Recursive Session Types Revisited

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Session types model structured communication-based programming. In particular, binary session types for the π-calculus describe communication between exactly two participants in a distributed scenario. Adding sessions to the π-calculus means augmenting it with type and term constructs. In a previous paper, we tried to understand to which extent the session constructs are more complex and expressive than the standard π-calculus constructs. Thus, we presented an encoding of binary session π-calculus to the standard typed π-calculus by adopting *linear* and *variant* types and the *continuation-passing* principle. In the present paper, we focus on recursive session types and we present an encoding into recursive linear π-types. This encoding is a conservative extension of the former in that it preserves the results therein obtained. Most importantly, it adopts a new treatment of the duality relation, which in the presence of recursive types has been proven to be quite challenging.

1 Introduction

Session types are a type formalism used to model structured communication-based programming for distributed systems. In particular, binary session types for the π-calculus describe communication between exactly two participants in such scenario [6, 8, 11, 12]. When sessions are added to the standard typed π-calculus, the syntax of types and terms is augmented with ad-hoc constructs, added on top of the already existing ones. This yields a duplication of type and term constructs, e.g. restriction of session channels and restriction of standard π-channels or, recursive session types and recursive standard π-types [6]. Most importantly, this redundancy is also propagated in the theory of session types: various properties are proven for session types as well as for standard π-types. In a previous work [5], we focused on a subset of binary session types, namely the finite ones, and posed the following question:

To which extent session constructs are more complex and more expressive than the standard π-calculus constructs?

We answered this question by showing an encoding of finite binary session types into finite *linear* π-types and of finite session processes into finite standard π-processes. In the present paper, we extend the encoding to an infinite setting, namely to recursive session types and replicated processes, and pose the same question. We encode recursive session types into recursive linear π-types and replicated session processes into replicated standard π-processes. We show that the current encoding *i*) is sound and complete with respect to typing derivations, intuitively meaning: "a session process is well-typed if and only if its encoding is well-typed"; and *ii*) satisfies the operational correspondence property, intuitively meaning: "a session process and its encoding reduce to processes still related by the encoding".

The interest and benefits of this encoding are mainly in *expressivity* and *reusability* for a larger setting than the one adopted in [5]. The encoding is an expressivity result for recursive types. Its faithfulness, proved by *i*) and *ii*), permits reusability of already existing theory for standard typed π-calculus: e.g.,

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subject reduction or type safety for session π-calculus can be obtained as corollaries from the encoding and the corresponding properties in the standard typed π-calculus.

The present encoding is not just an extension of the former, it presents novelties and differences with respect to [5], as listed below. *Duality* is a fundamental notion of session types, as it describes compatible behaviours between communicating parties. The most used duality is the *inductive duality function* $\overline{\cdot}$ [8, 12]. Recent work [2] has shown the inadequacy of $\overline{\cdot}$ in the presence of recursive types, because it does not commute with unfolding. As a consequence, using relations like subtyping or type equivalence becomes challenging, because these relations explicitly use unfolding of recursive types. In the light of such discovery, the present encoding adopts the *complement function* $\mathsf{cplt}()$ defined in [2], which is shown to be adequate, instead of $\overline{\cdot}$, adopted in the former encoding. Since $\mathsf{cplt}()$ and $\overline{\cdot}$ coincide for finite session types, the encoding in [5] remains sound and the present encoding is a conservative extension of the former, in that it preserves all the properties that the former encoding satisfies. For completeness, the present one is extended to standard variables and hence non session π-processes: in this case the encoding is an homomorphism and no linearity is required. On top of $\mathsf{cplt}()$, we present the *co-inductive duality relation*, which is shown to contain the complement [1], and is used in the type system for session π-calculus [3, 6]. This permits us to give a definition of complement and co-inductive duality for linear π-calculus types, which is another contribution of the present paper.

Structure of the paper. In § 2 we present the syntax of types and terms for both the session π-calculus and the standard types π-calculus. In § 3 we present the encoding of recursive session types and session processes and we state the main results for the encoding. In § 4 we give a detailed example of the encoding of a well-typed replicated process which uses recursive session types. We conclude in § 5. The proofs of the result herein presented can be found in the online version of the paper [3].

2 The Model

2.1 Background on π-calculus with sessions

Syntax. The syntax of the π-calculus with session types [6, 12] is given in Figure 1.

$$
\begin{array}{rcllllllll}
P, Q & ::= & x!\langle v\rangle.P & \text{(output)} & \mid & x?(y).P & \text{(input)} & \mid & x \triangleleft l_j.P & \text{(selection)} \\
 & & x \triangleright \{l_i : P_i\}_{i\in I} & \text{(branching)} & \mid & P \mid Q & \text{(parallel)} & \mid & (\boldsymbol{\nu} xy)P & \text{(session res.)} \\
 & & *P & \text{(replication)} & \mid & (\boldsymbol{\nu} x)P & \text{(channel res.)} & \mid & \mathbf{0} & \text{(inaction)} \\
v & ::= & x & \text{(variable)} & \mid & \mathbf{1} & \text{(unit value)} & & &
\end{array}
$$

Figure 1: π-calculus with sessions, syntax.

P, Q range over processes, x, y over variables, v over values and l over labels. A value is a variable or $\mathbf{1}$. A process is an output $x!\langle v\rangle.P$ which sends v on x with continuation P; an input $x?(y).P$ which receives a value on x and proceeds as P; a selection $x \triangleleft l_j.P$ which selects l_j on x and proceeds as P; a branching $x \triangleright \{l_i : P_i\}_{i\in I}$ which offers a set of labelled processes on x, with labels being all different; a parallel composition $P \mid Q$ of P, Q; replicated $*P$ which spawns copies of P; a session restriction $(\boldsymbol{\nu} xy)P$ or a standard channel restriction $(\boldsymbol{\nu} x)P$; or $\mathbf{0}$, the terminated process. Session restriction differs from the standard one: $(\boldsymbol{\nu} xy)$ states that x and y, called *co-variables*, are the opposite endpoints of a session channel and are bound in P. It models session creation and the connection phase [8, 11].

Session types. The syntax of types for the π-calculus with sessions [6] is given in Figure 2.

S	::=	$!T.S$	(send)	|	$?T.S$	(receive)	|	$\oplus\{l_i : S_i\}_{i\in I}$	(select)
		$\&\{l_i : S_i\}_{i\in I}$	(branch)	|	X	(type var.)	|	$\overline{X}$	(dual type var.)
		end	(termination)	|	$\mu X.S$	(rec. session type)			
T	::=	S	(session type)	|	$\sharp T$	(channel type)	|	X	(type variable)
		$\mu X.T$	(recursive type)	|	$\mathtt{Unit}$	(unit type)			

Figure 2: π-calculus with sessions, types.

S ranges over session types and T over types. A session type can be $!T.S$ or $?T.S$ which respectively, sends or receives a value of type T and continuation S; select $\oplus\{l_i : S_i\}_{i\in I}$ or branch $\&\{l_i : S_i\}_{i\in I}$ which are sets of labelled session types indicating respectively, internal and external choice, with labels being all different; a (dualised) type variable $X, \overline{X}$, or recursive session type $\mu X.S$ or the terminated type end. A type can be a session type S; a standard channel type $\sharp T$; a type variable X or recursive type $\mu X.T$ or a unit type $\mathtt{Unit}$. Recursive (session) types are required to be *guarded*, meaning that in $\mu X.T$, variable X may occur free in T only under at least one of the other type constructs. To work with recursive types we need the *unfolding* function (UNF) which unfolds a recursive type until the first type constructor different from μX is reached (see [3]). Finally, we use $\mathtt{SType}$ to denote the set of *closed* (no free type variables) and *guarded* session types.

On duality for session types. Below we give an adaptation of the complement function [2] to $X, \overline{X}$.

Definition 2.1 (Complement function for session types)**.** *The complement function is defined as:*

$$
\begin{array}{lcllcl}
\mathsf{cplt}(?T.S) & = & !T.\mathsf{cplt}(S) & \mathsf{cplt}(X) & = & \overline{X} \\
\mathsf{cplt}(!T.S) & = & ?T.\mathsf{cplt}(S) & \mathsf{cplt}(\overline{X}) & = & X \\
\mathsf{cplt}(\&\{l_i : S_i\}_{i\in I}) & = & \oplus\{l_i : \mathsf{cplt}(S)_i\}_{i\in I} & \mathsf{cplt}(\mu X.S) & = & \mu X.\mathsf{cplt}(S\lfloor^{\mu X.S}/_X\rfloor) \\
\mathsf{cplt}(\oplus\{l_i : S_i\}_{i\in I}) & = & \&\{l_i : \mathsf{cplt}(S)_i\}_{i\in I} & \mathsf{cplt}(\mathsf{end}) & = & \mathsf{end}
\end{array}
$$

It uses a syntactic substitution $\lfloor^{-}/_{-}\rfloor$, which acts only on carried types and is formally defined in [1–3]. Below we give the definition of standard type substitution for (dualised) type variables [7].

$$
\begin{array}{lll}
X\{S/X\} = S & Y\{S/X\} = Y & \text{If } X \neq Y \\
\overline{X}\{S/X\} = \mathsf{cplt}(S) & \overline{Y}\{S/X\} = \overline{Y} & \text{If } X \neq Y
\end{array}
$$

However, when describing opposite behaviours between communicating parties, in this paper we adopt the *co-inductive duality relation* $\perp_{\mathsf{S}}$ by following [6]. The benefits of this approach are: *i*) $\perp_{\mathsf{S}}$ commutes with unfolding [2] and hence it is adequate; *ii*) as stated in [6], since it is a relation it captures dual behaviours that $\bar{\cdot}, \mathsf{cplt}()$ do not capture, like $\mu X.?\mathtt{Unit}.X$ and $!\mathtt{Unit}.\mu X.!\mathtt{Unit}.X$. *iii*) as stated in [1], it contains $\mathsf{cplt}()$. Before defining $\perp_{\mathsf{S}}$, we need the notion of type equivalence and hence subtyping. For simplicity, we omit subtyping on base types and on standard channel types, which are given in [2, 6].

