# MAXIMUM FLOW

- How to do it...
- Why you want it...
- Where you find it...

## The Tao of Flow:

"Let your body go with the flow."
-Madonna, *Vogue*

"Maximum flow...because you're worth it."
-L'oreal

"Go with the flow, Joe."
-Paul Simon, *50 ways to leave your lover*

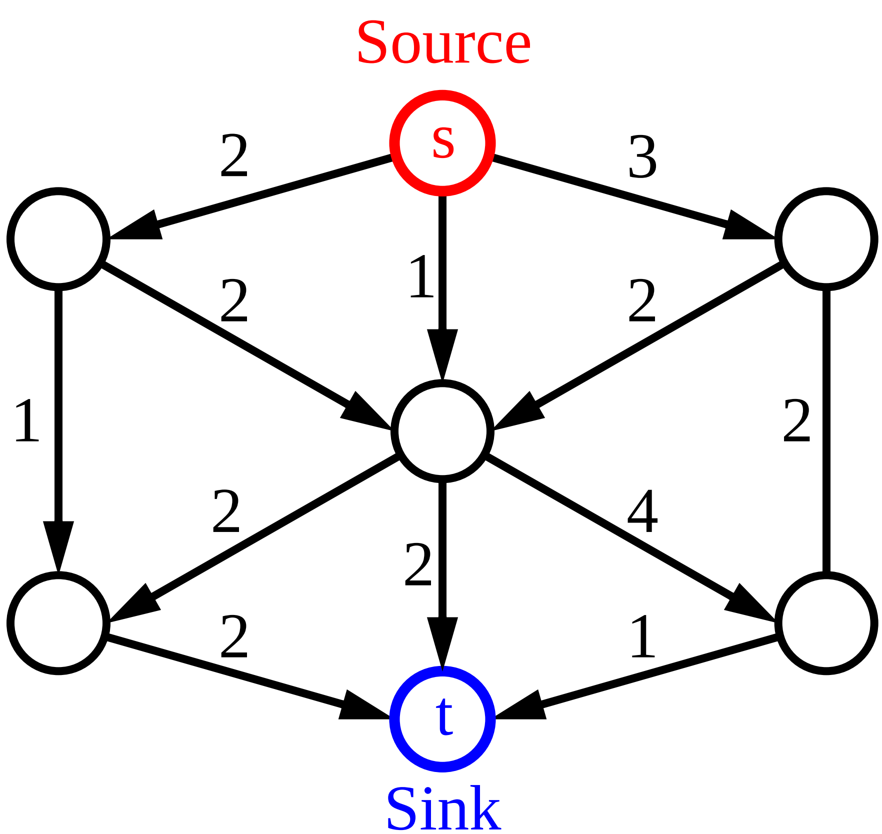
"Life is flow; flow is life."
-Ford & Fulkerson, *Ford & Fulkerson Algorithm*

