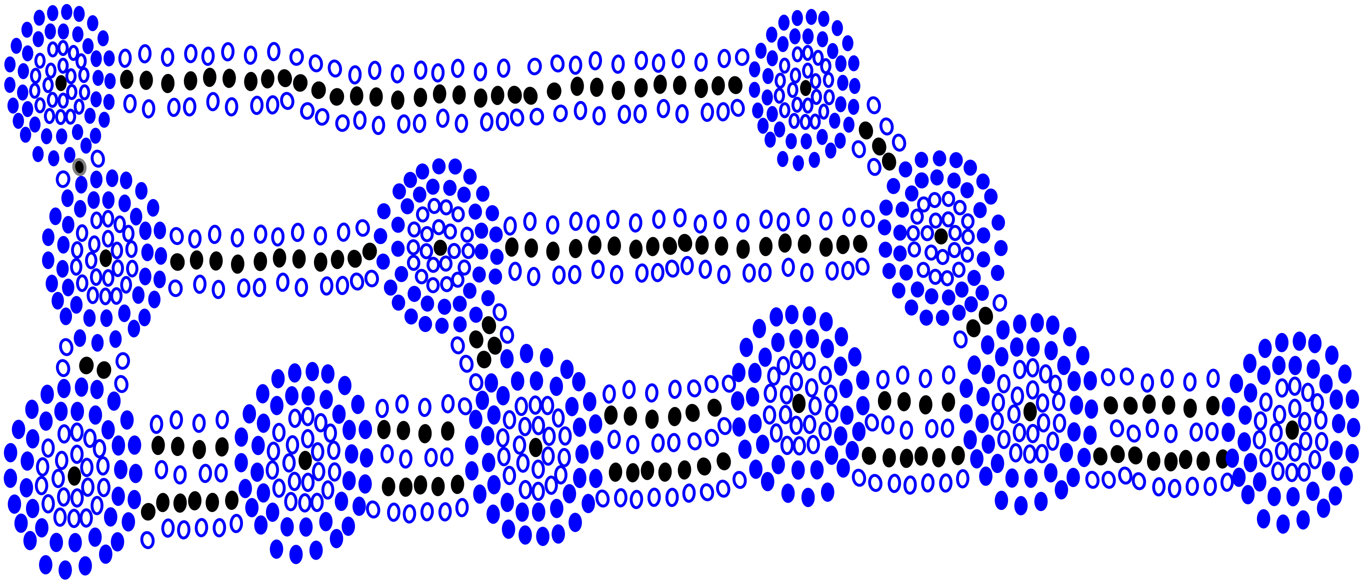
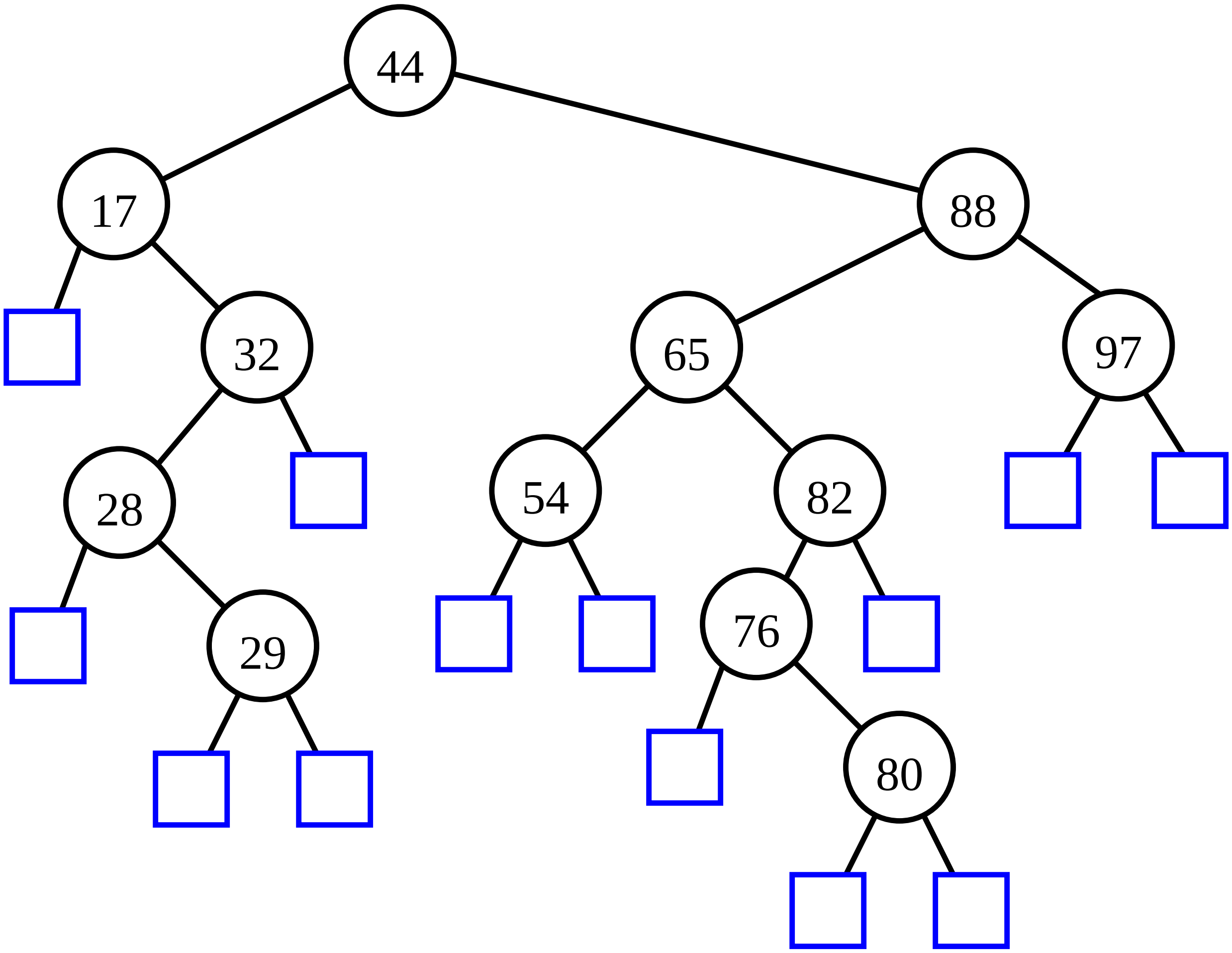
AVL TREES

- Binary Search Trees
- AVL Trees



Binary Search Trees

- A binary search tree is a binary tree T such that
  - each internal node stores an item (k, e) of a dictionary.
  - keys stored at nodes in the left subtree of v are less than or equal to k.
  - Keys stored at nodes in the right subtree of v are greater than or equal to k.
  - External nodes do not hold elements but serve as place holders.



