . Again by contradiction as in the previous case.

Hence, we proved  $\text{prnt}_{B_0}^{\triangleleft} = \text{prnt}_{B_1}^{\triangleleft}$ . This and (25) imply

$$(\text{prnt}_{B_0} \setminus \text{prnt}_{B_0}^{\triangleleft})(\text{prnt}_B \setminus \text{prnt}_B^{\triangleright}) = (\text{prnt}_{B_1} \setminus \text{prnt}_{B_1}^{\triangleleft})(\text{prnt}_B \setminus \text{prnt}_B^{\triangleright}) \quad . \quad (26)$$

By hypothesis  $\text{prnt}_B \setminus \text{prnt}_B^{\triangleright}$  is a surjective relation (i.e. no root is idle in  $B$ ). Then by (26) it follows that

$$\text{prnt}_{B_0} \setminus \text{prnt}_{B_0}^{\triangleleft} = \text{prnt}_{B_1} \setminus \text{prnt}_{B_1}^{\triangleleft} \quad (27)$$

because surjective relations are the epis in **Rel**. Finally

$$\begin{aligned} \text{prnt}_{B_0} &= \text{prnt}_{B_0}^{\triangleleft} \cup (\text{prnt}_{B_0} \setminus \text{prnt}_{B_0}^{\triangleleft}) \\ &= \text{prnt}_{B_1}^{\triangleleft} \cup (\text{prnt}_{B_1} \setminus \text{prnt}_{B_1}^{\triangleleft}) \\ &= \text{prnt}_{B_1} \quad . \end{aligned}$$

This concludes the proof.

**Proposition 11 (monos).** *A concrete place graph with sharing is mono iff no two sites are siblings.*

*Proof.* Recall that  $B : n \rightarrow h$  is mono if  $BB_0 = BB_1$  implies  $B_0 = B_1$  for any  $B_0, B_1 : m \rightarrow n$ . As in the previous proof, we only have to prove that  $prnt_{B_0} = prnt_{B_1}$ . By hypothesis,

$$prnt_B^{\triangleleft} \cup prnt_{B_0}^{\triangleright} \cup R_{B \circ B_0} = prnt_B^{\triangleleft} \cup prnt_{B_1}^{\triangleright} \cup R_{B \circ B_1} .$$

Since  $prnt_{B_0}^{\triangleright}$  and  $prnt_{B_1}^{\triangleright}$  can be proved equal (see argument for  $prnt_{B_0}^{\triangleleft} = prnt_{B_1}^{\triangleleft}$  in the previous proof), we have

$$(prnt_B \setminus prnt_B^{\triangleleft})(prnt_{B_0} \setminus prnt_{B_0}^{\triangleright}) = (prnt_B \setminus prnt_B^{\triangleleft})(prnt_{B_0} \setminus prnt_{B_0}^{\triangleright}) . \quad (28)$$

By hypothesis  $prnt_B \setminus prnt_B^{\triangleleft}$  is an injective relation (i.e. no two sites are siblings in  $B$ ). Then by (28) it follows that

$$prnt_{B_0} \setminus prnt_{B_0}^{\triangleright} = prnt_{B_1} \setminus prnt_{B_1}^{\triangleright} \quad (29)$$

because injective relations are the monos in **Rel**. This concludes the proof.

## 5 Axioms

In this section we show that every place graph with sharing in  $\mathbf{Pg}(\mathcal{K})$  can be constructed, using composition and tensor product, from a small set of elementary place graphs.

**Definition 7 (elementary place graphs).** *An elementary place graph is a place graph in one of the following forms:*

$\text{id}_1 : 1 \rightarrow 1$	<i>map a site to one root</i>
$1 : 0 \rightarrow 1$	<i>a barren root</i>
$0 : 1 \rightarrow 0$	<i>an orphaned site</i>
$\text{merge} : 2 \rightarrow 1$	<i>map two sites to one root</i>
$\text{split} : 1 \rightarrow 2$	<i>map one site to two roots</i>
$\gamma_{1,1} : 2 \rightarrow 2$	<i>swap 2 sites</i>
$\mathbf{K} : 1 \rightarrow 1$	<i>for every <math>\mathbf{K} \in \mathcal{K}</math>.</i>

*These place graphs are depicted in Figure 7.*

*Placings* (ranged over by  $\phi, \psi, \dots$ ) are node-free place graphs. They can be build form the elementary place graphs listed above, except  $\mathbf{K}$ . Intuitively any placing  $\phi : m \rightarrow n$  is a relation from  $m$  to  $n$  in **Rel**. In [9, Theorem 7] the author proves that category **Rel** can be presented by the equational theory of qualitative bicommutative bialgebras. If we define a bialgebra  $(\{0\}, \text{merge}, 1, \text{split}, 0, \gamma_{1,1})$  over  $\mathbf{Pg}(\mathcal{K})$ , such a theory is given by the following axioms:

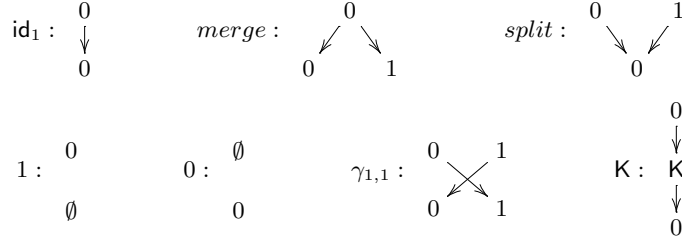


Fig. 7: Elementary place graphs.