"Use the flow, Luke!"
-Obi-wan Kenobi, *Star Wars*

"Learn flow, or flunk the course"
-CS16 Despot, as played by Roberto Tamassia

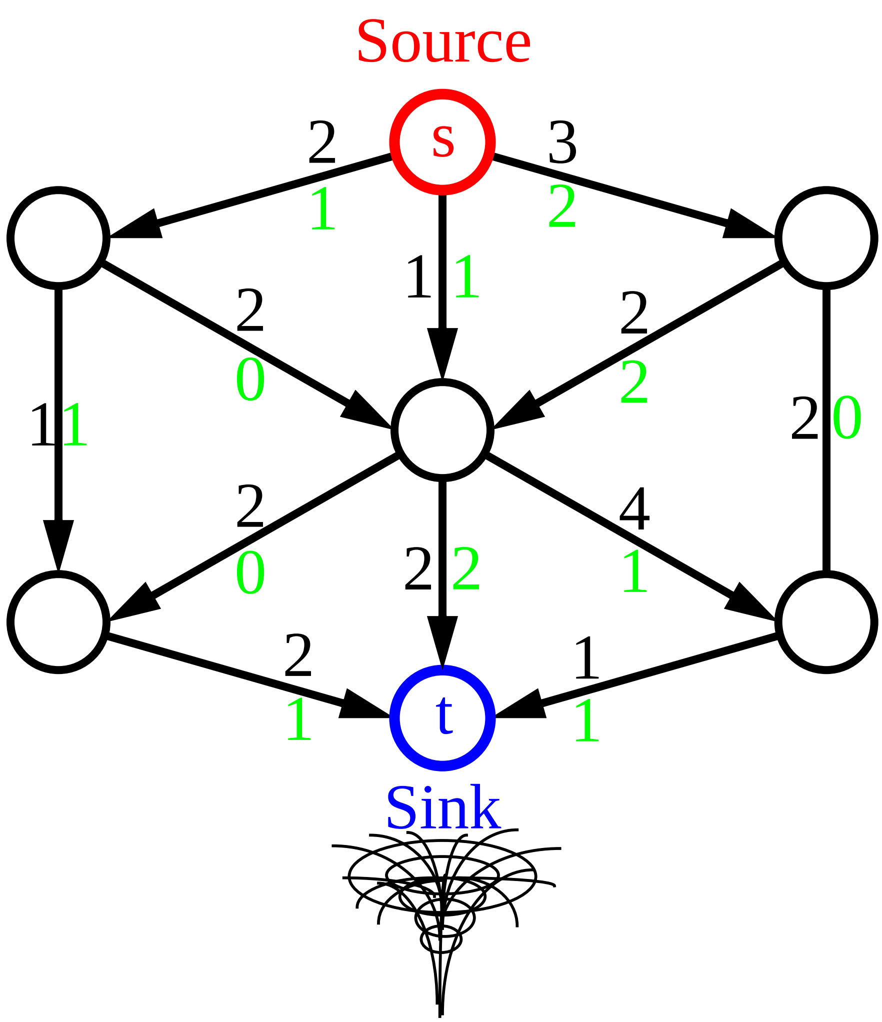
# Flow Networks

- Flow Network:
  - digraph
  - weights, called capacities on edges
  - two distinguishes vertices, namely
    - Source, “s”:
      Vertex with no incoming edges.
    - Sink, “t”:
      Vertex with no outgoing edges.



# Capacity and Flow

- Edge Capacities:
  Nonnegative weights on network edges
- Flow:
  - Function on network edges:
    $0 \leq$ flow $\leq$ **capacity**
    flow into vertex = flow out of vertex
    **value**: combined flow into the sink



# The Logic of Flow

- Flow:

  **flow(u,v)** $\forall$ **edge(u,v)**

  -Capacity rule: $\forall$ edge (u,v)

  $$0 \leq \text{flow}(u,v) \leq \textbf{capacity}(u,v)$$

  -Conservation rule: $\forall$ vertex $v \neq s, t$

  $$\sum_{u \in in(v)} \text{flow}(u,v) = \sum_{w\text{Œ}out(v)} \text{flow}(v,w)$$

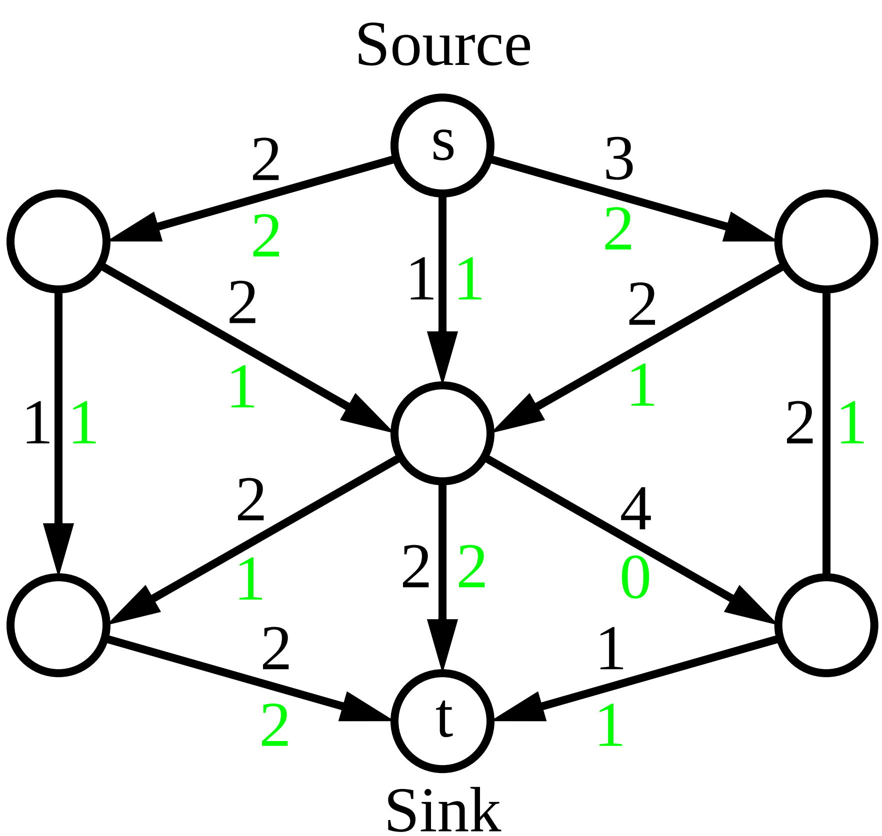
  -Value of flow:

  $$|f| = \sum_{w \in out(s)} \text{flow}(s,w) = \sum_{u \in in(t)} \text{flow}(u,t)$$

- **Note:**
  - $\forall$ means “for all”
  - in(v) is the set of vertices u such that there is and edge from u to v
  - out(v) is the set of vertices w such that there is an edge from v to w

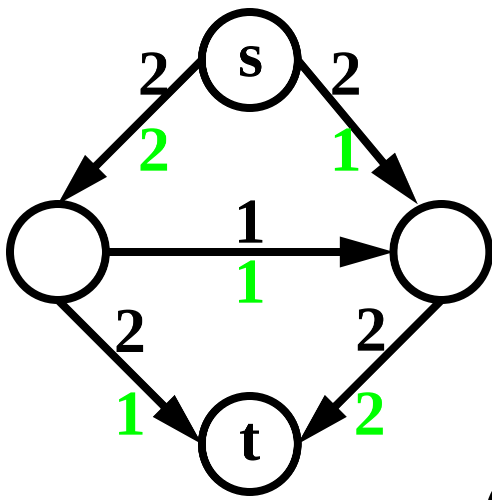
# Maximum Flow Problem

- "Given a network N, find a flow f of maximum value."
- Applications:
  - Traffic movement
  - Hydraulic systems
  - Electrical circuits
  - Layout

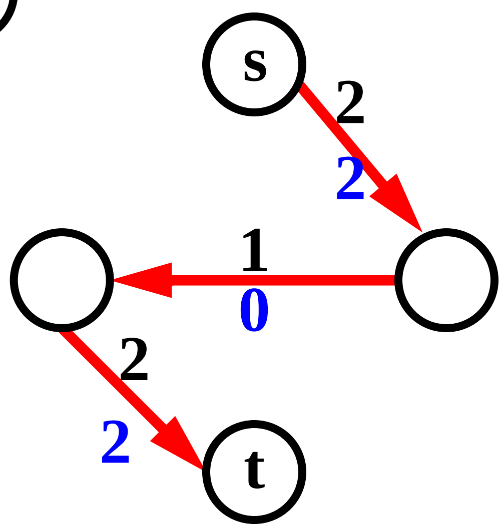



Example of Maximum Flow

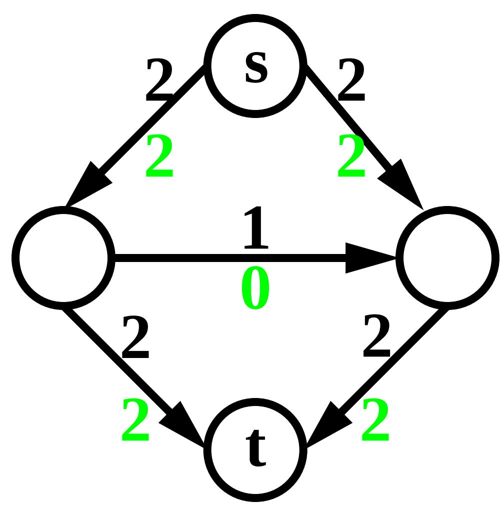
# Augmenting Flow




A network with a flow of value **3**




Augmenting path



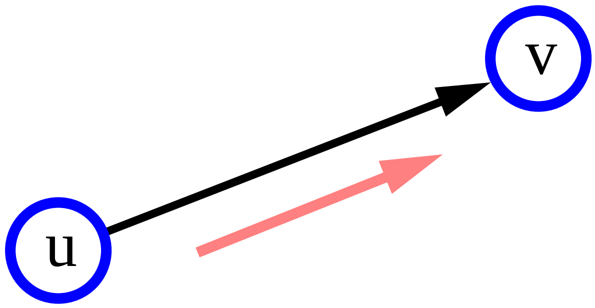

- Voila! We have increased the flow value to 4! But wait! What's an augmenting path?!?