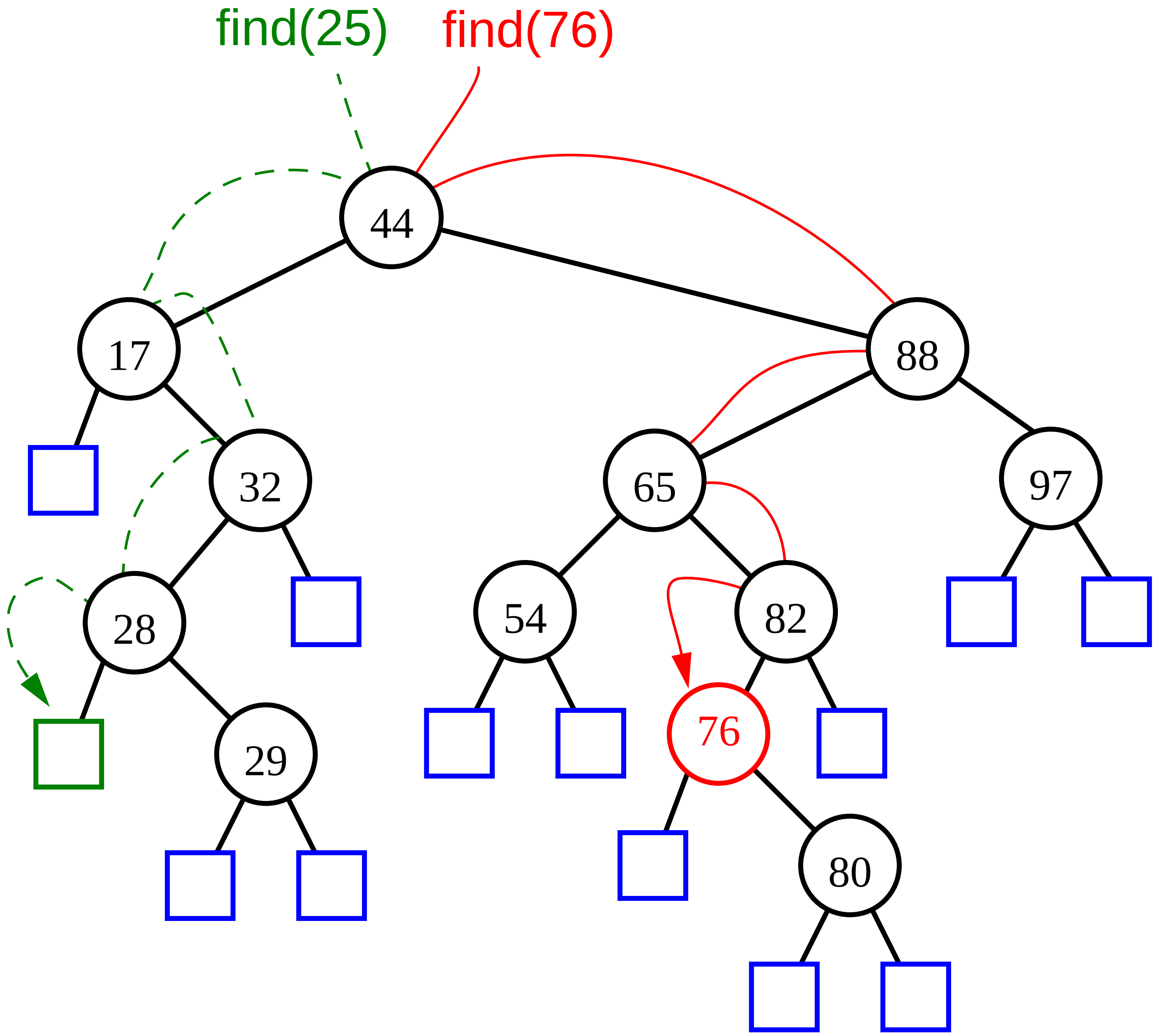
Search

- The binary search tree *T* is a decision tree, where the question asked at an internal node *v* is whether the search key *k* is less than, equal to, or greater than the key stored at *v*.
- Pseudocode:

```
Algorithm TreeSeach(k, v):
  Input: A search key k and a node v of a binary search
         tree T.
  Ouput: A node w of the subtree T(v) of T rooted at v,
         such that either w is an internal node storing
         key k or w is the external node encountered in
         the inorder traversal of T(v) after all the inter
         nal nodes with keys smaller than k and before
         all the internal nodes with keys greater than k.
  if v is an external node then
    return v
  if k = key(v) then
    return v
  else if k < key(v) then
    return TreeSearch(k, T.leftChild(v))
  else
    { k > key(v) }
    return TreeSearch(k, T.rightChild(v))
```

Search (cont.)

- A picture:



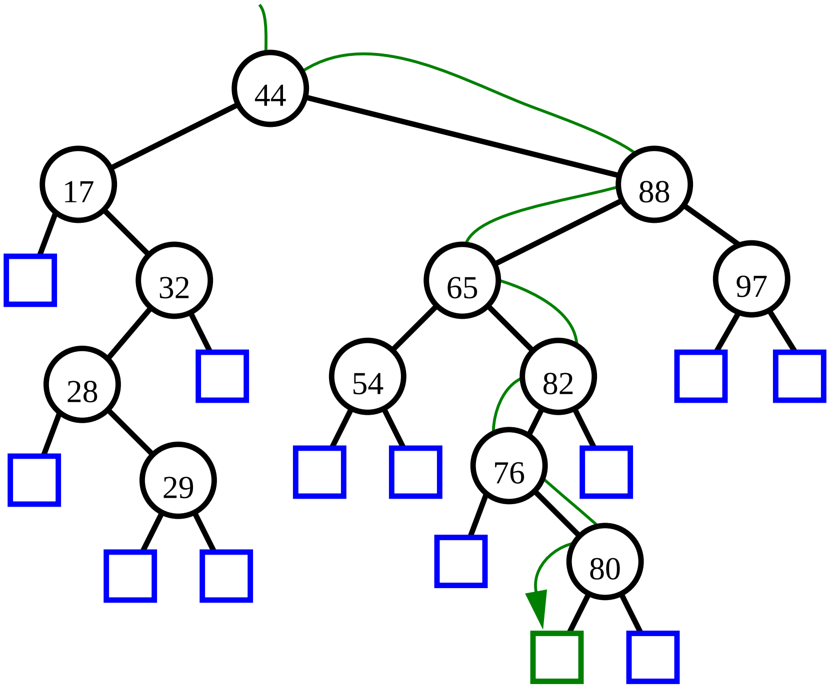
Insertion in a Binary Search Tree

- Start by calling TreeSearch(*k*, *T*.root()) on *T*. Let *w* be the node returned by TreeSearch
- If *w* is external, we know no item with key *k* is stored in *T*. We call expandExternal(*w*) on *T* and have *w* store the item (*k*, *e*)
- If *w* is internal, we know another item with key *k* is stored at *w*. We call TreeSearch(*k*, rightChild(*w*)) and recursively apply this alorithm to the node returned by TreeSearch.

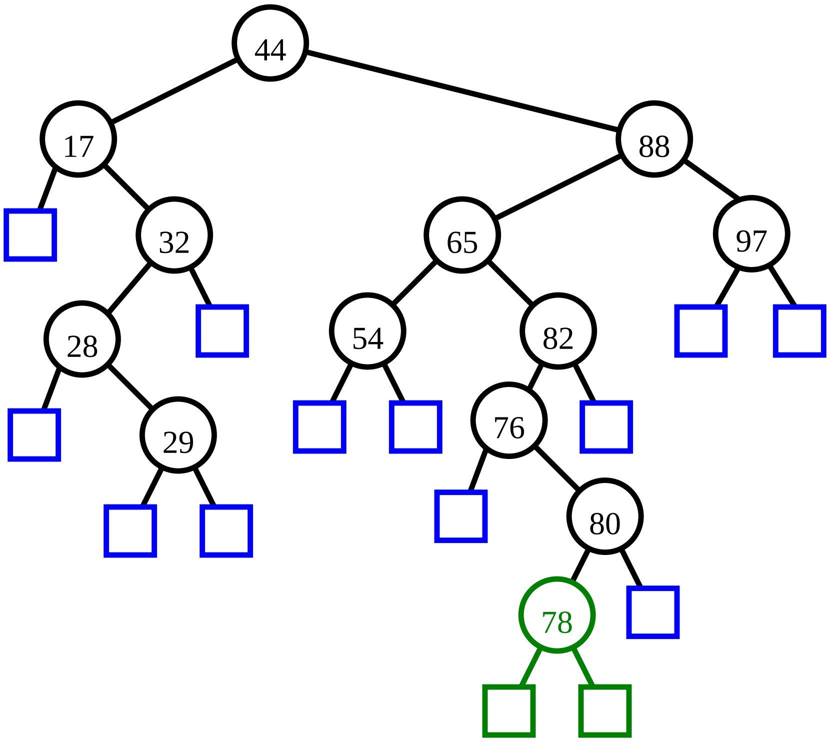
Insertion in a Binary Search Tree (cont.)