### category

$$\begin{aligned} id_n B = B = B id_m & \quad \text{for } B : m \rightarrow n \\ A(BC) = (AB)C & \end{aligned}$$

### symmetric monoidal category

$$\begin{aligned} A \otimes (B \otimes C) &= (A \otimes B) \otimes C \\ id_0 \otimes B = B = B \otimes id_0 & \\ (B_1 \otimes A_1)(B_0 \otimes A_0) &= B_1 B_0 \otimes A_1 A_0 \\ id_m \otimes id_n = id_{m \otimes n} & \\ \gamma_{m,0} = id_m & \\ \gamma_{m,n} \gamma_{n,m} = id_{n \otimes m} & \\ \gamma_{m \otimes n, k} = (\gamma_{m,k} \otimes id_n)(id_m \otimes \gamma_{n,k}) & \end{aligned}$$

### commutative monoid

$$\begin{aligned} merge(merge \otimes id_1) &= merge(id_1 \otimes merge) \\ merge(1 \otimes id_1) = id_1 &= merge(id_1 \otimes 1) \\ merge \gamma_{1,1} &= merge \end{aligned}$$

### commutative comonoid

$$\begin{aligned} (split \otimes id_1)split &= (id_1 \otimes split)split \\ (0 \otimes id_1)split = id_1 &= (id_1 \otimes 0)split \\ \gamma_{1,1}split &= split \end{aligned}$$

### bialgebra

$$\begin{aligned} split \, merge &= (merge \otimes merge)(id_1 \otimes \gamma_{1,1} \otimes id_1)(split \otimes split) \\ 0 \, merge &= 0 \otimes 0 \\ split \, 1 &= 1 \otimes 1 \\ 0 \, 1 &= id_0 \\ merge \, split &= id_1 \end{aligned}$$

Note that these axioms are just instances of the commutative diagrams in Section 3 with

$$\mu = \text{merge} \quad \eta = 1 \quad \Delta = \text{split} \quad \epsilon = 0 \quad \gamma = \gamma_{1,1} .$$

Hence, we have a complete axiomatisation for placings. It is straightforward to extend this result to arbitrary places graphs with sharing.

**Proposition 12.** *Every place graph with sharing can be obtained as the value of an expression containing only elementary place graphs as constants and composition and tensor product as operators.*

*Proof.* We prove the proposition by induction on the number of nodes of the place graph with sharing. The base case is immediate because node-free place graphs are placings and the axiomatisation we presented above is complete. Now, let  $B : m \rightarrow n$  be a place graph with sharing with  $k + 1$  nodes. Then, there is a concrete place graph with sharing  $\widehat{B} : m \rightarrow n$  which is defined as  $B$  enriched with support  $V_B$ . Let  $v$  be a node in which none of its children are nodes. Such a node must exist by acyclicity of  $prnt$ . Note that  $v$  can still have  $m' \leq m$  sites as children. Formally,  $(u, v) \in prnt$  if and only if  $u \in m'$ . Without loss of generality we assume  $ctrl(v) = K$ . Let  $\widehat{B}_1 : m + 1 \rightarrow n$  be the place graph obtained from  $\widehat{B}$  by removing node  $v$  and substituting it with a site. Furthermore, let  $\widehat{B}_0 : m \rightarrow m + 1$  be the place graph containing the sites in  $\widehat{B}$  and  $v$ . Then we have that  $\widehat{B} = \widehat{B}_1 \widehat{B}_0$ . By dropping the supports we can write  $B = B_1 B_0$ . But  $B_0$  can be defined in terms of elementary place graphs and placings as follows:

$$B_0 = \psi(\text{id}_m \otimes K)\phi \quad \text{with} \quad \psi : m + 1 \rightarrow m + 1 \quad \phi : m \rightarrow m + 1 . \quad (30)$$

Therefore, the statement follows by inductive hypothesis on  $B_1$  since it has  $k$  nodes.

The construction presented in the proof above can be adopted to define a normal form for place graphs with sharing. Intuitively, the same procedure for the construction of  $\widehat{B}_0$  is applied recursively until all the nodes are consumed. The only difference is that all the leaf nodes are removed in one go instead of removing only a single leaf at each step. Formally, a place graph  $B : m \rightarrow n$  may be expressed as  $B = B_0 \cdots B_h$  where each  $B_i$  contains exactly  $k_i$  nodes. Therefore,  $|V_B| = \sum_{i \leq h} k_i$  holds. Similarly to (30), its definition is:

$$B_i = \psi(\text{id}_{m_i} \otimes \bigotimes_{j < k_i} K)\phi .$$

This normal form can also be used to represent bigraphs with sharing. Since nodes are the only structure shared between link graphs and place graphs, it suffices to modify the node generator as follows:  $K_{\mathbf{x}} : 1 \rightarrow \langle 1, \{\mathbf{x}\} \rangle$ , for each  $K \in \mathcal{K}$ . These elementary bigraphs are called *ions*. Recall that a complete axiomatisation for linkings (i.e. node-free link graphs) is given in [1]. Therefore, a

bigraph  $G : \langle m, X \rangle \rightarrow \langle n, Y \rangle$  may be expressed as  $G = (\text{id}_n \otimes \omega)G_0 \cdots G_h$ , where  $\omega$  is a linking with outer interface  $Y$  and each  $G_i$  is defined as

$$G_i = ((\psi \otimes \bigotimes_{j < k_i} \text{id}_{\mathbf{x}_j})(\text{id}_{m_i} \otimes \bigotimes_{j < k_i} \mathbf{K}_{\mathbf{x}_j})\phi) \otimes \text{id}_{X_i} .$$

Note that  $X_h = X$ .

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