Definition 2.2 (Subtyping and type equivalence for session types [6])**.** *A relation* $\mathcal{R} \subseteq \mathtt{SType} \times \mathtt{SType}$ *is a* type simulation *if* $(T, S) \in \mathcal{R}$ *implies the following:*
i) *if* $\mathrm{UNF}(T) = \mathsf{end}$ *then* $\mathrm{UNF}(S) = \mathsf{end}$
ii) *if* $\mathrm{UNF}(T) = ?T_m.T'$ *then* $\mathrm{UNF}(S) = ?S_m.S'$ *and* $T_m \;\mathcal{R}\; S_m$ *and* $T' \;\mathcal{R}\; S'$
iii) *if* $\mathrm{UNF}(T) = !T_m.T'$ *then* $\mathrm{UNF}(S) = !S_m.S'$ *and* $S_m \;\mathcal{R}\; T_m$ *and* $T' \;\mathcal{R}\; S'$
iv) *if* $\mathrm{UNF}(T) = \&\{l_i : T_i\}_{i\in I}$ *then* $\mathrm{UNF}(S) = \&\{l_j : S_j\}_{j\in J}$, $I \subseteq J$, $T_i \;\mathcal{R}\; S_i$, $\forall i \in I$
v) *if* $\mathrm{UNF}(T) = \oplus\{l_i : T_i\}_{i\in I}$ *then* $\mathrm{UNF}(S) = \oplus\{l_j : S_j\}_{j\in J}$, $J \subseteq I$, $T_j \;\mathcal{R}\; S_j$, $\forall j \in J$

The subtyping relation $\leq_{\mathsf{s}}$ *is defined by* $T \leq_{\mathsf{s}} S$ *if and only if there exists a type simulation* $\mathcal{R}$ *such that* $(T,S) \in \mathcal{R}$. *The* type equivalence relation $=_{\mathsf{s}}$ *is defined by* $T =_{\mathsf{s}} S$ *if and only if* $T \leq_{\mathsf{s}} S$ *and* $S \leq_{\mathsf{s}} T$.

Definition 2.3 (Co-inductive duality for session types [6])**.** *A relation* $\mathcal{R} \in \mathtt{SType} \times \mathtt{SType}$ *is a* duality relation *if* $(T,S) \in \mathcal{R}$ *implies the following conditions:*
i) If $\text{UNF}(T) = \mathsf{end}$ *then* $\text{UNF}(S) = \mathsf{end}$
ii) If $\text{UNF}(T) = ?T_m.T'$ *then* $\text{UNF}(S) = !S_m.S'$ *and* $T' \ \mathcal{R}\ S'$ *and* $T_m =_{\mathsf{s}} S_m$
iii) If $\text{UNF}(T) = !T_m.T'$ *then* $\text{UNF}(S) = ?S_m.S'$ *and* $T' \ \mathcal{R}\ S'$ *and* $T_m =_{\mathsf{s}} S_m$
iv) If $\text{UNF}(T) = \&\{l_i : T_i\}_{i\in I}$ *then* $\text{UNF}(S) = \oplus\{l_i : S_i\}_{i\in I}$ *and* $\forall i \in I,\ T_i \ \mathcal{R}\ S_i$
v) If $\text{UNF}(T) = \oplus\{l_i : T_i\}_{i\in I}$ *then* $\text{UNF}(S) = \&\{l_i : S_i\}_{i\in I}$ *and* $\forall i \in I,\ T_i \ \mathcal{R}\ S_i$
The co-inductive duality relation $\perp_{\mathsf{s}}$ *is defined by* $T \perp_{\mathsf{s}} S$ *iff* $\exists\ \mathcal{R}$*, a duality relation such that* $(T,S) \in \mathcal{R}$.

Proposition 2.4. *Let* $T,S \in \mathtt{SType}$, $\mathsf{cplt}(T) = S \implies T \perp_{\mathsf{s}} S$.

Proposition 2.5 (Idempotence)**.** *Let* $T,S,U \in \mathtt{SType}$. *If* $T \perp_{\mathsf{s}} S$ *and* $S \perp_{\mathsf{s}} U$ *then* $T =_{\mathsf{s}} U$.

By Proposition 2.4 and Proposition 2.5 we have the following.

Proposition 2.6. *Let* $T,S,U \in \mathtt{SType}$. *If* $\mathsf{cplt}(T) = S$ *and* $\mathsf{cplt}(S) = U$ *then* $T =_{\mathsf{s}} U$.

2.2 Background on standard π-calculus

Syntax. The syntax of the polyadic π-calculus [10] is given in Figure 3.

$$
\begin{array}{rcllllllll}
P,Q & ::= & x!\langle\tilde{v}\rangle.P & \text{(output)} & \mid & x?(\tilde{y}).P & \text{(input)} & \mid & P \mid Q & \text{(parallel)} \\
 & & (\nu x)P & \text{(channel res.)} & \mid & *P & \text{(repl.)} & \mid & \mathbf{case}\, v\, \mathbf{of}\, \{l_i\_x_i \triangleright P_i\}_{i\in I} & \text{(case)} \\
 & & \mathbf{0} & \text{(inaction)} & & & & & & \\
v & ::= & x & \text{(variable)} & \mid & \mathbf{1} & \text{(unit val.)} & \mid & l\_v & \text{(variant val.)}
\end{array}
$$

Figure 3: Standard π-calculus, syntax.

P,Q range over processes, x,y over variables, l over labels and v over values, i.e., variables, $\mathbf{1}$, or variant values. A process can be an output $x!\langle\tilde{v}\rangle.P$ which sends $\tilde{v}$ on x and proceeds as P; an input $x?(\tilde{y}).P$ which receives a sequence of values on x, substitutes them for $\tilde{y}$ in P; a parallel composition $P \mid Q$ of P,Q; replicated $*P$; a restriction $(\nu x)P$ which creates a new channel x and binds it in P; a $\mathbf{case}\, v\, \mathbf{of}\, \{l_i\_x_i \triangleright P_i\}_{i\in I}$ which offers a set of labelled processes, with labels being all different; or inaction $\mathbf{0}$.

Standard π-types. The syntax of π-types [9, 10] is defined in Figure 4.

$$
\begin{array}{rclllllll}
\tau & ::= & \emptyset[] & \text{(no capability)} & \mid & \sharp[\widetilde{T}] & \text{(connection)} & & \\
 & & \ell_{\mathsf{i}}\,[\widetilde{T}] & \text{(linear input)} & \mid & \ell_{\mathsf{o}}\,[\widetilde{T}] & \text{(linear output)} & \mid & \ell_{\sharp}\,[\widetilde{T}] \quad \text{(linear connection)} \\
T & ::= & \tau & \text{(channel type)} & \mid & \langle l_i : T_i\rangle_{i\in I} & \text{(variant type)} & \mid & X \quad \text{(type var.)} \\
 & & \overline{X} & \text{(dual type var.)} & \mid & \mathtt{Unit} & \text{(unit type)} & \mid & \mu X.T \quad \text{(recursive type)}
\end{array}
$$

Figure 4: Standard π-calculus, types.

τ ranges over channel types and T over types. A channel type is a type with no capability $\emptyset[]$, meaning it cannot be used further; a connection $\sharp[\widetilde{T}]$, indefinitely used; a linear input $\ell_{\mathsf{i}}\,[\widetilde{T}]$, a linear output $\ell_{\mathsf{o}}\,[\widetilde{T}]$ or the combination of both, i.e., a linear connection $\ell_{\sharp}\,[\widetilde{T}]$ used *exactly once* [9] according to its capability.

A type can be a channel type τ; a variant $\langle l_i : T_i \rangle_{i \in I}$ being a set of labelled types, with labels being all different; a (dualised) type variable $X, \overline{X}$ or recursive type $\mu X.T$ or Unit. Again, we require recursive types to be *guarded* and use PType to denote the set of *closed* (no free type variables) and *guarded* standard π-types. To conclude, the definition of *unfolding* is the same as in the previous section.

On duality for linear types. Inspired by duality on session types, below we give the definition of complement function $\mathsf{picplt}()$ and co-inductive duality relation $\perp_{\mathsf{p}}$ for linear π-types.

Definition 2.7 (Complement function for linear π-types)**.** *The complement function is defined as:*

$$\begin{array}{lclclcl}
\mathsf{picplt}(\ell_{\mathsf{i}}\ [\widetilde{T}]) & = & \ell_{\mathsf{o}}\ [\widetilde{T}] & & \mathsf{picplt}(X) & = & \overline{X} \\
\mathsf{picplt}(\ell_{\mathsf{o}}\ [\widetilde{T}]) & = & \ell_{\mathsf{i}}\ [\widetilde{T}] & & \mathsf{picplt}(\overline{X}) & = & X \\
\mathsf{picplt}(\mu X.T) & = & \mu X.\mathsf{picplt}(T\lfloor^{\mu X.T}/_X\rfloor) & & \mathsf{picplt}(\emptyset[]) & = & \emptyset[]
\end{array}$$

The definition of type substitution for linear types is the same as in the previous section, where $\mathsf{cplt}()$ is replaced by $\mathsf{picplt}()$. Before defining $\perp_{\mathsf{p}}$, we give a co-inductive definition of subtyping and type equivalence for linear π-types. For simplicity, we omit the subtyping on base types, standard channel types, or variant types which can be found in the literature [10].