- Insertion of an element with key 78:

a)

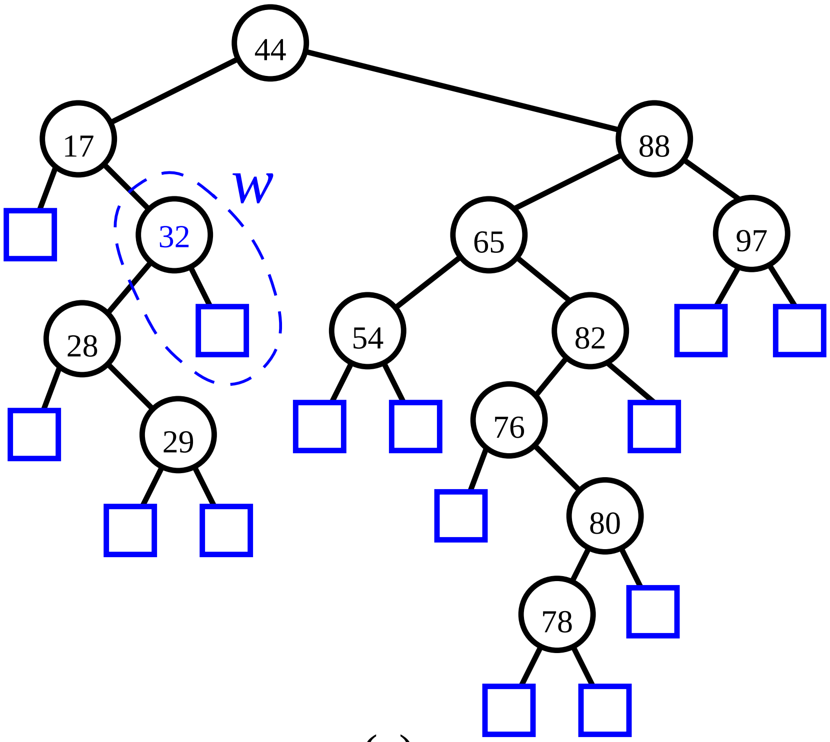


b)



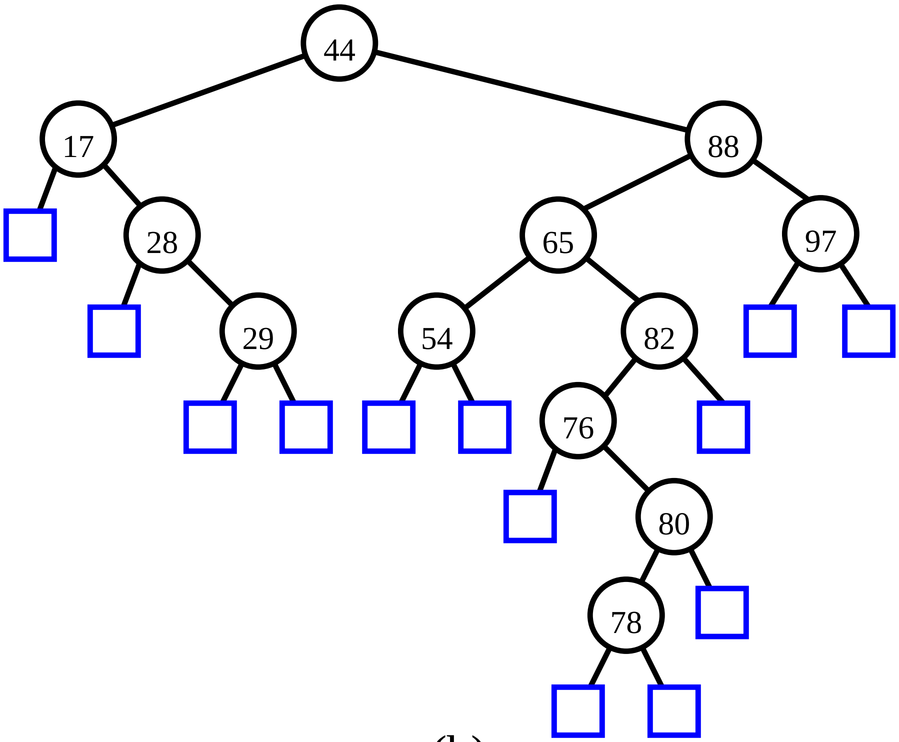
Removal from a Binary Search Tree

- Removal where the key to remove is stored at a node (w) with an external child:



(a)

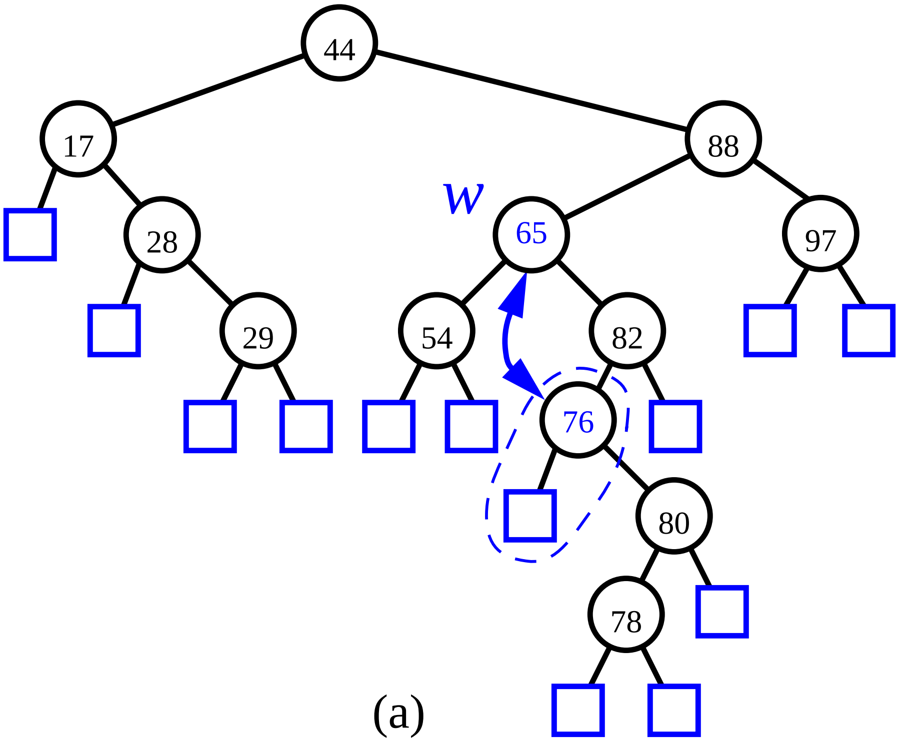
Removal from a Binary Search Tree (cont.)