# Augmenting Path

- **Forward Edges**

  flow(u,v) < **capacity**(u,v)

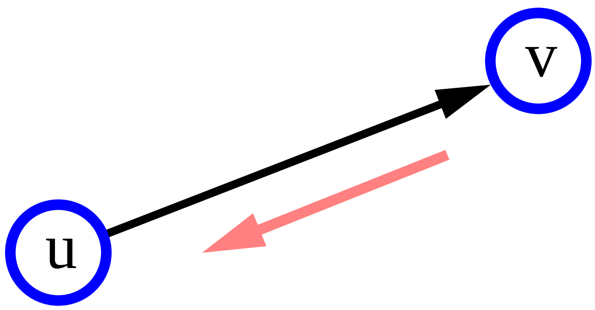
  flow can be increased!




- **Backward Edges**

  flow(u,v) > 0

  flow can be decreased!



# Maximum Flow Theorem

A flow has maximum value
if and only if
it has no augmenting path.

**Proof:**

Flow is maximum $\Rightarrow$ No augmenting path

(The ***only-if*** part is easy to prove.)

No augmenting path $\Rightarrow$ Flow is maximum

(Proving the ***if*** part is more difficult.)

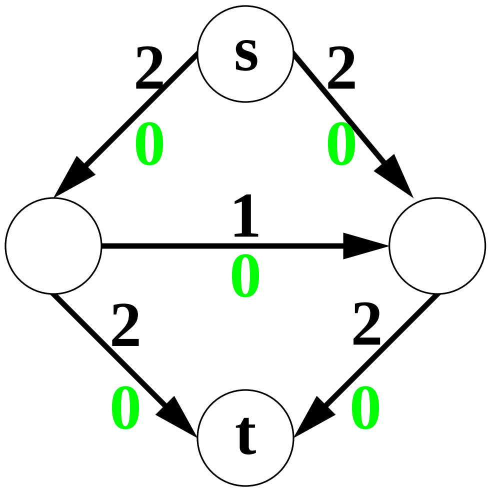
# Ford & Fulkerson Algorithm

- One day, Ford phoned his buddy Fulkerson and said, "Hey Fulk!  Let's formulate an algorithm to determine maximum flow."  Fulk responded in kind by saying, "Great idea, Ford!  Let's just do it!"  And so, after several days of abstract computation, they came up with the Ford Fulkerson Algorithm, affectionately known as the "Ford & Fulkerson Algorithm."

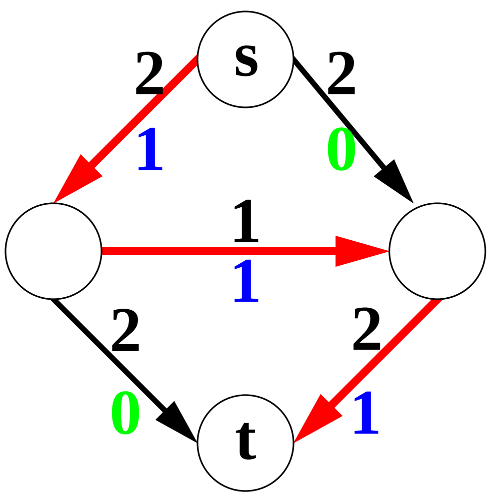
```
initialize network with null flow;
Method FindFlow
  if augmenting paths exist then
    find augmenting path;
    increase flow;
    recursive call to FindFlow;
```

- And now, for some algorithmic animation...

