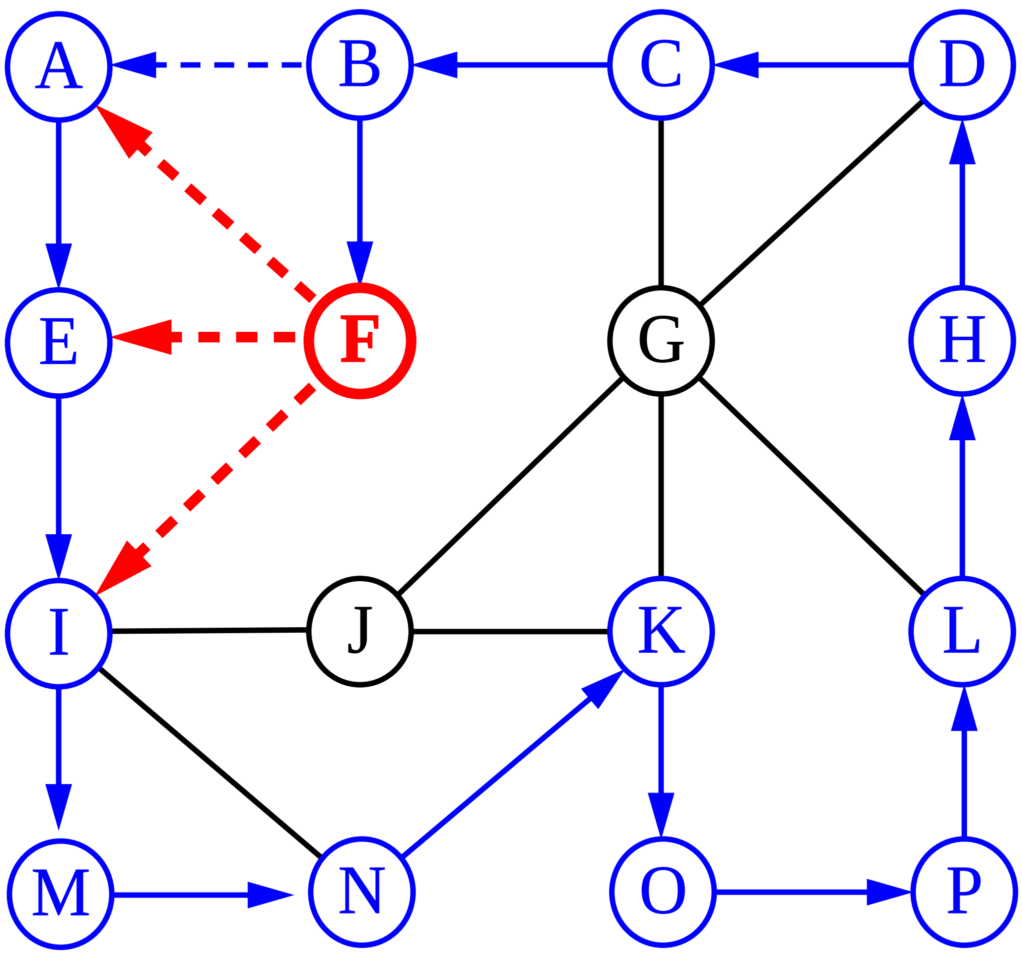
Graph Traversals

- Depth-First Search
- Breadth-First Search



Exploring a Labyrinth Without Getting Lost

- A **depth-first search (DFS)** in an undirected graph G is like wandering in a labyrinth with a string and a can of red paint without getting lost.
- We start at vertex *s*, tying the end of our string to the point and painting *s* "visited". Next we label *s* as our current vertex called *u*.
- Now we travel along an arbitrary edge (*u*,*v*).
- If edge (*u*,*v*) leads us to an already visited vertex *v* we return to *u*.
- If vertex *v* is unvisited, we unroll our string and move to *v*, paint *v* "visited", set *v* as our current vertex, and repeat the previous steps.
- Eventually, we will get to a point where all incident edges on *u* lead to visited vertices. We then backtrack by unrolling our string to a previously visited vertex *v*. Then *v* becomes our current vertex and we repeat the previous steps.