Definition 2.8 (Subtyping and type equivalence for linear π-types)**.** *A relation* $\mathcal{R} \subseteq \mathtt{PType} \times \mathtt{PType}$ *is a* type simulation *if* $(T,S) \in \mathcal{R}$ *implies the following:*
i) if $\textsc{unf}(T) = \emptyset[]$ *then* $\textsc{unf}(S) = \emptyset[]$
ii) if $\textsc{unf}(T) = \ell_{\mathsf{i}}\ [\widetilde{T}]$ *then* $\textsc{unf}(S) = \ell_{\mathsf{i}}\ [\widetilde{S}]$ *and* $\widetilde{T}\ \mathcal{R}\ \widetilde{S}$
iii) if $\textsc{unf}(T) = \ell_{\mathsf{o}}\ [\widetilde{T}]$ *then* $\textsc{unf}(S) = \ell_{\mathsf{o}}\ [\widetilde{S}]$ *and* $\widetilde{T}\ \mathcal{R}\ \widetilde{S}$
The subtyping relation $\leq_{\mathsf{s}}$ *is defined by* $T \leq_{\mathsf{p}} S$ *if and only if there exists a type simulation* $\mathcal{R}$ *such that* $(T,S) \in \mathcal{R}$. *The* type equivalence relation $=_{\mathsf{p}}$ *is defined by* $T =_{\mathsf{p}} S$ *if and only if* $T \leq_{\mathsf{p}} S$ *and* $S \leq_{\mathsf{p}} T$.

Definition 2.9 (Co-inductive duality for linear π- types)**.** *A relation* $\mathcal{R} \in \mathtt{PType} \times \mathtt{PType}$ *is a* duality relation *if* $(T,S) \in \mathcal{R}$ *implies the following conditions:*
i) If $\textsc{unf}(T) = \emptyset[]$ *then* $\textsc{unf}(S) = \emptyset[]$
ii) If $\textsc{unf}(T) = \ell_{\mathsf{i}}\ [\widetilde{T}]$ *then* $\textsc{unf}(S) = \ell_{\mathsf{o}}\ [\widetilde{S}]$ *and* $\widetilde{T} =_{\mathsf{p}} \widetilde{S}$
iii) If $\textsc{unf}(T) = \ell_{\mathsf{o}}\ [\widetilde{T}]$ *then* $\textsc{unf}(S) = \ell_{\mathsf{i}}\ [\widetilde{T}]$*and* $\widetilde{T} =_{\mathsf{p}} \widetilde{S}$
The co-inductive duality relation $\perp_{\mathsf{p}}$ *is defined by* $T \perp_{\mathsf{p}} S$ *iff* $\exists\ \mathcal{R}$*, a duality relation such that* $(T,S) \in \mathcal{R}$.

Proposition 2.10. *Let* $T,S \in \mathtt{PType}$, $\mathsf{picplt}(T) = S \implies T \perp_{\mathsf{p}} S$.

Proposition 2.11 (Idempotence)**.** *Let* $T,S,U \in \mathtt{PType}$. *If* $T \perp_{\mathsf{p}} S$ *and* $S \perp_{\mathsf{p}} U$ *then* $T =_{\mathsf{p}} U$.

Proposition 2.12. *Let* $T,S,U \in \mathtt{PType}$. *If* $\mathsf{picplt}(T) = S$ *and* $\mathsf{picplt}(S) = U$ *then* $T =_{\mathsf{p}} U$.

3 Encoding recursive session types

Below we give the encoding of recursive session types into recursive linear π-types and of session π-processes into standard π-processes. It is based on the notions of: *linearity*, *variant types* and *continuation-passing* principle. To preserve communication safety and privacy of session types, we use linear channels. To encode internal and external choice, we adopt variant types and the **case** process. To preserve the sequentiality of session types and hence session fidelity, we adopt the continuation-passing principle.

Types encoding. The encoding of session types is presented in Figure 5. Type end is encoded as the channel type with no capability $\emptyset[]$; output $!T.S$ and input $?T.S$ session types are encoded as linear output $\ell_{\mathsf{o}}\ [[\![T]\!], [\![\mathsf{cplt}(S)]\!]]$ and linear input $\ell_{\mathsf{i}}\ [[\![T]\!], [\![S]\!]]$ channel types carrying the encoding of type T and of

$$
\begin{array}{lll}
\llbracket \mathsf{end} \rrbracket & \stackrel{\text{def}}{=} & \emptyset[] \\
\llbracket !T.S \rrbracket & \stackrel{\text{def}}{=} & \ell_{\mathsf{o}}\ [\llbracket T \rrbracket, \llbracket \mathsf{cplt}(S) \rrbracket] \\
\llbracket ?T.S \rrbracket & \stackrel{\text{def}}{=} & \ell_{\mathsf{i}}\ [\llbracket T \rrbracket, \llbracket S \rrbracket] \\
\llbracket \oplus\{l_i : S_i\}_{i\in I} \rrbracket & \stackrel{\text{def}}{=} & \ell_{\mathsf{o}}\ [\langle l_i : \llbracket \mathsf{cplt}(S_i) \rrbracket \rangle_{i\in I}] \\
\llbracket \&\{l_i : S_i\}_{i\in I} \rrbracket & \stackrel{\text{def}}{=} & \ell_{\mathsf{i}}\ [\langle l_i : \llbracket S_i \rrbracket \rangle_{i\in I}] \\
\llbracket X \rrbracket & \stackrel{\text{def}}{=} & X \\
\llbracket \overline{X} \rrbracket & \stackrel{\text{def}}{=} & \overline{X} \\
\llbracket \mu X.S \rrbracket & \stackrel{\text{def}}{=} & \mu X.\llbracket S \rrbracket
\end{array}
\qquad
\begin{array}{lll}
\llbracket x \rrbracket_f & \stackrel{\text{def}}{=} & f_x \\
\llbracket x!\langle v \rangle.P \rrbracket_f & \stackrel{\text{def}}{=} & (\boldsymbol{\nu} c) f_x!\langle v, c \rangle.\llbracket P \rrbracket_{f,\{x\mapsto c\}} \\
\llbracket x?(y).P \rrbracket_f & \stackrel{\text{def}}{=} & f_x?(y,c).\llbracket P \rrbracket_{f,\{x\mapsto c\}} \\
\llbracket x \triangleleft l_j.P \rrbracket_f & \stackrel{\text{def}}{=} & (\boldsymbol{\nu} c) f_x!\langle l_j\_c \rangle.\llbracket P \rrbracket_{f,\{x\mapsto c\}} \\
\llbracket x \triangleright \{l_i : P_i\}_{i\in I} \rrbracket_f & \stackrel{\text{def}}{=} & f_x?(y).\ \mathbf{case}\ y\ \mathbf{of}\ \{l_i\_c \triangleright \llbracket P_i \rrbracket_{f,\{x\mapsto c\}}\}_{i\in I} \\
\llbracket (\boldsymbol{\nu} xy)P \rrbracket_f & \stackrel{\text{def}}{=} & (\boldsymbol{\nu} c)\llbracket P \rrbracket_{f,\{x,y\mapsto c\}} \\
\llbracket P \mid Q \rrbracket_f & \stackrel{\text{def}}{=} & \llbracket P \rrbracket_f \mid \llbracket Q \rrbracket_f \\
\llbracket (\boldsymbol{\nu} x)P \rrbracket_f & \stackrel{\text{def}}{=} & (\boldsymbol{\nu} x)\llbracket P \rrbracket_f \\
\llbracket *P \rrbracket_f & \stackrel{\text{def}}{=} & *\llbracket P \rrbracket_f \\
\llbracket \mathbf{0} \rrbracket_f & \stackrel{\text{def}}{=} & \mathbf{0}
\end{array}
$$

Figure 5: Encoding of session types and terms

continuation type $\mathsf{cplt}(S)$ and S, respectively. $\mathsf{cplt}(S)$ is adopted in the output since it is the type of a channel as seen by the receiver, namely the communicating counterpart. Select $\oplus\{l_i : S_i\}_{i\in I}$ and branch $\&\{l_i : S_i\}_{i\in I}$ are encoded as linear output $\ell_{\mathsf{o}}\ [\langle l_i : \llbracket \mathsf{cplt}(S_i) \rrbracket \rangle_{i\in I}]$ and linear input $\ell_{\mathsf{i}}\ [\langle l_i : \llbracket S_i \rrbracket \rangle_{i\in I}]$ types carrying a variant type with the encoded continuation types; the reason for $\mathsf{cplt}(S_i)$ is the same as before. The encoding of a (dualised) type variable and a recursive session type is an homomorphism.

Terms encoding. The encoding of session π-terms, presented in Figure 5, uses a partial function f from variables to variables which performs a renaming of linear variables into new linear variables to respect their nature of being used exactly once and it is the identity function over standard variables. The formal definition can be found in [3]. A variable x is encoded as f_x; an output $x!\langle v \rangle.P$ is encoded as an output on f_x of v and the freshly created channel c which replaces x in the encoding of P. The encoding of an input $x?(y).P$ is an input on f_x with placeholders y and the continuation channel c used in $\llbracket P \rrbracket_{f,\{x\mapsto c\}}$. The encodings of selection $x \triangleleft l_j.P$ and branching $x \triangleright \{l_i : P_i\}_{i\in I}$ are the output and input processes on f_x, respectively. The output carries a variant value l_j_c where l_j is the selected label and c the new channel to be used in the continuation. The input has a continuation of a **case**, offering the encoded processes of the branching. The session restriction process $(\boldsymbol{\nu} xy)P$ is encoded as the restriction on c which replaces both of the endpoint x, y in the encoding of P. The rest of the equations states that the encoding of parallel composition, standard channel restriction and replication is an homomorphism and the encoding of the inaction process is the identity function.