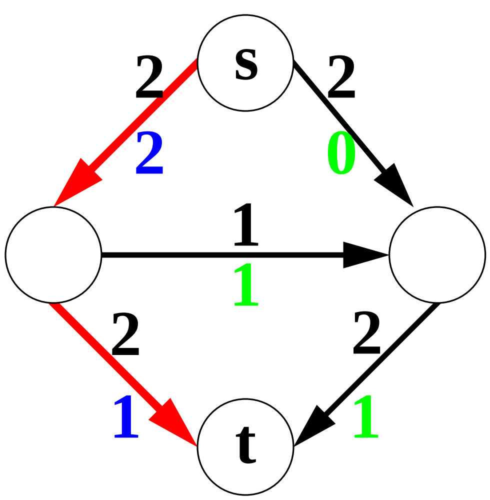
# Finding the Maximum Flow




Initialize the network with a null flow. Note the **capacities above the edges**, and the flow below the edges.

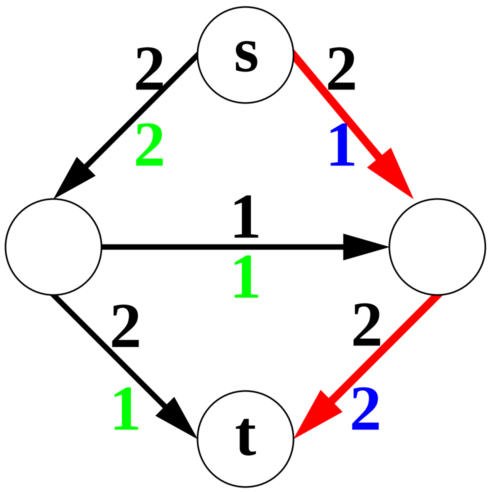



Send one unit of flow through the network. Note the path of the flow unit traced in red. The incremented flow values are in blue.

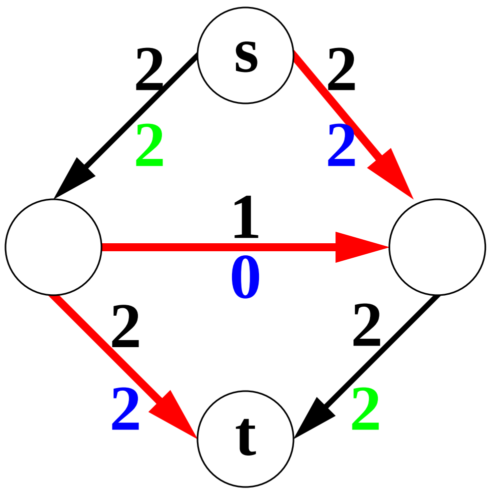



Send another unit of flow through the network.

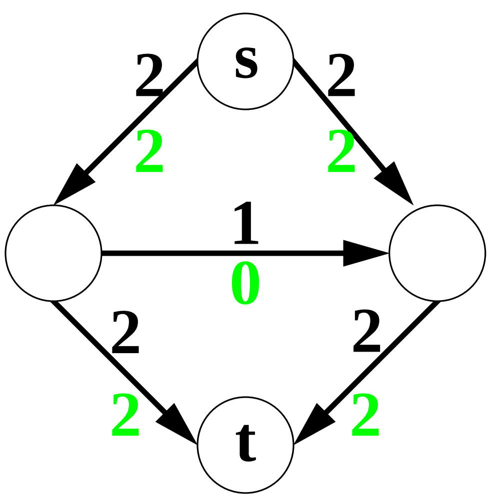
# Finding the Maximum Flow




Send another unit of flow through the network. Note that there still exists an augmenting path, that can proceed *backward* against the directed central edge.




Send one unit of flow through the augmenting path. Note that there are no more augmenting paths. That means...