Exploring a Labyrinth Without Getting Lost (cont.)

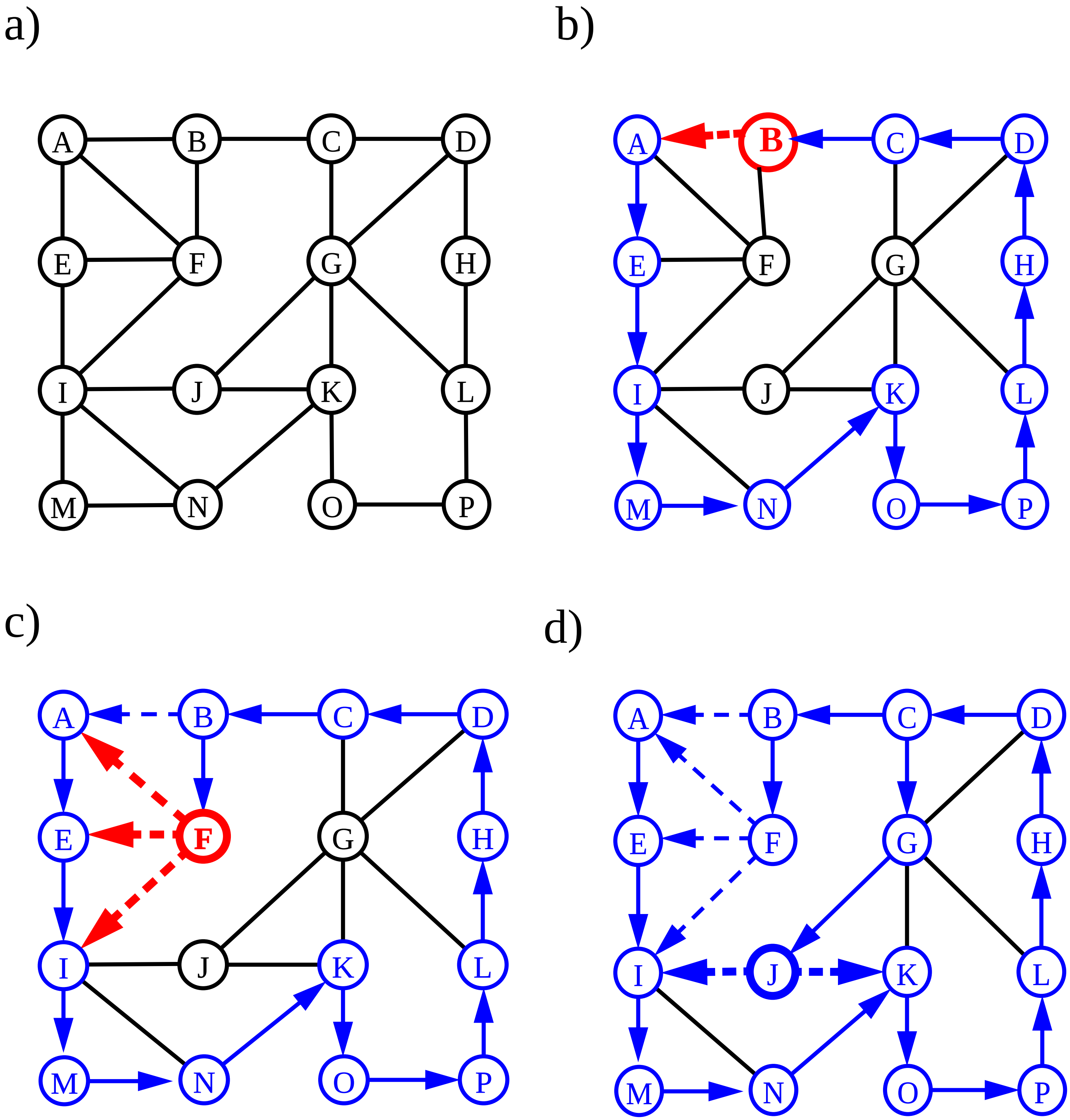
- Then, if we all incident edges on v lead to visited vertices, we backtrack as we did before. We continue to backtrack along the path we have traveled, finding and exploring unexplored edges, and repeating the procedure.
- When we backtrack to vertex *s* and there are no more unexplored edges incident on *s*, we have finished our DFS search.