Results of the encoding. The proofs of the following results can be found in [3]. The following two lemmas relate the encoding of equal and dual session types to equal and dual linear π-types.

Lemma 3.1 (Encoding $=_{\mathsf{s}}$). *$T, S \in \mathtt{SType}$ and $T =_{\mathsf{s}} S$. If $\llbracket T \rrbracket = \tau$, then $\llbracket S \rrbracket = \sigma$ and $\tau =_{\mathsf{p}} \sigma$.*

Lemma 3.2 (Encoding $\perp_{\mathsf{s}}$). *$T, S \in \mathtt{SType}$ and $T \perp_{\mathsf{s}} S$. If $\llbracket \textsc{unf}(T) \rrbracket = \tau$ then, $\llbracket \textsc{unf}(S) \rrbracket = \sigma$ and $\tau \perp_{\mathsf{p}} \sigma$.*

Lemma 3.3 (Value Typing). *$\Gamma \vdash v : T$ if and only if $\llbracket \Gamma \rrbracket_f \vdash \llbracket v \rrbracket_f : \llbracket T \rrbracket$.* [1]

Theorem 3.4 (Process Typing). *$\Gamma \vdash P$ if and only if $\llbracket \Gamma \rrbracket_f \vdash \llbracket P \rrbracket_f$.*

Theorem 3.5 (Operational Correspondence). *Let P be a session process. The following hold.*

1. *If $P \to P'$ then $\llbracket P \rrbracket_f \to\hookrightarrow \llbracket P' \rrbracket_f$,*

[1] The encoding is extended to typing environments Γ and the details can be checked in [3].

2. *If* $[\![P]\!]_f \to Q$ *then,* $\exists\, P', \mathcal{E}[\cdot]$ *such that* $\mathcal{E}[P] \to \mathcal{E}[P']$ *and* $Q \hookrightarrow [\![P']\!]_{f'}$*, where* f' *is the updated* f *after reduction and* $f_x = f_y$ *for all* $(\nu xy) \in \mathcal{E}[\cdot]$.

$\hookrightarrow$ denotes $\equiv$ possibly extended with a **case** reduction; $\mathcal{E}[\cdot]$ is an *evaluation context*.

4 Example of encoding

We present an error-free process which requires recursive session types. We let a, b range over standard channels and x, y, z, v, w range over session channels; we associate a type to a variable in an object position in input or restriction as in [6]. The typing rules and the operational semantics can be found in [3].

Let $P = *(a?(x:T).x \triangleleft l.a!\langle x\rangle.\mathbf{0})$ be a replicated process that on a standard channel a receives a session channel x on which selects l and proceeds as $a!\langle x\rangle$. We have the following typing derivation for P:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{a:\sharp T \vdash \mathbf{0}}\ \text{T-Nil} \qquad S \preceq_{\mathsf{s}} T}{a:\sharp T, x:S \vdash a!\langle x\rangle.\mathbf{0}}\ \text{T-Out}}{a:\sharp T, x:\oplus\{l:S\} \vdash x \triangleleft l.a!\langle x\rangle.\mathbf{0} \qquad T \preceq_{\mathsf{s}} \oplus\{l:S\}}\ \text{T-Select}}{a:\sharp T \vdash a?(x:T).x \triangleleft l.a!\langle x\rangle.\mathbf{0} \qquad \mathsf{un}(a:\sharp T)}\ \text{T-In}}{a:\sharp T \vdash *(a?(x:T).x \triangleleft l.a!\langle x\rangle.\mathbf{0})}\ \text{T-Rep}$$

For this derivation to hold, T and S need to be such that $T \preceq_{\mathsf{s}} \oplus\{l:S\}$ and $S \preceq_{\mathsf{s}} T$. The simplest way to solve this system of subtyping in-equations is to have $S = T$, which requires $T = \mu X.\oplus\{l:X\}$.

Let $Q = *(b?(x:U).x \triangleright \{l: b!\langle x\rangle.\mathbf{0}\})$; it has dual behaviour to P and the typing derivation is similar to above, with T-Select replaced by T-Branch. We now have the in-equations $U \preceq_{\mathsf{s}} \&\{l:S'\}$ and $S' \preceq_{\mathsf{s}} U$. We let $U = \mu X.\&\{l:X\}$. By Definition 2.3, we have $U \perp_{\mathsf{s}} T$. We now close P and Q with two auxiliary output processes, $a!\langle v\rangle.\mathbf{0}$ and $b!\langle w\rangle.\mathbf{0}$, where v, w are to be co-variables. Then, we have:

$$\begin{aligned}
\emptyset \vdash Sys =& (\nu a:\sharp T)(\nu b:\sharp U)(\nu vw:T)(a!\langle v\rangle.\mathbf{0} \mid b!\langle w\rangle.\mathbf{0} \mid P \mid Q)\\
\equiv\to& (\nu a:\sharp T)(\nu b:\sharp U)(\nu vw:T)(v \triangleleft l.a!\langle v\rangle.\mathbf{0} \mid w \triangleright \{l: b!\langle w\rangle.\mathbf{0}\} \mid P \mid Q)\\
\to& (\nu a:\sharp T)(\nu b:\sharp U)(\nu vw:T)(a!\langle v\rangle.\mathbf{0} \mid b!\langle w\rangle.\mathbf{0} \mid P \mid Q) = Sys \to^*
\end{aligned}$$

The encoding of types is as follows. Since $U \perp_{\mathsf{s}} T$ by Lemma 3.2 we have $\upsilon \perp_{\mathsf{p}} \tau$.

$$\begin{aligned}
[\![U]\!] &= [\![\mu X.\&\{l:X\}]\!] = \mu X.[\![\&\{l:X\}]\!] = \mu X.\ell_{\mathsf{i}}\,[\langle l:[\![X]\!]\rangle] = \mu X.\ell_{\mathsf{i}}\,[\langle l:X\rangle] = \upsilon\\
[\![T]\!] &= [\![\mu X.\oplus\{l:X\}]\!] = \mu X.[\![\oplus\{l:X\}]\!] = \mu X.\ell_{\mathsf{o}}\,[\langle l:[\![\mathsf{cplt}(X)]\!]\rangle] = \mu X.\ell_{\mathsf{o}}\,[\langle l:\overline{X}\rangle] = \tau
\end{aligned}$$

Duality of session types boils down to opposite capabilities in the outermost level $(\ell_{\mathsf{i}}, \ell_{\mathsf{o}})$ and the *same* carried type, where in [5] same means *syntactic identity* and in the present means *type equivalence*. Unfolding is performed in order to test linear type duality and the type equivalence of the carried type.

$$\begin{aligned}
[\![P]\!]_f &= [\![*(a?(x).x \triangleleft l.a!\langle x\rangle.\mathbf{0})]\!]_f = *[\![(a?(x).x \triangleleft l.a!\langle x\rangle.\mathbf{0})]\!]_f = *(a?(x).[\![x \triangleleft l.a!\langle x\rangle.\mathbf{0}]\!]_f)\\
&= *(a?(x).(\nu c)x!\langle l_c\rangle.[\![a!\langle x\rangle.\mathbf{0}]\!]_{f,\{x\mapsto c\}}) = *(a?(x).(\nu c)x!\langle l_c\rangle.a!\langle c\rangle.\mathbf{0})\\
[\![Q]\!]_f &= [\![*(b?(x).x \triangleright \{l: b!\langle x\rangle.\mathbf{0}\})]\!]_f = *[\![(b?(x).x \triangleright \{l: b!\langle x\rangle.\mathbf{0}\})]\!]_f = *(b?(x).[\![x \triangleright \{l: b!\langle x\rangle.\mathbf{0}\})]\!]_f\\
&= *(b?(x).x?(y).\mathbf{case}\, y\, \mathbf{of}\, \{l_c \triangleright [\![b!\langle x\rangle.\mathbf{0}\}]\!]_{f,\{x\mapsto c\}}\}) = *(b?(x).x?(y).\mathbf{case}\, y\, \mathbf{of}\, \{l_c \triangleright b!\langle c\rangle.\mathbf{0}\}\})
\end{aligned}$$

$$\begin{aligned}
\emptyset \vdash [\![Sys]\!]_f =& (\nu a)(\nu b)(\nu z)[\![(a!\langle v\rangle.\mathbf{0} \mid b!\langle w\rangle.\mathbf{0} \mid P \mid Q)]\!]_{f,\{v,w\mapsto z\}} = (\nu a)(\nu b)(\nu z)(a!\langle z^+\rangle.\mathbf{0} \mid b!\langle z^-\rangle.\mathbf{0} \mid [\![P]\!]_f \mid [\![Q]\!]_f)\\
\equiv& (\nu a)(\nu b)(\nu z)(a!\langle z^+\rangle.\mathbf{0} \mid b!\langle z^-\rangle.\mathbf{0} \mid a?(x).(\nu c)x!\langle l_c\rangle.a!\langle c\rangle.\mathbf{0} \mid b?(x).x?(y).\mathbf{case}\, y\, \mathbf{of}\, \{l_c \triangleright b!\langle c\rangle.\mathbf{0}\}\} \mid\\
& *(a?(x).(\nu c)x!\langle l_c\rangle.a!\langle c\rangle.\mathbf{0}) \mid *(b?(x).x?(y).\mathbf{case}\, y\, \mathbf{of}\, \{l_c \triangleright b!\langle c\rangle.\mathbf{0}\}\}))
\end{aligned}$$

$$\rightarrow (\nu a)(\nu b)(\nu z)((\nu c)z^+!\langle l\_c^-\rangle.a!\langle c^+\rangle.\mathbf{0} \mid z^-?(y).\mathbf{case}\,y\,\mathbf{of}\,\{l\_c^- \triangleright b!\langle c^-\rangle.\mathbf{0}\}\} \mid [\![P]\!]_f \mid [\![Q]\!]_f)$$
$$\rightarrow (\nu a)(\nu b)(\nu z)((\nu c).a!\langle c^+\rangle.\mathbf{0} \mid \mathbf{case}\,l\_c^-\,\mathbf{of}\,\{l\_c^- \triangleright b!\langle c^-\rangle.\mathbf{0}\}\} \mid [\![P]\!]_f \mid [\![Q]\!]_f)$$
$$\hookrightarrow (\nu a)(\nu b)((\nu c).a!\langle c^+\rangle.\mathbf{0} \mid b!\langle c^-\rangle.\mathbf{0}\}\} \mid [\![P]\!]_f \mid [\![Q]\!]_f) \equiv [\![Sys]\!]_f \rightarrow^*$$