(b)

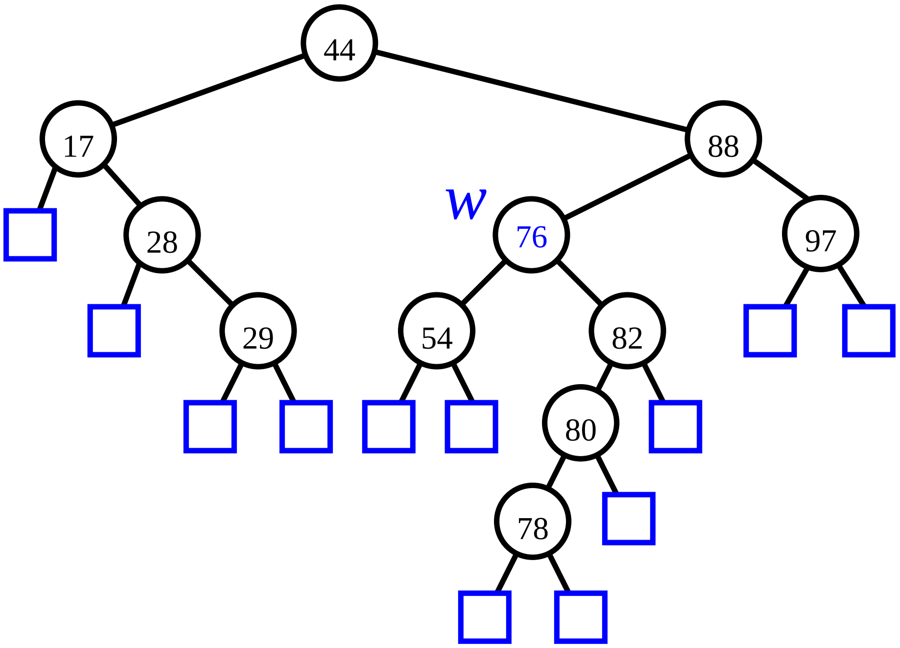
Removal from a Binary Search Tree (cont.)

- Removal where the key to remove is stroed at a node whose children are both internal:



(a)

Removal from a Binary Search Tree (cont.)



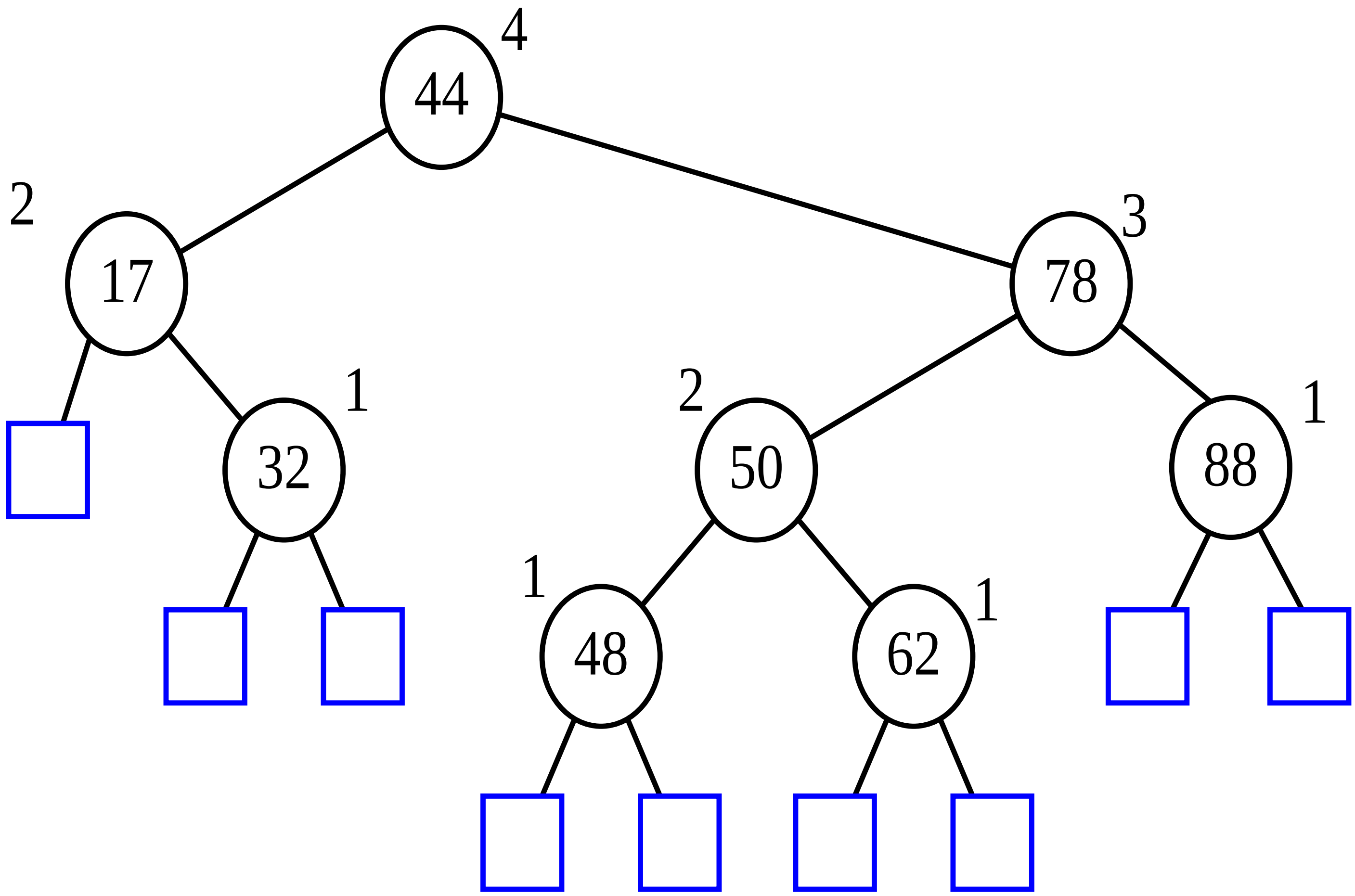
(b)

Time Complexity

- Searching, insertion, and removal in a binary search tree is ***O***(*h*), where *h* is the height of the tree.
- However, in the worst-case search, insertion, and removal time is ***O***(*n*), if the height of the tree is equal to *n*. Thus in some cases searching, insertion, and removal is no better than in a sequence.
- Thus, to prevent the worst case, we need to develop a rebalancing scheme to bound the height of the tree to log *n*.

AVL Tree

- An AVL Tree is a binary search tree such that for every internal node *v* of *T*, the heights of the children of *v* can differ by at most 1.
- An example of an AVL tree where the heights are shown next to the nodes:



Height of an AVL Tree

- **Proposition**: The height of an AVL tree T storing n keys is $\boldsymbol{O}(\log n)$.
- **Justification**: The easiest way to approach this problem is to try to find the minimum number of internal nodes of an AVL tree of height h: $n(h)$.
- We see that $n(1) = 1$ and $n(2) = 2$
- for n 3, an AVL tree of height h with $n(h)$ minimal contains the root node, one AVL subtree of height n-1 and the other AVL subtree of height n-2.
- i.e. $n(h) = 1 + n(h\text{-}1) + n(h\text{-}2)$
- Knowing $n(h\text{-}1) > n(h\text{-}2)$, we get $n(h) > 2\mathrm{n}(h\text{-}2)$
  - $n(h) > 2\mathrm{n}(h\text{-}2)$
  - $n(h) > 4\mathrm{n}(h\text{-}4)$