Depth-First Search

Algorithm DFS(v);
  Input: A vertex v in a graph
  Output: A labeling of the edges as “discovery” edges and “backedges”

```
for each edge e incident on v do
   if edge e is unexplored then
      let w be the other endpoint of e
      if vertex w is unexplored then
         label e as a discovery edge
         recursively call DFS(w)
      else
         label e as a backedge
```

Depth-First Search(cont.)



Depth-First Search(cont.)

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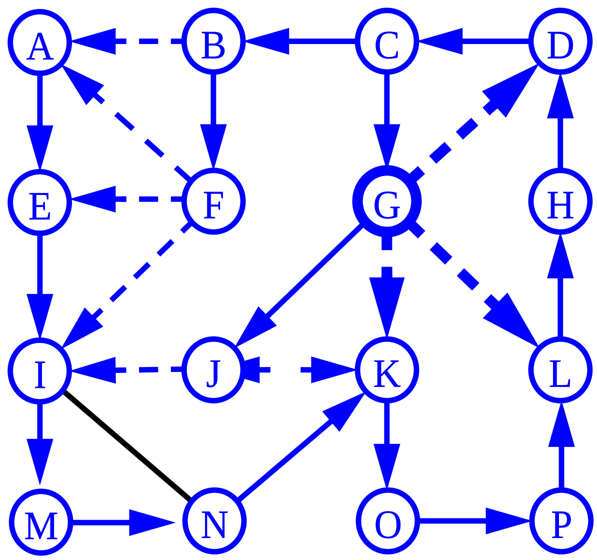

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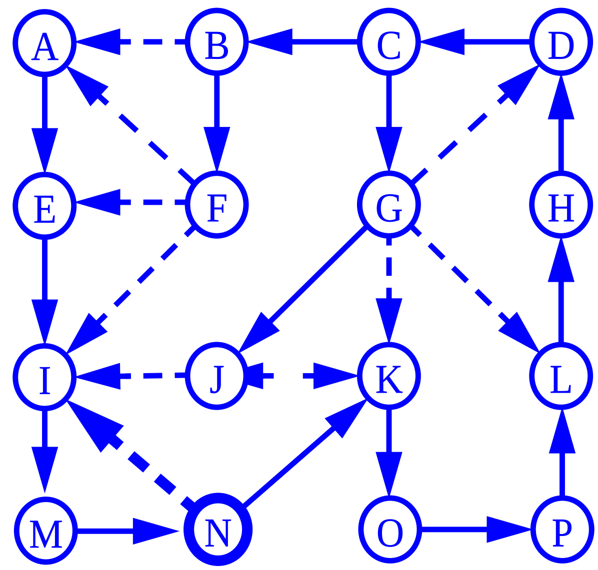


DFS Properties

- Proposition 9.12 : Let G be an undirected graph on which a DFS traversal starting at a vertex *s* has been preformed. Then:
  1) The traversal visits all vertices in the connected component of *s*
  2) The discovery edges form a spanning tree of the connected component of *s*

- Justification of 1):
  - Let's use a contradiction argument: suppose there is at least on vertex *v* not visited and let *w* be the first unvisited vertex on some path from *s* to *v*.
  - Because *w* was the first unvisited vertex on the path, there is a neighbor *u* that has been visited.
  - But when we visited *u* we must have looked at edge(*u*, *w*). Therefore *w* must have been visited.
  - and justification