5 Conclusions and Future Work

In this paper we present an encoding of recursive session types into recursive linear types and session processes into corresponding π- processes. The encoding is a conservative extension of the one given in [5]. It uses cplt() instead of $\bar{\cdot}$, because the latter is inadequate in the presence of recursive types [1,2]. Since these two functions coincide for finite session types, the encoding in [5] remains sound. We prove the faithfulness of the present encoding with respect to typing derivations and operational semantics, following the same line of [4,5]. As long as future work is concerned, we would like to test our encoding under different dualities for sessions presented in [1]. Moreover, as in [5] we would like to extend the present encoding to advanced features like polymorphism or higher-order, or multiparty session types.
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A Appendix

A.1 Operational Semantics

Session π- calculus. The operational semantics of the π-calculus with sessions is given in terms of a binary reduction relation $\rightarrow$ over processes defined by the rules in Figure 6.

$$
\begin{array}{rl}
\text{(R- S{\scriptsize TND}C{\scriptsize OM})} & x!\langle v\rangle.P \mid x?(z).Q \rightarrow P \mid Q[v/z] \\
\text{(R-C{\scriptsize OM})} & (\boldsymbol{\nu} xy)(x!\langle v\rangle.P \mid y?(z).Q) \rightarrow (\boldsymbol{\nu} xy)(P \mid Q[v/z]) \\
\text{(R-S{\scriptsize EL})} & (\boldsymbol{\nu} xy)(x \triangleleft l_j.P \mid y \triangleright \{l_i : P_i\}_{i\in I}) \rightarrow (\boldsymbol{\nu} xy)(P \mid P_j) \quad j \in I
\end{array}
$$

Figure 6: π-calculus with sessions, semantics.

Rule (R-StndCom) is the communication rule on a standard channel x. Rule (R-Com) is the rule for communication on session channel endpoints: the output process sends a value v on x, whether the input process receives it on y and substitutes the placeholder z with it. A key difference with the previous communication rule is that the subject of the output (x) and the subject of the input (y) are two co-variables, created and bound together by the ($\boldsymbol{\nu}$) construct. A consequence of this is that communication happens only in bound variables. After the communication the restriction still persists in order to enable further possible communications. Rule (R-Sel) is similar to the previous one: communication occurs on co-variables being under a session restriction. The selection process selects a label among the ones offered by the branching process and then continues as P, whether the branching process continues as the P_j of the selected label. The reduction relation is closed under the usual contexts: session restriction, channel restriction, parallel composition and usual structural congruence $\equiv$ [6, 12].

Standard π-calculus. The operational semantics of the standard π-calculus is given in terms of a binary reduction relation $\rightarrow$ over processes defined by the rules in Figure 7.

$$
\begin{array}{rl}
\text{(R}\pi\text{-C{\scriptsize OM})} & x!\langle \tilde{v}\rangle.P \mid x?(\tilde{z}).Q \rightarrow P \mid Q[\tilde{v}/\tilde{z}] \\
\text{(R}\pi\text{-C{\scriptsize ASE})} & \textbf{case}\, l_j\_v\, \textbf{of}\, \{l_i\_x_i \triangleright P_i\}_{i\in I} \rightarrow P_j[v/x_j] \quad j \in I
\end{array}
$$

Figure 7: Standard π-calculus, semantics.

Rule (Rπ-Com) is the communication rule: the output process sends a sequence of values $\tilde{v}$ on x and the input process receives it and substitutes the sequence of placeholders $\tilde{z}$. Rule (Rπ-Case), is often called a *case normalization* since it does not require a counterpart to reduce. The case process reduces to P_j substituting x_j with the value v, if the label l_j, among the offered ones, is selected. As usual, the reduction relation is closed under channel restriction, parallel composition and usual structural congruence $\equiv$ [10].

A.2 Typing Rules

Session π-calculus. Typing rules for the π-calculus with session types are given in Figure 8.

Standard π-calculus. Typing rules for the π-calculus with linear types are given in Figure 9.

$$\textsc{T-In}\ \frac{\Gamma_1 \vdash x : ?T.S \quad \Gamma_2, x : S, y : T \vdash P}{\Gamma_1 \circ \Gamma_2 \vdash x?(y).P} \qquad \textsc{T-Out}\ \frac{\Gamma_1 \vdash x : !T.S \quad \Gamma_2 \vdash v : T \quad \Gamma_3, x : S \vdash P}{\Gamma_1 \circ \Gamma_2 \circ \Gamma_3 \vdash x!\langle v\rangle.P}$$

$$\textsc{T-Branch}\ \frac{\Gamma_1 \vdash x : \&\{l_i : S_i\}_{i\in I} \quad \Gamma_2, x : S_i \vdash P_i\ \forall i \in I}{\Gamma_1 \circ \Gamma_2 \vdash x \triangleright \{l_i : P_i\}_{i\in I}} \qquad \textsc{T-Select}\ \frac{\Gamma_1 \vdash x : \oplus\{l_i : S_i\}_{i\in I} \quad \Gamma_2, x : S_j \vdash P\ j \in I}{\Gamma_1 \circ \Gamma_2 \vdash x \triangleleft l_j.P}$$

$$\textsc{T-Par}\ \frac{\Gamma_1 \vdash P \quad \Gamma_2 \vdash Q}{\Gamma_1 \circ \Gamma_2 \vdash P \mid Q} \qquad \textsc{T-SessRes}\ \frac{\Gamma, x : T, y : S \vdash P \quad T \perp_{\mathsf{s}} S}{\Gamma \vdash (\boldsymbol{\nu} xy)P} \qquad \textsc{T-Res}\ \frac{\Gamma, x : T \vdash P}{\Gamma \vdash (\boldsymbol{\nu} x)P} \qquad \textsc{T-Rep}\ \frac{\Gamma \vdash P \quad \mathsf{un}(\Gamma)}{\Gamma \vdash *P} \qquad \textsc{T-Nil}\ \frac{\mathsf{un}(\Gamma)}{\Gamma \vdash \mathbf{0}}$$

Figure 8: π-calculus with sessions, typing rules.

$$\textsc{T}\pi\textsc{-Inp}\ \frac{\Gamma_1 \vdash x : \ell_{\mathsf{i}}\ [\widetilde{T}] \qquad \Gamma_2, \tilde{y} : \widetilde{T} \vdash P}{\Gamma_1 \uplus \Gamma_2 \vdash x?(\tilde{y}).P} \qquad \textsc{T}\pi\textsc{-Out}\ \frac{\Gamma_1 \vdash x : \ell_{\mathsf{o}}\ [\widetilde{T}] \quad \widetilde{\Gamma_2} \vdash \tilde{v} : \widetilde{T} \quad \Gamma_3 \vdash P}{\Gamma_1 \uplus \widetilde{\Gamma_2} \uplus \Gamma_3 \vdash x!\langle \tilde{v}\rangle.P}$$

$$\textsc{T}\pi\textsc{-Case}\ \frac{\Gamma_1 \vdash v : \langle l_i : T_i\rangle_{i\in I} \quad \Gamma_2, x_i : T_i \vdash P_i \quad \forall i \in I}{\Gamma_1 \uplus \Gamma_2 \vdash \mathbf{case}\, v\, \mathbf{of}\, \{l_i\_x_i \triangleright P_i\}_{i\in I}} \qquad \textsc{T}\pi\textsc{-Variant}\ \frac{\Gamma \vdash v : T_j \quad j \in I}{\Gamma \vdash l_j\_v : \langle l_i : T_i\rangle_{i\in I}}$$

$$\textsc{T}\pi\textsc{-Par}\ \frac{\Gamma_1 \vdash P \quad \Gamma_2 \vdash Q}{\Gamma_1 \uplus \Gamma_2 \vdash P \mid Q} \qquad \textsc{T}\pi\textsc{-Res}\ \frac{\Gamma, x : \ell_\alpha\ [\widetilde{T}] \vdash P}{\Gamma \vdash (\boldsymbol{\nu} x)P} \qquad \textsc{T}\pi\textsc{-Rep}\ \frac{\Gamma \vdash P \quad \mathsf{un}(\Gamma)}{\Gamma \vdash *P} \qquad \textsc{T}\pi\textsc{-Nil}\ \frac{\mathsf{un}(\Gamma)}{\Gamma \vdash \mathbf{0}}$$

Figure 9: π-calculus with linear types, typing rules.