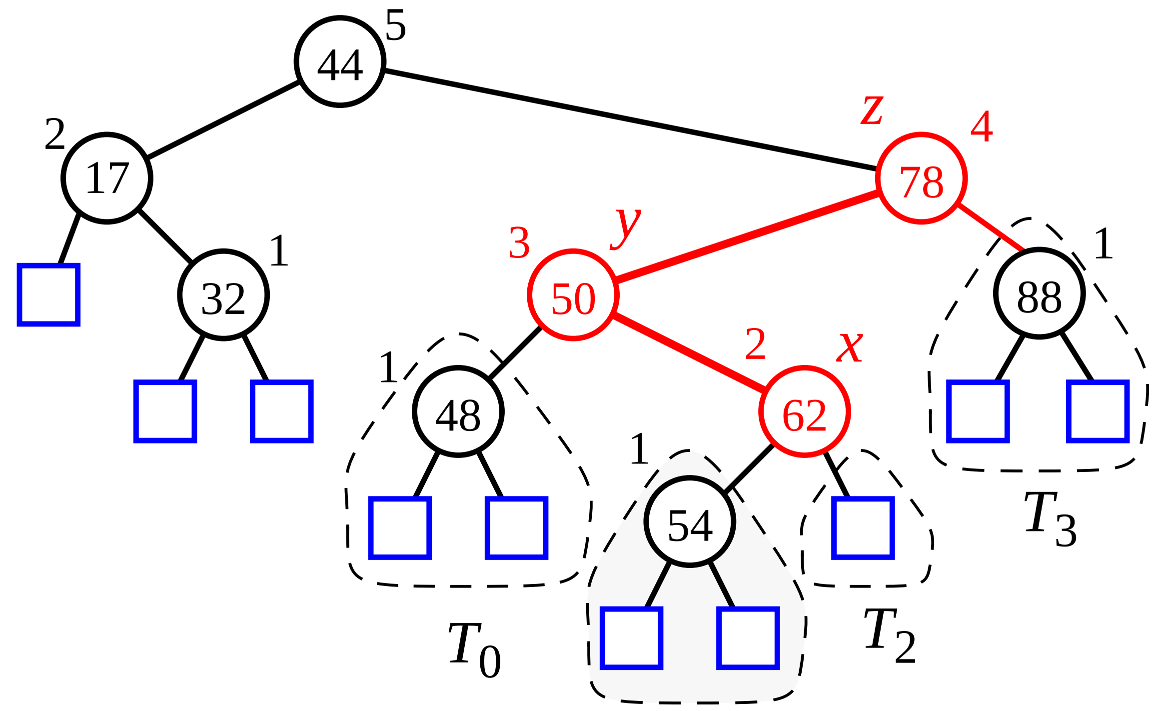
    ...
  - $n(h) > 2^i n(h\text{-}2i)$
- Solving the base case we get: $n(h)$ $2^{h/2\text{-}1}$
- Taking logarithms: $h < 2\log n(h) + 2$
- Thus the height of an AVL tree is $\boldsymbol{O}(\log n)$

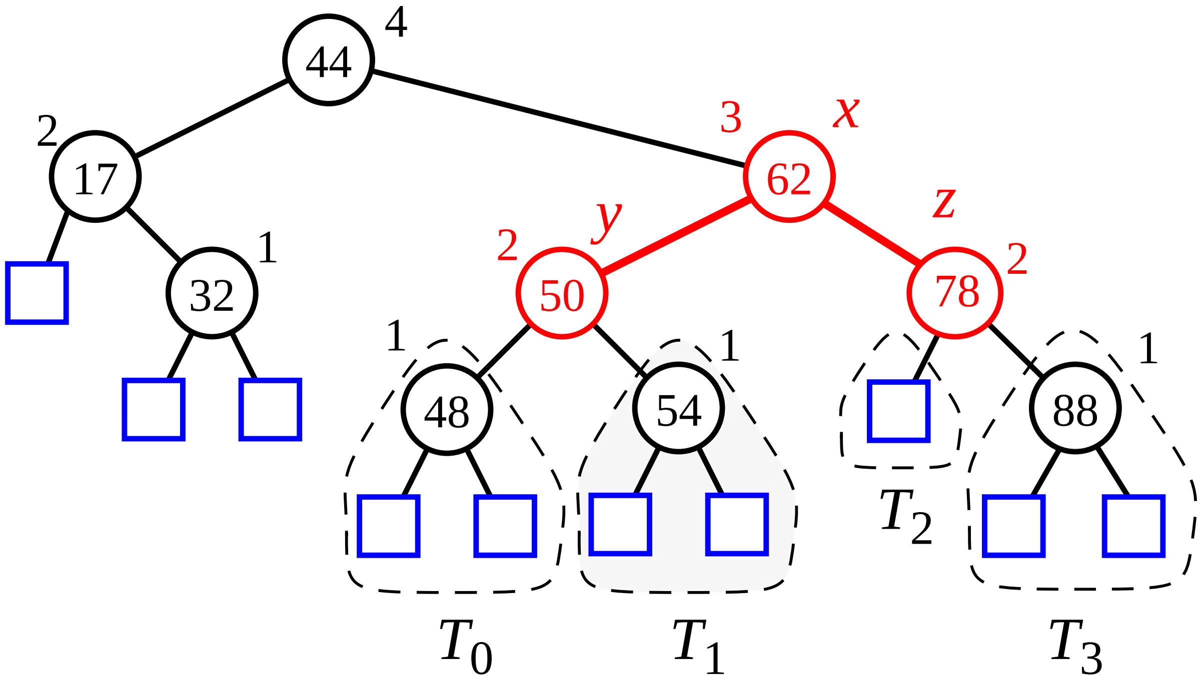
Insertion

- A binary search tree *T* is called balanced if for every node *v*, the height of *v*'s children differ by at most one.
- Inserting a node into an AVL tree involves performing an expandExternal(*w*) on *T*, which changes the heights of some of the nodes in *T*.
- If an insertion causes *T* to become unbalanced, we travel up the tree from the newly created node until we find the first node *x* such that its grandparent *z* is unbalanced node.
- Since *z* became unbalanced by an insertion in the subtree rooted at its child *y*,
height(*y*) = height(sibling(*y*)) + 2
- To rebalance the subtree rooted at *z*, we must perform a *restructuring*
  - we rename *x*, *y*, and *z* to *a*, *b*, and *c* based on the order of the nodes in an in-order traversal.
  - *z* is replaced by *b*, whose children are now *a* and *c* whose children, in turn, consist of the four other subtrees formerly children of *x*, *y*, and *z*.

Insertion (contd.)

- Example of insertion into an AVL tree.