- Justification of 2):
  - We only mark edges from when we go to unvisited vertices. So we never form a cycle of discovery edges, i.e. discovery edges form a tree.
  - This is a spanning tree because DFS visits each vertex in the connected component of *s*

Running Time Analysis

- Remember:
  - DFS is called on each vertex exactly once.
  - For every edge is examined exactly twice, once from each of its vertices
- For n_s vertices and m_s edges in the connected component of the vertex s, a DFS starting at s runs in $O(n_s + m_s)$ time if:
  - The graph is represented in a data structure, like the adjacency list, where vertex and edge methods take constant time
  - Marking the vertex as explored and testing to see if a vertex has been explored takes O(1)
  - We have a way of systematically considering the edges incident on the current vertex so we do not examine the same edge twice.

Marking Vertices

- Let’s look at ways to mark vertices in a way that satisfies the above condition.
- Extend vertex positions to store a variable for marking
- Use a hash table mechanism which satisfies the above condition is the probabilistic sense, because is supports the mark and test operations in O(1) expected time

The Template Method Pattern

- the **template method** pattern provides a ***generic computation mechanism*** that can be specialized by redefining certain steps
- to apply this pattern, we design a class that
  - implements the ***skeleton*** of an algorithm
  - invokes auxiliary methods that can be redefined by its subclasses to perform useful computations
- **Benefits**
  - makes the correctness of the specialized computations rely on that of the skeleton algorithm
  - demonstrates the power of class inheritance
  - provides code reuse
  - encourages the development of generic code
- **Examples**
  - ***generic traversal of a binary tree*** (which includes preorder, inorder, and postorder) and its applications
  - ***generic depth-first search of an undirected graph*** and its applications

Generic Depth First Search

```
public abstract class DFS {
 protected Object dfsVisit(Vertex v) {
   protected InspectableGraph graph;
   protected Object visitResult;
   initResult();
   startVisit(v);
   mark(v);
   for (Enumeration inEdges = graph.incidentEdges(v);
         inEdges.hasMoreElements();) {
     Edge nextEdge = (Edge) inEdges.nextElement();
     if (!isMarked(nextEdge)) { // found an unexplored edge
      mark(nextEdge);
      Vertex w = graph.opposite(v, nextEdge);
      if (!isMarked(w)) { // discovery edge
        mark(nextEdge);
        traverseDiscovery(nextEdge, v);
        if (!isDone())
          visitResult = dfsVisit(w); }
      else  // back edge
       traverseBack(nextEdge, v);
     }
   }
   finishVisit(v);
   return result();
  }
```

Auxiliary Methods of the Generic DFS

```
public Object execute(InspectableGraph g, Vertex start,
                                  Object info) {
  graph = g;
  return null;
}

protected void initResult() {}

protected void startVisit(Vertex v) {}

 protected void traverseDiscovery(Edge e, Vertex from) {}

protected void traverseBack(Edge e, Vertex from) {}

protected boolean isDone() { return false; }

protected void finishVisit(Vertex v) {}

protected Object result() { return new Object(); }
```

Specializing the Generic DFS

- class FindPath specializes DFS to return a path from vertex start to vertex target.

```
public class FindPathDFS extends DFS {
  protected Sequence path;
  protected boolean done;
  protected Vertex target;
  public Object execute(InspectableGraph g, Vertex start,
                                    Object info) {
    super.execute(g, start, info);
    path = new NodeSequence();
    done = false;
    target = (Vertex) info;
    dfsVisit(start);
    return path.elements();
  }
  protected void startVisit(Vertex v) {
    path.insertFirst(v);
    if (v == target) { done = true; }
  }
  protected void finishVisit(Vertex v) {
    if (!done) path.remove(path.first());
  }
  protected boolean isDone() { return done; }
}
```

Other Specializations of the Generic DFS

- FindAllVertices specializes DFS to return an enumeration of the vertices in the connecteed component containing the start vertex.

```
public class FindAllVerticesDFS extends DFS {
   protected Sequence vertices;
   public Object execute(InspectableGraph g, Vertex start,
                                             Object info) {
      super.execute(g, start, info);
      vertices = new NodeSequence();
      dfsVisit(start);
      return vertices.elements();
   }

   public void startVisit(Vertex v) {
      vertices.insertLast(v);
   }
}
```