A.3 Results on duality

This part of the appendix is dedicated to duality and results related to this notion. The complement function, as previously stated, uses a syntactic substitution $\lfloor {}^{-}/_{-} \rfloor$, which acts only on carried types and is formally defined in [1,2]. We report it in the following:

$$\begin{array}{lcl}
?T.S'\lfloor {}^{S}/_{X} \rfloor & = & !T.(S'\lfloor {}^{S}/_{X} \rfloor) \ \text{ if } T \text{ is a base type} \\
!T.S'\lfloor {}^{S}/_{X} \rfloor & = & !T.(S'\lfloor {}^{S}/_{X} \rfloor) \ \text{ if } T \text{ is a base type} \\
?T.S'\lfloor {}^{S}/_{X} \rfloor & = & ?T\{{}^{S}/_{X}\}.(S'\lfloor {}^{S}/_{X} \rfloor) \\
!T.S'\lfloor {}^{S}/_{X} \rfloor & = & !T\{{}^{S}/_{X}\}.(S'\lfloor {}^{S}/_{X} \rfloor) \\
\&\{l_i : S_i\}_{i\in I}\lfloor {}^{S}/_{X} \rfloor & = & \&\{l_i : S_i\lfloor {}^{S}/_{X} \rfloor\}_{i\in I} \\
\oplus\{l_i : S_i\}_{i\in I}\lfloor {}^{S}/_{X} \rfloor & = & \oplus\{l_i : S_i\lfloor {}^{S}/_{X} \rfloor\}_{i\in I}
\end{array}
\qquad
\begin{array}{lcl}
Y\lfloor {}^{S}/_{X} \rfloor & = & X \ \text{ if } Y \neq X \\
\mathsf{end}\lfloor {}^{S}/_{X} \rfloor & = & \mathsf{end} \\
(\mu Y.S')\lfloor {}^{S}/_{X} \rfloor & = & \mu Y.(S'\lfloor {}^{S}/_{X} \rfloor) \ \text{ if } Y \neq X \\
(\mu X.S')\lfloor {}^{S}/_{X} \rfloor & = & \mu X.S'
\end{array}$$

We give now the proofs of the results presented in the main part of the paper.

Proposition A.1. *Let* $T, S \in \mathsf{SType}$, $\mathsf{cplt}(T) = S \implies T \perp_{\mathsf{s}} S$.

Proof. Follows from [1]. Intuitively, it holds since the former is a function and the latter a relation and both are defined by following the same idea. □

Proposition A.2 (Idempotence)**.** *Let* $T,S,U \in \mathsf{SType}$. *If* $T \perp_{\mathsf{s}} S$ *and* $S \perp_{\mathsf{s}} U$ *then* $T =_{\mathsf{s}} U$.

Proof. Suppose $T \perp_{\mathsf{s}} S$ and $S \perp_{\mathsf{s}} U$. To prove that $T =_{\mathsf{s}} U$, we need to construct a relation $\mathcal{R}$ such that $(T,U) \in \mathcal{R}$ and $\mathcal{R}$ and $\mathcal{R}^{-1}$ are type simulation relations, which by Definition 2.2, it means to show that $\mathcal{R}$ is a type equivalence relation. Let $\mathcal{R} = \{ (T',U') \mid \exists S'.T' \perp_{\mathsf{s}} S', S' \perp_{\mathsf{s}} U' \}$. We show that $\mathcal{R}$ is a type equivalence relation. Let $(T,U) \in \mathcal{R}$, so by definition of $\mathcal{R}$, $\exists S$ such that $T \perp_{\mathsf{s}} S$ and $S \perp_{\mathsf{s}} U$.

If $\textsc{unf}(T) = \mathsf{end}$ then, since $T \perp_{\mathsf{s}} S$ we have $\textsc{unf}(S) = \mathsf{end}$ and since $S \perp_{\mathsf{s}} U$ we have $\textsc{unf}(U) = \mathsf{end}$ which concludes the case.

If $\textsc{unf}(T) = ?T_m.T_n$ then, since $T \perp_{\mathsf{s}} S$ we have $\textsc{unf}(S) = !S_m.S_n$ and $T_m =_{\mathsf{s}} S_m$, $T_n \perp_{\mathsf{s}} S_n$. On the other hand, since $S \perp_{\mathsf{s}} U$ we have $\textsc{unf}(U) = ?U_m.U_n$ and $S_m =_{\mathsf{s}} U_m$ and $S_n \perp_{\mathsf{s}} U_n$. By transitivity of $=_{\mathsf{s}}$ we have $T_m =_{\mathsf{s}} U_m$ and since $T_n \perp_{\mathsf{s}} S_n$ and $S_n \perp_{\mathsf{s}} U_n$ by definition of $\mathcal{R}$ we have that $T_n\ \mathcal{R}\ U_n$ which concludes this case. The case of $\textsc{unf}(T) = !T_m.T_n$ is symmetric to the previous one, where ? is exchanged with !.

If $\textsc{unf}(T) = \&\{l_i : T_i\}_{i \in I}$ then, since $T \perp_{\mathsf{s}} S$ we have $\textsc{unf}(S) = \oplus\{l_i : S_i\}_{i \in I}$ and $\forall i \in I$, $T_i \perp_{\mathsf{s}} S_i$, and since $S \perp_{\mathsf{s}} U$ we have $\textsc{unf}(U) = \&\{l_i : U_i\}_{i \in I}$ and $\forall i \in I. S_i \perp_{\mathsf{s}} U_i$. By definition of $\mathcal{R}$ since $\forall i \in I$ we have that $T_i \perp_{\mathsf{s}} S_i$ and $S_i \perp_{\mathsf{s}} U_i$ then $\forall i \in I$, $T_i \perp_{\mathsf{s}} U_i$ which concludes this case. The case of $\textsc{unf}(T) = \oplus\{l_i : T_i\}_{i \in I}$ is symmetric to the previous one, where & is exchanged with $\oplus$. ◻

Proposition A.3. *Let* $T,S \in \mathsf{PType}$, $\mathsf{picplt}(T) = S \implies T \perp_{\mathsf{p}} S$.

Proof. The proof follows the same line as Proposition A.1. ◻

Proposition A.4 (Idempotence)**.** *Let* $T,S,U \in \mathsf{PType}$. *If* $T \perp_{\mathsf{p}} S$ *and* $S \perp_{\mathsf{p}} U$ *then* $T =_{\mathsf{p}} U$.

Proof. The proof follows the same line as Proposition A.2. ◻

A.4 Results on encoding

Formally, the unfolding function is defined as follows:

$$\frac{S\{\mu X.S/X\}\ \textsc{unf}\ S'}{\mu X.S\ \textsc{unf}\ S'} \qquad\qquad \frac{}{S\ \textsc{unf}\ S}\ S \neq \mu X.S'$$

Formally function f is defined as follows:

$$f, \{x \mapsto c\} \stackrel{\mathsf{def}}{=} \begin{cases} f \cup \{x \mapsto c\} & \text{if } x \notin \mathsf{dom}(f) \\ (f \setminus \{x \mapsto f(x)\}) \cup \{x \mapsto c\} & \text{otherwise} \end{cases}$$

In order to prove the following results, the encoding is extended to typing contexts in the expected way. It is presented in Figure 10. Notice that, the ‘,’ operator on session typing contexts is interpreted as the

$$\begin{array}{rcll} [\![\emptyset]\!]_f & \stackrel{\mathsf{def}}{=} & \emptyset & \text{(E-Empty)} \\ [\![\Gamma, x : T]\!]_f & \stackrel{\mathsf{def}}{=} & [\![\Gamma]\!]_f \uplus f_x : [\![T]\!] & \text{(E-Gamma)} \end{array}$$

Figure 10: Encoding of typing contexts

‘$\uplus$’ operator on linear typing contexts. This is the case because the (dual) co-variables are interpreted as the same (linear) channel, which in order to be used for communication, must have connection capability. Hence, by using the $\uplus$ the dual capabilities of linear channels can be “summed-up” into the connection capability: $\ell_{\mathsf{i}} \uplus \ell_{\mathsf{o}} = \ell_{\sharp}$.

Lemma A.5 (Encoding equal types)**.** *Let $T,S \in \mathsf{SType}$ be such that $T =_{\mathsf{s}} S$. If $[\![T]\!] = \tau$, then $[\![S]\!] = \sigma$ and $\tau =_{\mathsf{p}} \sigma$.*

Proof. Suppose $T =_{\mathsf{s}} S$ and $[\![T]\!] = \tau$ and $[\![S]\!] = \sigma$. To prove $\tau =_{\mathsf{p}} \sigma$, we need to construct a relation $\mathcal{R}$ such that $(\tau,\sigma) \in \mathcal{R}$ and $\mathcal{R}$ and $\mathcal{R}^{-1}$ are type simulation relations, which by Definition 2.8, it means to show that $\mathcal{R}$ is a type equivalence relation. Let $\mathcal{R} = \{(\tau',\sigma') \mid \exists T',S'.\ T' =_{\mathsf{s}} S', [\![T]\!] = \tau', [\![S]\!] = \sigma'\}$. We show that $\mathcal{R}$ is a type simulation. Let $(\tau,\sigma) \in \mathcal{R}$, by definition of $\mathcal{R}$ it means that $\exists T,S$ such that $T =_{\mathsf{s}} S$ and $[\![T]\!] = \tau$ and $[\![S]\!] = \sigma$.