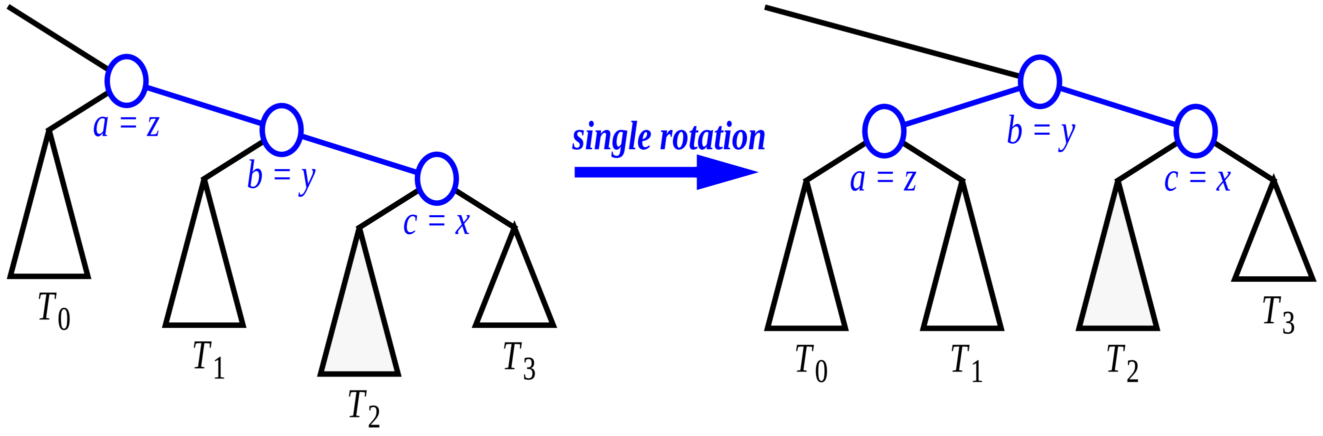


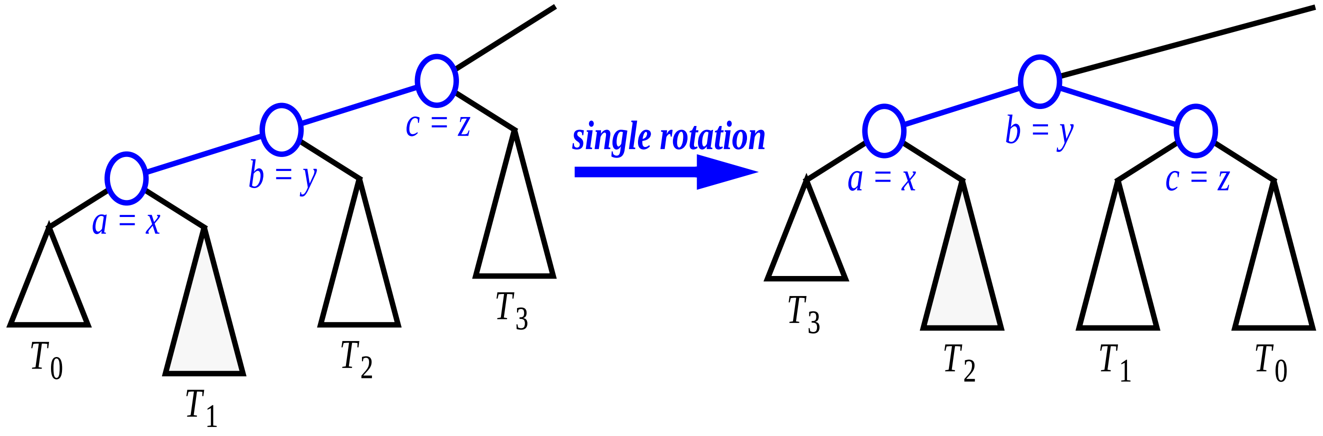


Restructuring

- The four ways to rotate nodes in an AVL tree, graphically represented:

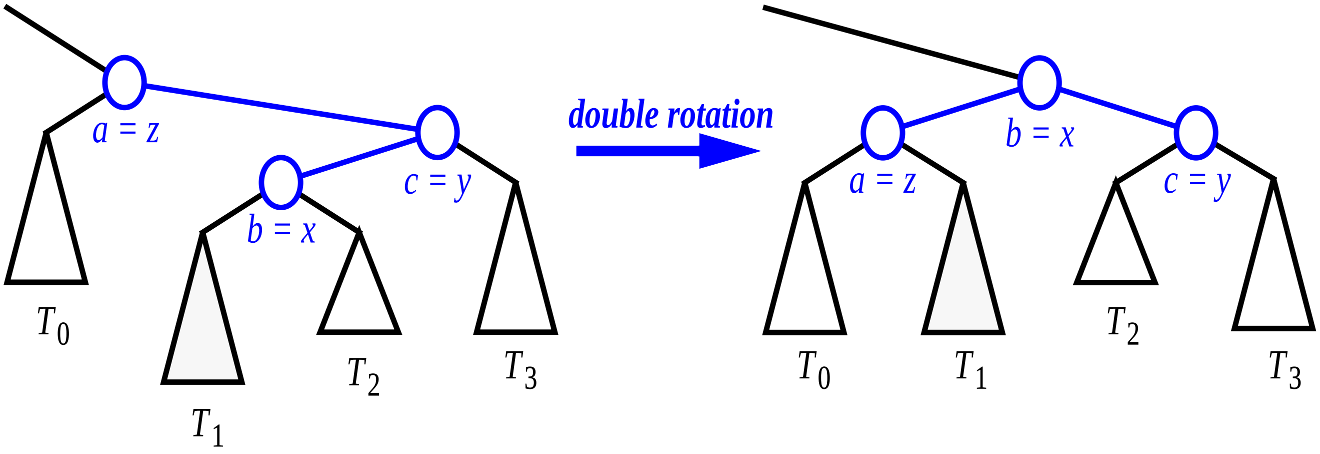
  - Single Rotations:

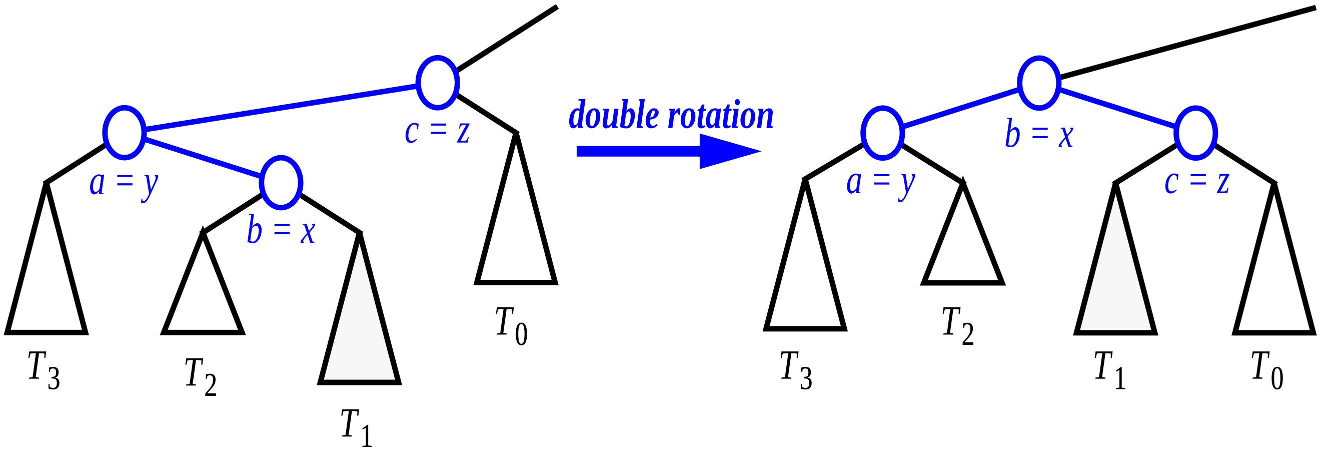




Restructuring (contd.)

- double rotations:





Restructuring (contd.)

- In Pseudo-Code:

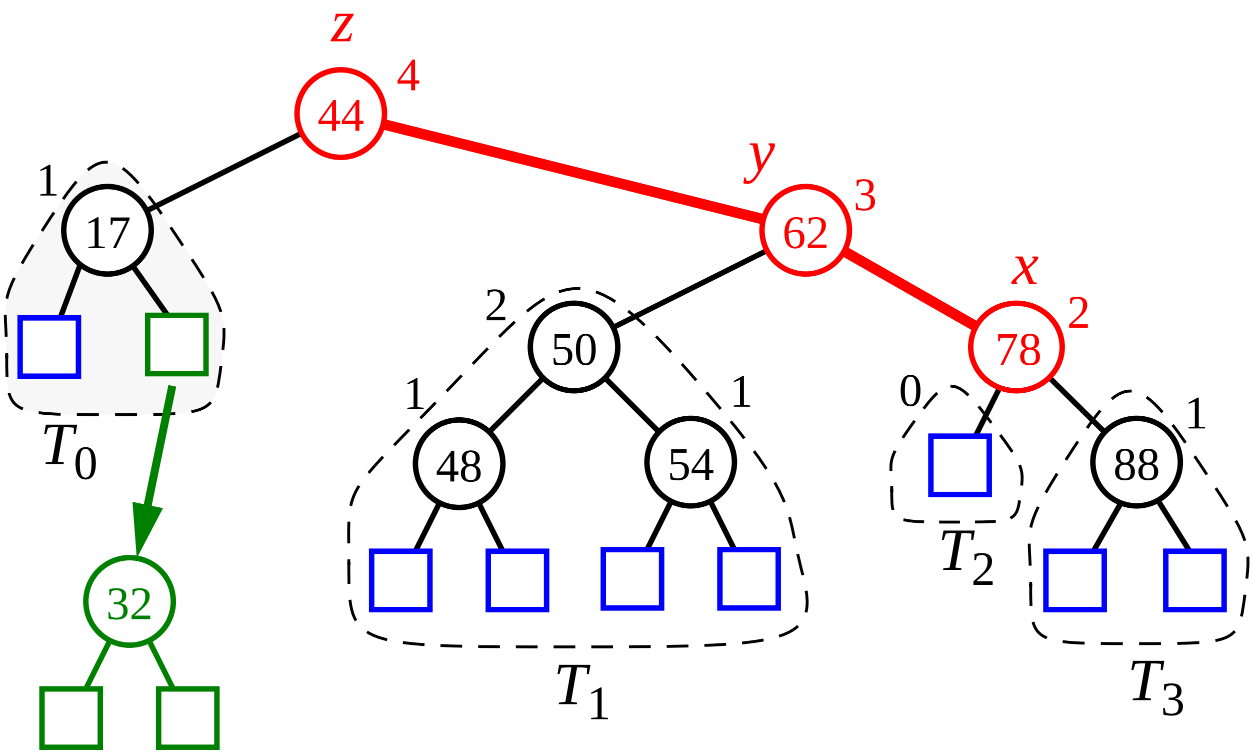
```
Algorithm restructure($x$):
  Input: A node $x$ of a binary search tree $T$ that has both
     a parent $y$ and a grandparent $z$
  Output: Tree $T$ restructured by a rotation (either
     single or double) involving nodes $x$, $y$, and $z$.

1:  Let ($a$, $b$, $c$) be an inorder listing of the nodes $x$, $y$,
    and $z$, and let ($T_0$, $T_1$, $T_2$, $T_3$) be an inorder listing
    of the the four subtrees of $x$, $y$, and $z$ not rooted at $x$,
    $y$, or $z$
2.  Replace the subtree rooted at $z$ with a new subtree
    rooted at $b$
3.  Let $a$ be the left child of $b$ and let $T_0$, $T_1$ be the left
    and right subtrees of $a$, respectively.
4.  Let $c$ be the right child of $b$ and let $T_2$, $T_3$ be the left
    and right subtrees of $c$, respectively.
```

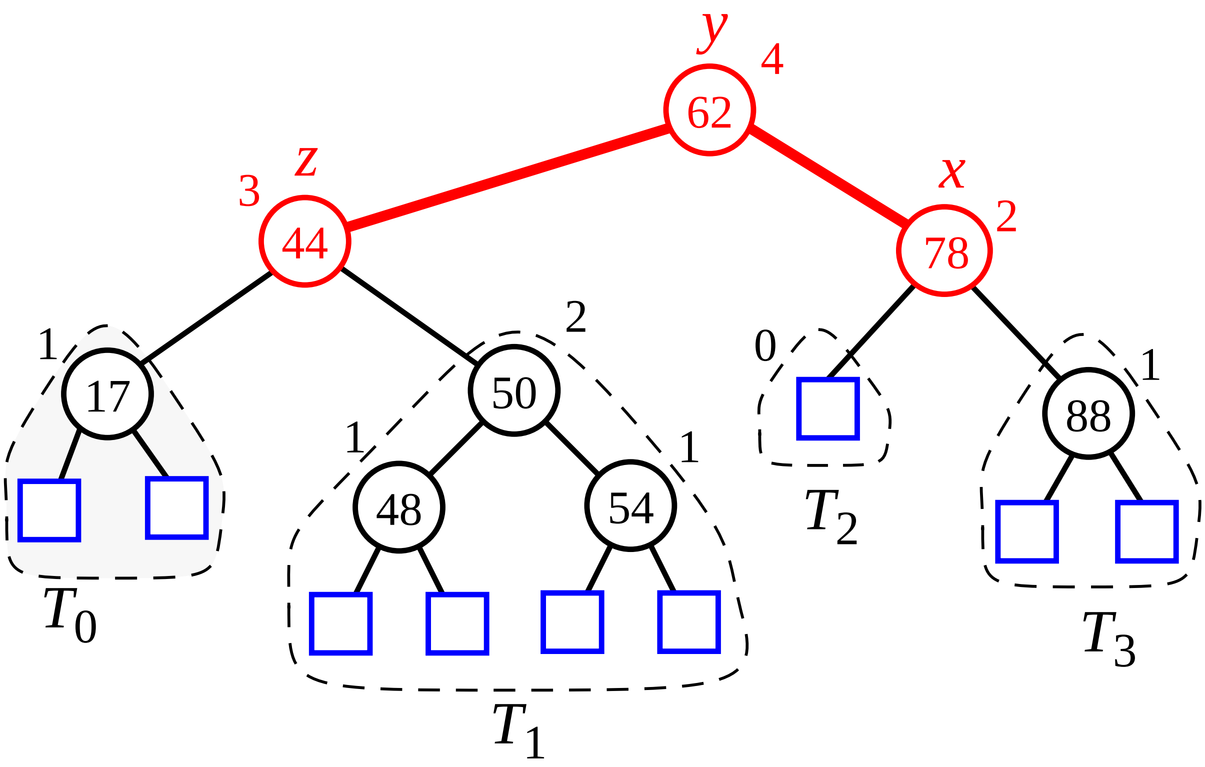
Removal

- We can easily see that performing a removeAboveExternal(*w*) can cause *T* to become unbalanced.
- Let *z* be the first unbalanced node encountered while travelling up the tree from *w*. Also, let y be the child of *z* with the larger height, and let *x* be the child of *y* with the larger height.
- We can perform operation restructure(*x*) to restore balance at the subtree rooted at *z*.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of *T* is reached.

Removal (contd.)

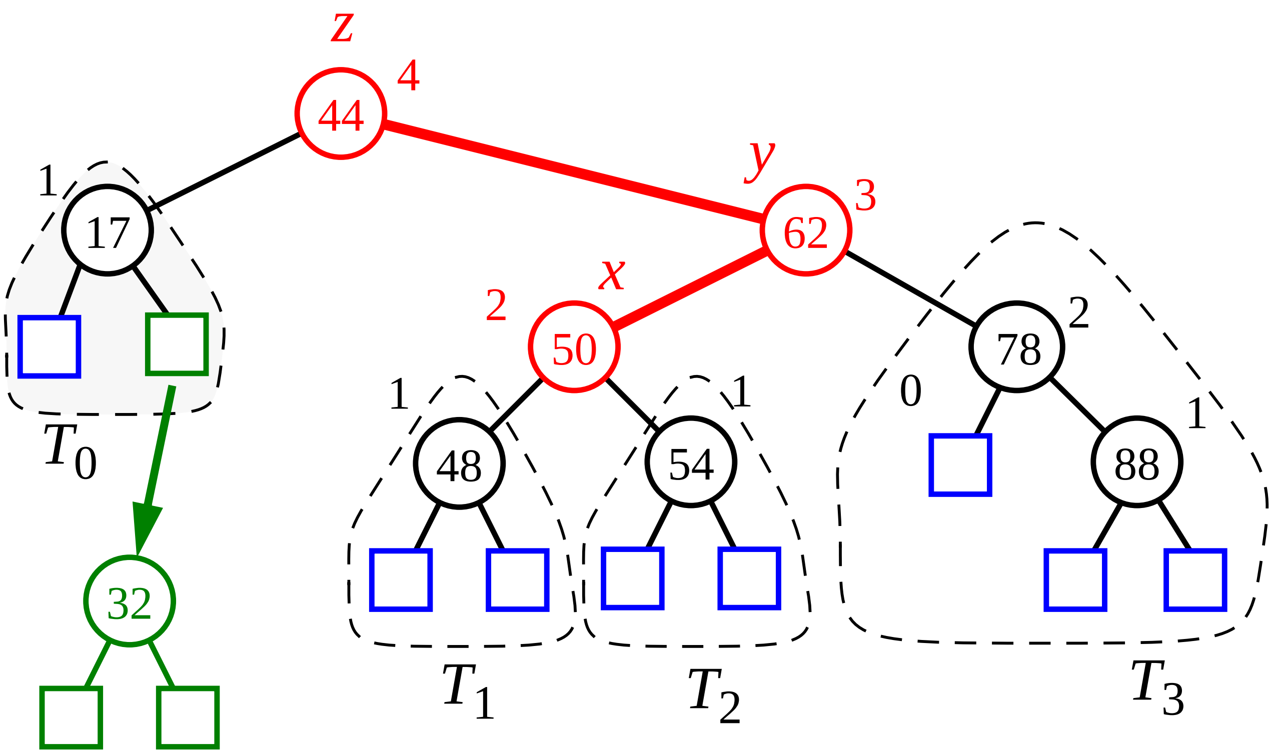
- example of deletion from an AVL tree:

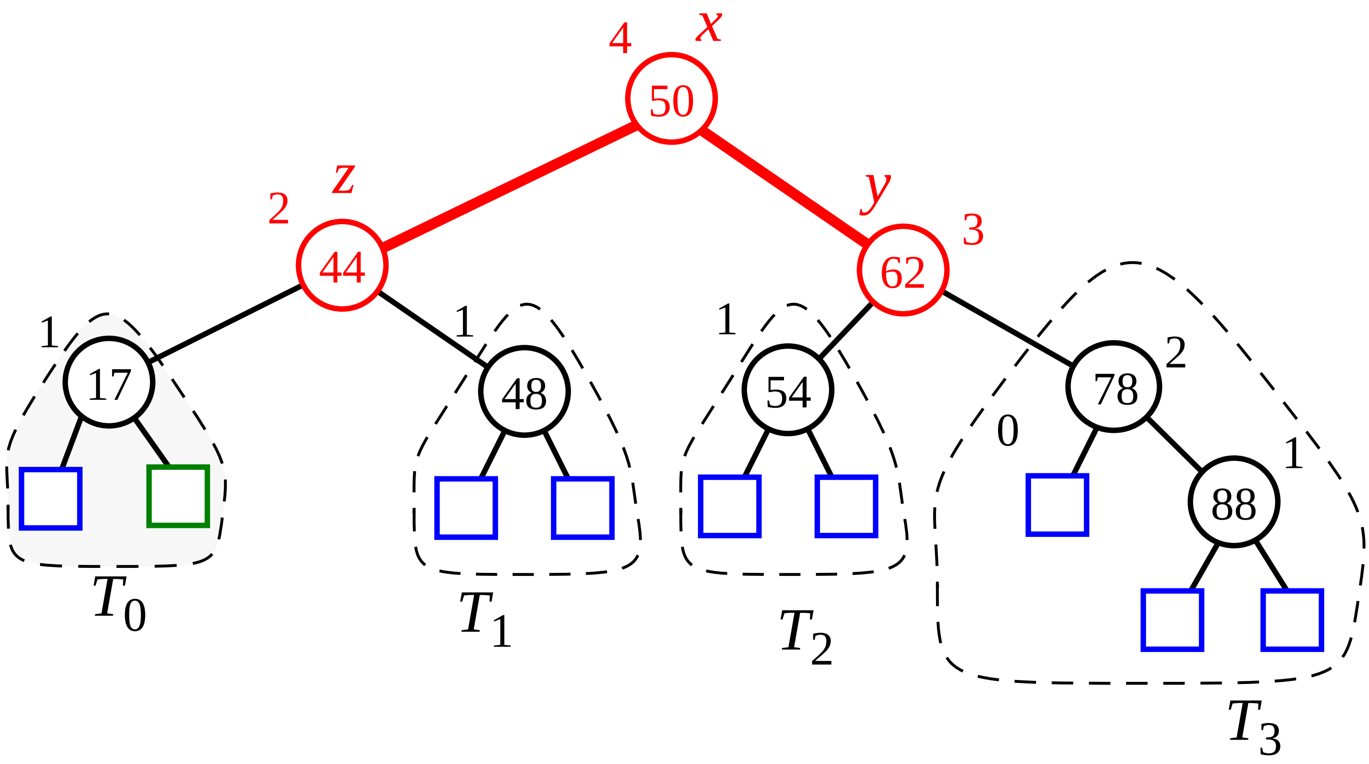




Removal (contd.)

- example of deletion from an AVL tree





Implementation

- A Java-based implementation of an AVL tree requires the following node class:

```
public class AVLItem extends Item  {
  int height;

  AVLItem(Object k, Object e, int h)  {
    super(k, e);
    height = h;
  }

  public int height()  {
    return height;
  }

  public int setHeight(int h)  {
    int oldHeight = height;
    height = h;
    return oldHeight;
  }
}
```

Implementation (contd.)

```
public class SimpleAVLTree
  extends SimpleBinarySearchTree
  implements Dictionary  {

public SimpleAVLTree(Comparator c)  {
    super(c);
    T = new RestructurableNodeBinaryTree();
   }
   private int height(Position p)  {
    if (T.isExternal(p))
       return 0;
    else
       return ((AVLItem) p.element()).height();
   }
   private void setHeight(Position p)  { // called only
                                         // if p is internal
    ((AVLItem) p.element()).setHeight
       (1 + Math.max(height(T.leftChild(p)),
                     height(T.rightChild(p))));
   }
```

Implementation (contd.)

```
private boolean isBalanced(Position p) {
      // test whether node p has balance factor
      // between -1 and 1
    int bf = height(T.leftChild(p)) - height(T.rightChild(p));
    return ((-1 <= bf) && (bf <= 1));
  }


private Position tallerChild(Position p) {
       // return a child of p with height no
       // smaller than that of the other child
    if(height(T.leftChild(p)) >= height(T.rightChild(p)))
      return T.leftChild(p);
    else
      return T.rightChild(p);
  }
```

Implementation (contd.)

```
private void rebalance(Position zPos) {
//traverse the path of T from zPos to the root;
//for each node encountered recompute its
//height and perform a rotation if it is
//unbalanced
   while (!T.isRoot(zPos)) {
     zPos = T.parent(zPos);
     setHeight(zPos);
     if (!isBalanced(zPos)) { // perform a rotation
       Position xPos =  tallerChild(tallerChild(zPos));
       zPos = ((RestructurableNodeBinaryTree)
                 T).restructure(xPos);
       setHeight(T.leftChild(zPos));
       setHeight(T.rightChild(zPos));
       setHeight(zPos);
     }
   }
}
```

Implementation (contd.)

```
public void insertItem(Object key, Object element)
      throws InvalidKeyException {
   super.insertItem(key, element);// may throw an
                                   // InvalidKeyException
   Position zPos = actionPos; // start at the
                                   // insertion position
   T.replace(zPos, new AVLItem(key, element, 1));
   rebalance(zPos);
   }

public Object remove(Object key)
         throws InvalidKeyException {
     Object toReturn = super.remove(key); // may throw
                                    // an InvalidKeyException
      if (toReturn != NO_SUCH_KEY) {
         Position zPos = actionPos; // start at the
                                    // removal position
         rebalance(zPos);
      }
      return toReturn;
   }
}
```