More Specializations of the Generic DFS

- ConnectivityTest uses a specialized DFS to test if a graph is connected.

```
public class ConnectivityTest {
   protected static DFS tester = new FindAllVerticesDFS();
   public static boolean isConnected(InspectableGraph g)
   {
      if (g.numVertices() == 0) return true; //empty is
                                              //connected
      Vertex start = (Vertex)g.vertices().nextElement();
      Enumeration compVerts =
                (Enumeration)tester.execute(g, start, null);
      // count how many elements are in the enumeration
      int count = 0;
      while (compVerts.hasMoreElements()) {
         compVerts.nextElement();
         count++;
      }
      if (count == g.numVertices()) return true;
      return false;
   }
}
```

Another Specialization of the Generic DFS

- FindCycle specializes DFS to determine if the connected component of the start vertex contains a cycle, and if so return it.

```
public class FindCycleDFS extends DFS {
   protected Sequence path;
   protected boolean done;
   protected Vertex cycleStart;
   public Object execute(InspectableGraph g, Vertex start,
                                          Object info) {
      super.execute(g, start, info);
      path = new NodeSequence();
      done = false;
      dfsVisit(start);
      //copy the vertices up to cycleStart from the path to
      //the cycle sequence.
      Sequence theCycle = new NodeSequence();
      Enumeration pathVerts = path.elements();
```

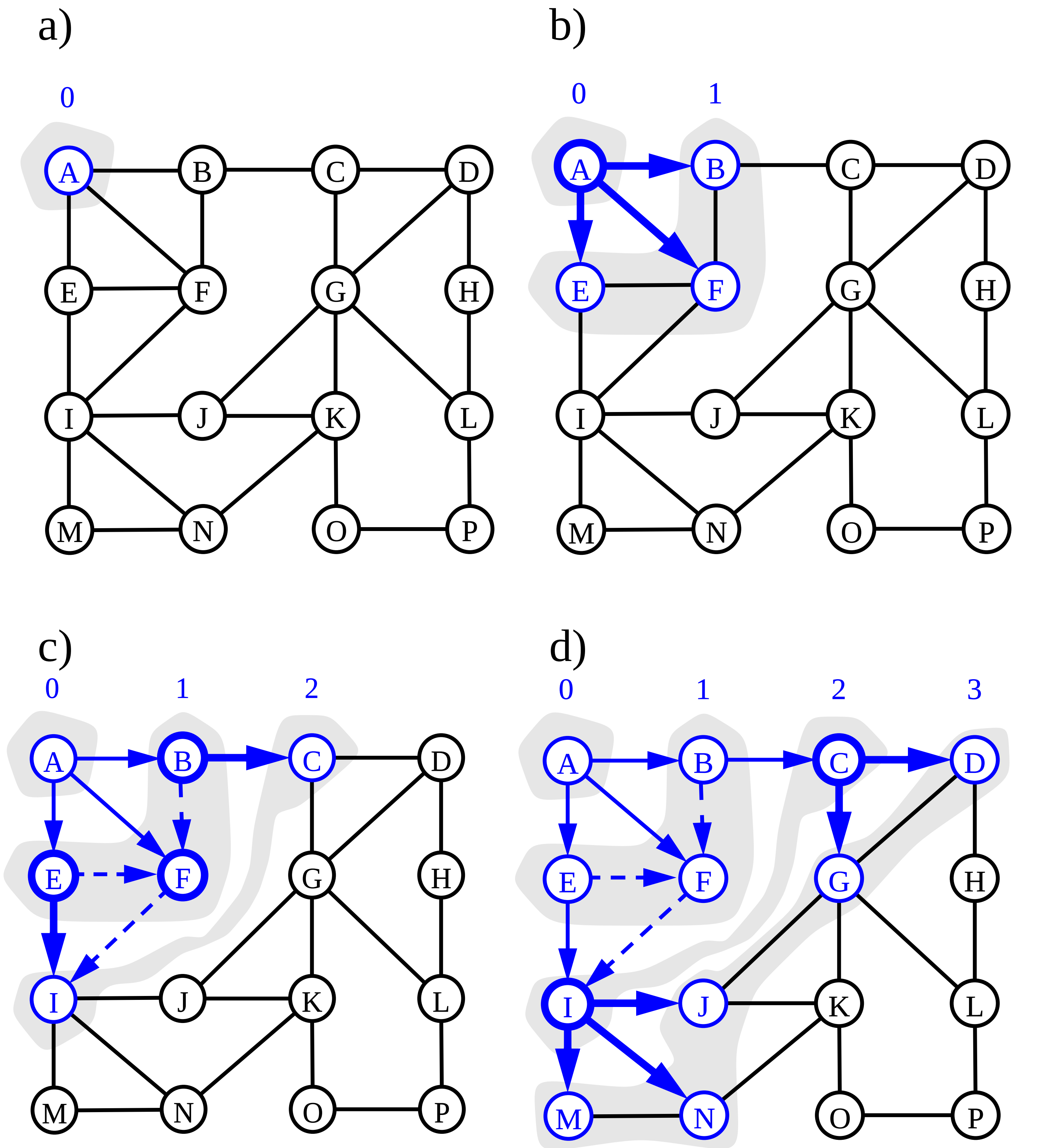
```
    while (pathVerts.hasMoreElements()) {
      Vertex v = (Vertex)pathVerts.nextElement();
      theCycle.insertFirst(v);
      if ( v == cycleStart) {
        break;
      }
    }
    return theCycle.elements();
  }
  protected void startVisit(Vertex v) {path.insertFirst(v);}
  protected void finishVisit(Vertex v) {
    if (done) {path.remove(path.first());}
  }
  //When a back edge is found, the graph has a cycle
  protected void traverseBack(Edge e, Vertex from) {
    Enumeration pathVerts = path.elements();
    cycleStart = graph.opposite(from, e);
    done = true;
  }
  protected boolean isDone() {return done;}
}
```