If $\textsc{unf}(T) = \mathsf{end}$ then $\textsc{unf}(S) = \mathsf{end}$. By encoding we have $[\![\mathsf{end}]\!] = \emptyset[]$. We conclude by Definition 2.8 on type equivalence for linear π-types.

If $\textsc{unf}(T) = ?T_m.T'$ then $\textsc{unf}(S) = ?S_m.S'$ and $T_m =_{\mathsf{s}} S_m$ and $T' =_{\mathsf{s}} S'$. By encoding we have $[\![?T_m.T']\!] = \ell_{\mathsf{i}}\ [[\![T_m]\!],[\![T']\!]]$ and let $[\![T_m]\!] = \tau_m$, $[\![T']\!] = \tau'$ and $[\![?S_m.S']\!] = \ell_{\mathsf{i}}\ [[\![S_m]\!],[\![S']\!]]$ and let $[\![S_m]\!] = \sigma_m$, $[\![S']\!] = \sigma'$. Since $T_m =_{\mathsf{s}} S_m$ and $T' =_{\mathsf{s}} S'$, by definition of $\mathcal{R}$ it means that $(\tau_m,\sigma_m) \in \mathcal{R}$ and $(\tau',\sigma') \in \mathcal{R}$ which concludes this case by Definition 2.8. The case of $\textsc{unf}(T) = !T_m.T'$ and $\textsc{unf}(S) = !S_m.S'$ is symmetric to the previous one, where ? is replaced by !.

If $\textsc{unf}(T) = \&\{l_i : T_i\}_{i\in I}$ then $\textsc{unf}(S) = \&\{l_i : S_i\}_{i\in I}$, and $\forall i \in I$, $T_i =_{\mathsf{s}} S_i$. By encoding we have $[\![\&\{l_i : T_i\}_{i\in I}]\!] = \ell_{\mathsf{i}}\ [\langle l_i : [\![T_i]\!]\rangle_{i\in I}]$ and $[\![\&\{l_i : S_i\}_{i\in I}]\!] = \ell_{\mathsf{i}}\ [\langle l_i : [\![S_i]\!]\rangle_{i\in I}]$. Let $\forall i \in I.\ [\![T_i]\!] = \tau_i$ and $[\![S_i]\!] = \sigma_i$. By definition of $\mathcal{R}$ we have $\forall i \in I.(\tau_i,\sigma_i) \in \mathcal{R}$ and we conclude by Definition 2.8. The case of $\textsc{unf}(T) = \oplus\{l_i : T_i\}_{i\in I}$ and $\textsc{unf}(S) = \oplus\{l_i : S_i\}_{i\in I}$ is symmetric to the previous one, where & is replaced by $\oplus$. $\square$

Lemma A.6 (Encoding of dual types)**.** *Let $T,S \in \mathsf{SType}$, such that $T \perp_{\mathsf{s}} S$. If $[\![\textsc{unf}(T)]\!] = \tau$ then, $[\![\textsc{unf}(S)]\!] = \sigma$ and $\tau \perp_{\mathsf{p}} \sigma$.*

Proof. The proof is done by case analysis on the unfolding of session types T,S.

- If $\textsc{unf}(T) = \mathsf{end}$ then $\textsc{unf}(S) = \mathsf{end}$.
  By the encoding we have $[\![\mathsf{end}]\!] = \emptyset[]$. We conclude by the duality of $\emptyset[]$ in Definition 2.9.
- If $\textsc{unf}(T) = ?T_m.T'$ then $\textsc{unf}(S) = !S_m.S'$ and $T' \perp_{\mathsf{s}} S'$ and $T_m =_{\mathsf{s}} S_m$.
  By the encoding we have $[\![?T_m.T']\!] = \ell_{\mathsf{i}}\ [[\![T_m]\!],[\![T']\!]]$ and $[\![!S_m.S']\!] = \ell_{\mathsf{o}}\ [[\![S_m]\!],[\![\mathsf{cplt}(S')]\!]]$ and let $\mathsf{cplt}(S') = U$. By Proposition 2.4, $S' \perp_{\mathsf{s}} U$, by assumption $T' \perp_{\mathsf{s}} S'$ then by Proposition 2.5 we have that $T' =_{\mathsf{s}} U$. By Lemma 3.1 we have $[\![T_m]\!] =_{\mathsf{p}} [\![S_m]\!]$ and $[\![T']\!] =_{\mathsf{p}} [\![U]\!] = [\![\mathsf{cplt}(S')]\!]$ and by duality on π-types, Definition 2.9, we have $\ell_{\mathsf{i}}\ [[\![T_m]\!],[\![T']\!]] \perp_{\mathsf{p}} \ell_{\mathsf{o}}\ [[\![S_m]\!],[\![U]\!]]$, which concludes the case.
- If $\textsc{unf}(T) = !T_m.T'$ then $\textsc{unf}(S) = ?S_m.S'$ and $T' \perp_{\mathsf{s}} S'$ and $T_m =_{\mathsf{s}} S_m$.
  By the encoding we have $[\![!T_m.T']\!] = \ell_{\mathsf{o}}\ [[\![T_m]\!],[\![\mathsf{cplt}(T')]\!]]$ and let $\mathsf{cplt}(T') = U$ and $[\![?S_m.S']\!] = \ell_{\mathsf{i}}\ [[\![S_m]\!],[\![S']\!]]$. By Proposition 2.4, $T' \perp_{\mathsf{s}} U$; and since $T' \perp_{\mathsf{s}} S'$ then by Proposition 2.5 we have that $U =_{\mathsf{s}} S'$. Hence, by Lemma 3.1 we have $[\![U]\!] =_{\mathsf{p}} [\![S']\!]$. By assumption $T_m =_{\mathsf{s}} S_m$ hence by Lemma 3.1 we have $[\![T_m]\!] =_{\mathsf{p}} [\![S_m]\!]$. By Definition 2.9 we have $\ell_{\mathsf{o}}\ [[\![T_m]\!],[\![U]\!]] \perp_{\mathsf{p}} \ell_{\mathsf{i}}\ [[\![S_m]\!],[\![S']\!]]$, which concludes the case.
- If $\textsc{unf}(T) = \&\{l_i : T_i\}_{i\in I}$ then $\textsc{unf}(S) = \oplus\{l_i : S_i\}_{i\in I}$ and $\forall i \in I$, $T_i \perp_{\mathsf{s}} S_i$.
  By encoding we have $[\![\&\{l_i : T_i\}_{i\in I}]\!] = \ell_{\mathsf{i}}\ [\langle l_i : [\![T_i]\!]\rangle_{i\in I}]$ and $[\![\oplus\{l_i : S_i\}_{i\in I}]\!] = \ell_{\mathsf{o}}\ [\langle l_i : [\![\mathsf{cplt}(S_i)]\!]\rangle_{i\in I}]$ and let $\forall i \in I, \mathsf{cplt}(S_i) = U_i$. By Proposition 2.4 we have $\forall i \in I$, $S_i \perp_{\mathsf{s}} U_i$ and since by assumption $\forall i \in I$, $T_i \perp_{\mathsf{s}} S_i$, then by Proposition 2.5 we have that $\forall i \in I$, $T_i =_{\mathsf{s}} U_i$. By Lemma 3.1 we have $\forall i \in I, [\![T_i]\!] =_{\mathsf{p}} [\![U_i]\!]$. By Definition 2.9 we have $\ell_{\mathsf{i}}\ [\langle l_i : [\![T_i]\!]\rangle_{i\in I}] \perp_{\mathsf{p}} \ell_{\mathsf{o}}\ [\langle l_i : [\![U_i]\!]\rangle_{i\in I}]$.
- If $\textsc{unf}(T) = \oplus\{l_i : T_i\}_{i\in I}$ then $\textsc{unf}(S) = \&\{l_i : S_i\}_{i\in I}$ and $\forall i \in I$, $T_i \perp_{\mathsf{s}} S_i$.
  By encoding we have that $[\![\oplus\{l_i : T_i\}_{i\in I}]\!] = \ell_{\mathsf{o}}\ [\langle l_i : [\![\mathsf{cplt}(T_i)]\!]\rangle_{i\in I}]$ and let $\forall i \in I, \mathsf{cplt}(T_i) = U_i$ and

$[\![\&\{l_i : S_i\}_{i\in I}]\!] = \ell_{\mathsf{i}}\ [\langle l_i : [\![S_i]\!]\rangle_{i\in I}]$. By Proposition 2.4 we have $\forall i \in I,\ T_i \bot_{\mathsf{s}}\ U_i$ and since by assumption $\forall i \in I,\ T_i \bot_{\mathsf{s}}\ S_i$, by Proposition 2.5 we have that $\forall i \in I,\ S_i =_{\mathsf{s}} U_i$. By Lemma 3.1 we have $\forall i \in I, [\![S_i]\!] =_{\mathsf{p}} [\![U_i]\!]$. By Definition 2.9 we have $\ell_{\mathsf{o}}\ [\langle l_i : [\![U_i]\!]\rangle_{i\in I}] \bot_{\mathsf{p}}\ \ell_{\mathsf{i}}\ [\langle l_i : [\![S_i]\!]\rangle_{i\in I}]$.

□

The following results state that soundness and completeness of the encoding with respect to typing values (Lemma A.7) and typing processes (Theorem A.8), as well as the operational correspondence (Theorem A.10).

The encoding of typing environments Γ is given in Figure 10. The following proofs follow exactly the same line as in [4].

Lemma A.7 (Value Typing)**.** *$\Gamma \vdash v : T$ if and only if $[\![\Gamma]\!]_f \vdash [\![v]\!]_f : [\![T]\!]$.*

Proof. We split the proof as follows. We consider only the case for variables, the case for value **1** is similar.

- (*only if*): The proof is done by induction on the derivation $\Gamma \vdash v : T$, by analysing the last rule applied.
  Case (T-Var):
  $$\frac{\mathsf{un}(\Gamma)}{\Gamma, x : T \vdash x : T}$$
  To prove $[\![\Gamma, x : T]\!]_f \vdash [\![x]\!]_f : [\![T]\!]$. By (E-Gamma) and the encoding of variables it means $[\![\Gamma]\!]_f \uplus f_x : [\![T]\!] \vdash f_x : [\![T]\!]$. By rule (T$\pi$-Var) we obtain the result.
- (*if*): The proof is done by induction on the structure of the value v
  Case $v = x$:
  By the encoding of variables we have $[\![x]\!]_f = f_x$ and assume $[\![\Gamma]\!]_f \vdash f_x : [\![T]\!]$. By (T$\pi$-Var), typing rule for variables in the standard π-calculus, this means that $(f_x : [\![T]\!]) \in [\![\Gamma]\!]_f$ and hence $[\![\Gamma]\!]_f = \Gamma_1^\pi, f_x : [\![T]\!]$ which by (E-Gamma) means that $\Gamma = \Gamma_1, x : T$, where $\Gamma_1^\pi = [\![\Gamma_1]\!]_f$. By (T$\pi$-Var) we have $\mathsf{un}([\![\Gamma']\!]_f)$. By the encoding of types also $\mathsf{un}(\Gamma')$ holds. By rule (T-Var) we obtain the result.

□

Theorem A.8 (Process Typing)**.** *$\Gamma \vdash P$ if and only if $[\![\Gamma]\!]_f \vdash [\![P]\!]_f$.*

Proof. We split the proof as follows.

- (*only if*): The proof is done by induction on the the derivation $\Gamma \vdash P$, by analysing the last typing rule applied. We consider the most important case, where the duality relation is checked explicitly. The rest of the cases can be seen in [4].
  Case (T-Res):
  $$\frac{\Gamma, x : T, y : S \vdash P \quad T \bot_{\mathsf{s}}\ S}{\Gamma \vdash (\nu xy)P}\ (\text{T-Res})$$
  To prove that $[\![\Gamma]\!]_f \vdash [\![(\nu xy)P]\!]_f$, which by encoding of restriction means $[\![\Gamma]\!]_f \vdash (\nu c)[\![P]\!]_{f,\{x,y\mapsto c\}}$. We distinguish the following two cases:
  - Suppose $T \neq \mathsf{end}$, and hence $S \neq \mathsf{end}$. By induction hypothesis $[\![\Gamma, x : T, y : S]\!]_{f'} \vdash [\![P]\!]_{f'}$, for some function f' such that $\mathsf{dom}(f') = \mathsf{dom}(\Gamma) \cup \{x, y\}$ and let $f'(x) = f'(y) = c$ and let $f = f' - \{x \mapsto c, y \mapsto c\}$. By applying (E-Gamma), the typing judgement becomes $[\![\Gamma]\!]_f \uplus c : [\![T]\!] \uplus c : [\![S]\!] \vdash [\![P]\!]_{f,\{x,y\mapsto c\}}$, namely $[\![\Gamma]\!]_f \uplus c : [\![T]\!] \uplus [\![S]\!] \vdash [\![P]\!]_{f,\{x,y\mapsto c\}}$. By the typing rules,

[4, 6], we notice that the types adopted are unfolded, i.e., no recursive type of the form $\mu X.T$ occurs in the rules. Hence, by Lemma A.6, and by the idempotence of unfolding, namely $\textsc{unf}(\textsc{unf}(T)) = \textsc{unf}(T)$ we have $[\![T]\!] = \tau$ and $[\![S]\!] = \sigma$ and $\tau \perp_{\mathsf{p}} \sigma$. -The unfolding function is given at the beginning of this section.- Since $T \neq \mathsf{end}$, $S \neq \mathsf{end}$, we have that $[\![T]\!] = \ell_\alpha\ [W]$ and $[\![S]\!] = \ell_{\overline{\alpha}}\ [W]$ and hence $c : \ell_\sharp\ [W]$, where W denote the (tuple of) carried type which is irrelevant and if $a = \mathsf{i}$ then $\overline{\alpha} = \mathsf{o}$ and opposite otherwise. Since variable c owns both capabilities it means that $c \notin \mathsf{dom}([\![\Gamma]\!]_f)$. Hence, $[\![\Gamma]\!]_f, c : [\![T]\!] \uplus [\![S]\!] \vdash [\![P]\!]_{f,\{x,y \mapsto c\}}$. By applying rule (T$\pi$-Res1) we obtain $[\![\Gamma]\!]_f \vdash (\boldsymbol{\nu} c)[\![P]\!]_{f,\{x,y\mapsto c\}}$ which concludes this case.

– Suppose $T = S = \mathsf{end}$. By induction hypothesis $[\![\Gamma, x : \mathsf{end}, y : \mathsf{end}]\!]_{f'} \vdash [\![P]\!]_{f'}$, for some function f' such that $\mathsf{dom}(f') = \mathsf{dom}(\Gamma) \cup \{x, y\}$ and let $f'(x) = f'(y) = c$ and let $f = f' - \{x \mapsto c, y \mapsto c\}$. By applying equation (E-Gamma), the typing judgement becomes $[\![\Gamma]\!]_f \uplus c : [\![\mathsf{end}]\!] \uplus c : [\![\mathsf{end}]\!] \vdash [\![P]\!]_{f,\{x,y\mapsto c\}}$, namely $[\![\Gamma]\!]_f \uplus c : \emptyset[] \uplus c : \emptyset[] \vdash [\![P]\!]_{f,\{x,y\mapsto c\}}$ which by the combination of unrestricted π-types means $[\![\Gamma]\!]_f \uplus c : \emptyset[] \vdash [\![P]\!]_{f,\{x,y\mapsto c\}}$. Notice that $c \notin \mathsf{dom}([\![\Gamma]\!]_f)$, otherwise $\uplus$ would not have been defined. Hence we obtain $[\![\Gamma]\!]_f, c : \emptyset[] \vdash [\![P]\!]_{f,\{x,y\mapsto c\}}$. Notice that $c \notin [\![P]\!]_{f,\{x,y\mapsto c\}}$, since $x, y \notin FV(P)$ being x, y terminated channels. Then, by rule (Tπ-Res2) we obtain $[\![\Gamma]\!]_f \vdash (\boldsymbol{\nu} c)[\![P]\!]_{f,\{x,y\mapsto c\}}$ which concludes this case.

- (*if*): The proof is done by induction on the structure of P. We show a different case from restriction. The details of other cases can be checked in [4].
  Case $x?(y).P$:
  By equation (E-Input) we have $[\![x?(y).P]\!]_f \stackrel{\mathsf{def}}{=} f_x?(y, c).[\![P]\!]_{f,\{x\mapsto c\}}$ and assume $[\![\Gamma]\!]_f \vdash f_x?(y, c).[\![P]\!]_{f,\{x\mapsto c\}}$ which by rule (Tπ-Inp) means:

$$\frac{\Gamma_1^\pi \vdash f_x : \ell_{\mathsf{i}}[T^\pi, U^\pi] \qquad \Gamma_2^\pi, y : T^\pi, c : U^\pi \vdash [\![P]\!]_{f,\{x\mapsto c\}}}{[\![\Gamma]\!]_f \vdash f_x?(y, c).[\![P]\!]_{f,\{x\mapsto c\}}}$$

  where $[\![\Gamma]\!]_f = \Gamma_1^\pi \uplus \Gamma_2^\pi$. By Auxiliary Lemma in [4] $\Gamma_1^\pi = [\![\Gamma_1]\!]_f$ and $\Gamma_2^\pi = [\![\Gamma_2]\!]_f$, such that $\Gamma = \Gamma_1 \circ \Gamma_2$, and $\mathsf{dom}(f) = \mathsf{dom}(\Gamma_1) \cup \mathsf{dom}(\Gamma_2)$. By Auxiliary Lemma in [4] we have $\Gamma_1 \vdash x : ?T.U$. By induction hypothesis $\Gamma_2, y : T, x : U \vdash P$ where $T^\pi = [\![T]\!]$, $U^\pi = [\![U]\!]$ and by the encoding of P we notice that c substitutes x. By applying (T-Inp) we obtain $\Gamma_1 \circ \Gamma_2 \vdash x?(y).P$.

□

Before giving the Operational Correspondence we give the following definition.

Definition A.9 (Evaluation Context)**.** *An* evaluation context *is a process with a hole* $[\cdot]$ *and is produced by the following grammar:*

$$\mathcal{E}[\cdot] \stackrel{\mathsf{def}}{=} [\cdot] \quad | \quad (\boldsymbol{\nu} xy)\, \mathcal{E}[\cdot]$$

Theorem A.10 (Operational Correspondence)**.** *Let P be a session process. The following hold.*

1. *If* $P \to P'$ *then* $[\![P]\!]_f \to\hookrightarrow [\![P']\!]_f$,
2. *If* $[\![P]\!]_f \to\equiv Q$ *then,* $\exists\, P', \mathcal{E}[\cdot]$ *such that* $\mathcal{E}[P] \to \mathcal{E}[P']$ *and* $Q \to^*\equiv [\![P']\!]_{f'}$, *where* f' *is the updated* f *after the communication and* $f_x = f_y$ *for all* $(\boldsymbol{\nu} xy) \in \mathcal{E}[\cdot]$.

Proof. The proof of this theorem, for both cases, follows exactly the same line as in the author's Ph.D. thesis. We refer the reader to [4] for the details. □