Breadth-First Search

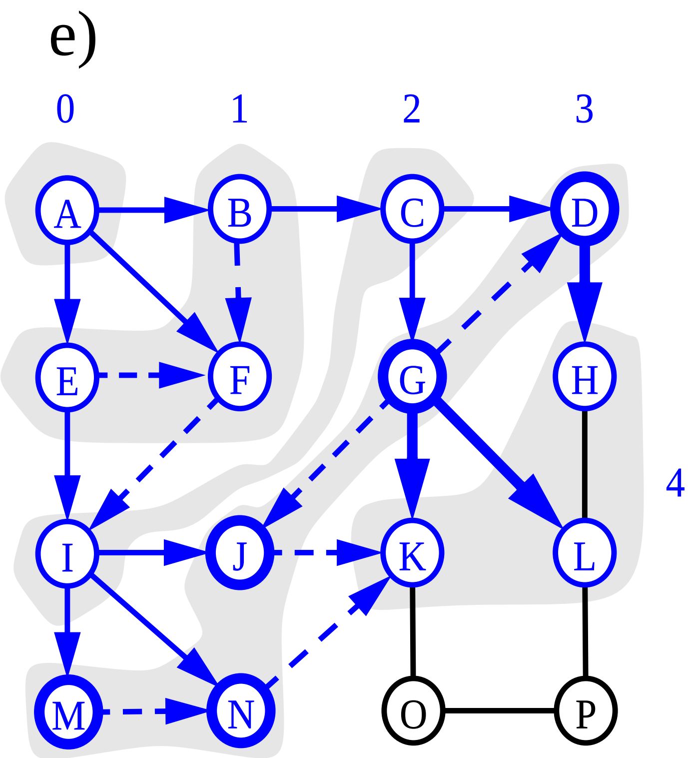
- Like DFS, a Breadth-First Search (BFS) traverses a connected component of a graph, and in doing so defines a spanning tree with several useful properties

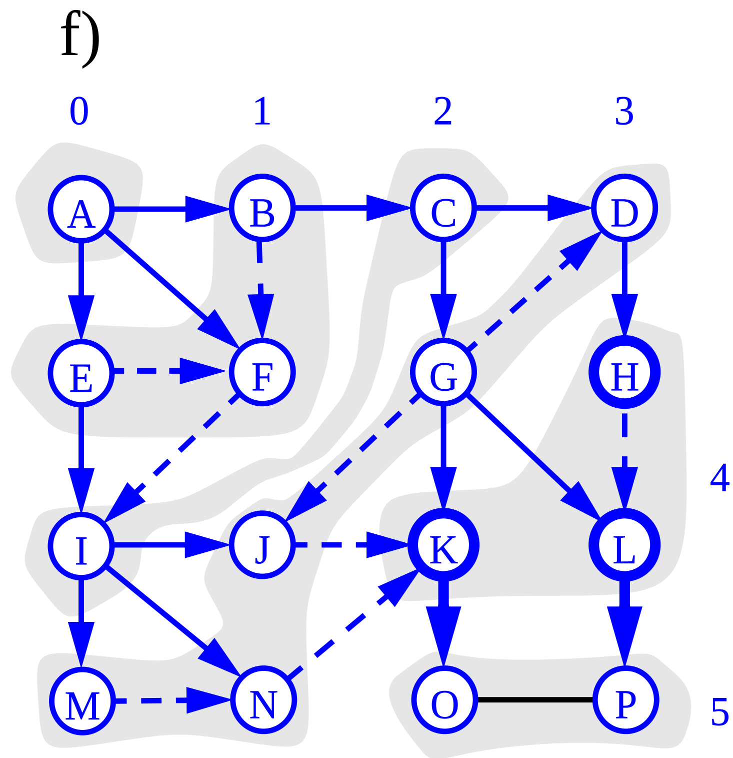
  - The starting vertex *s* has level 0, and, as in DFS, defines that point as an "anchor."
  - In the first round, the string is unrolled the length of one edge, and all of the edges that are only one edge away from the anchor are visited.
  - These edges are placed into level 1
  - In the second round, all the new edges that can be reached by unrolling the string 2 edges are visited and placed in level 2.
  - This continues until every vertex has been assigned a level.
  - The label of any vertex *v* corresponds to the length of the shortest path from *s* to *v*.

BFS - A Graphical Representation



More BFS





BFS Pseudo-Code

```
Algorithm BFS(s):
  Input: A vertex s in a graph
  Output: A labeling of the edges as "discovery" edges
    and "cross edges"
  initialize container L0 to contain vertex s
  i ← 0
  while Li is not empty do
    create container Li+1 to initially be empty
    for each vertex v in Li do
      if edge e incident on v do
        let w be the other endpoint of e
        if vertex w is unexplored then
          label e as a discovery edge
          insert w into Li+1
        else
          label e as a cross edge
  i ← i + 1
```

Properties of BFS

- Proposition: Let G be an undirected graph on which a a BFS traversal starting at vertex s has been performed. Then
  - The traversal visits all vertices in the connected component of s.
  - The discovery-edges form a spanning tree T, which we call the BFS tree, of the connected component of s
  - For each vertex v at level i, the path of the BFS tree T between s and v has i edges, and any other path of G between s and v has at least i edges.
  - If (u, v) is an edge that is not in the BFS tree, then the level numbers of u and v differ by at most one.

- Proposition: Let G be a graph with n vertices and m edges. A BFS traversal of G takes time $O(n + m)$. Also, there exist $O(n + m)$ time algorithms based on BFS for the following problems:
  - Testing whether G is connected.
  - Computing a spanning tree of G
  - Computing the connected components of G
  - Computing, for every vertex v of G, the minimum number of edges of any path between s and v.