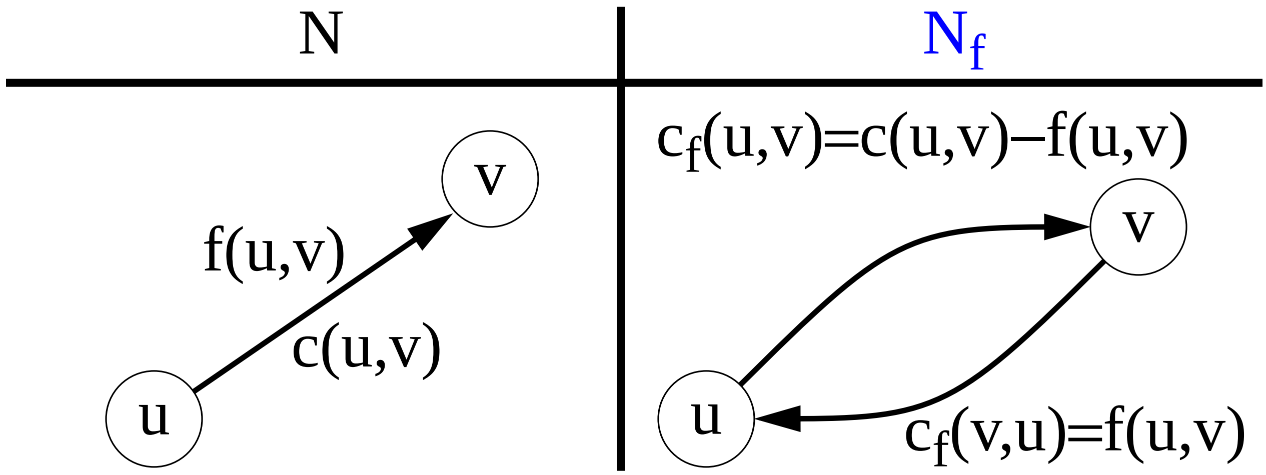



With the help of both Ford & Fulkerson, we have achieved this network's ***maximum flow***.

Is this power, or what?!?

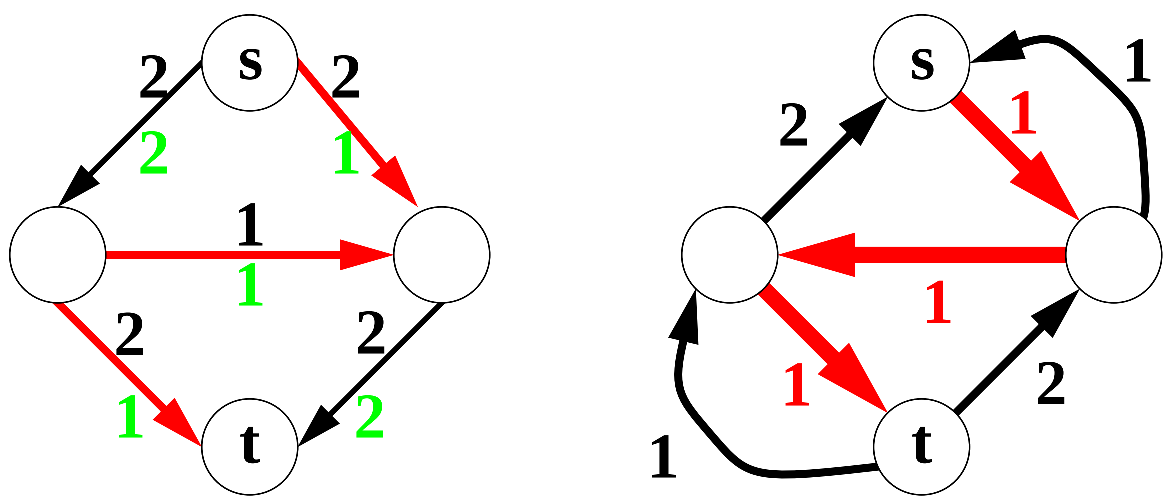
# Residual Network

- Residual Network $N_f$ = (V, Ef, cf, s, t)




- In the residual network $N_f$, all edges (w,z) with capacity cf(w,z) = 0 are removed.

Augmenting path in network N ↔ Directed path in the residual network $N_f$




Augmenting paths can be found performing a depth-first search on the residual network $N_f$

# Maximum Flow Algorithm

***Part I: Setup***

Start with null flow:

  $f(u,v) = 0 \ \forall\ (u,v) \in E$;

Initialize residual network:

  $N_f = N$;

***Part II: Loop***

```
repeat
  search for directed path p in N_f from s to t
  if (path p found)
       Df = min {c_f(u,v), (u,v) ∈ p};
       for (each (u,v) ∈ p) do
          if (forward (u,v))
             f(u,v) = f(u,v) + Df;
          if (backward (u,v))
             f(u,v) = f(u,v) - Df;
       update N_f;
until (no augmenting path);
```

# Maximum Flow: Time Complexity

- And now, the moment you've all been waiting for...the time complexity of Ford & Fulkerson's Maximum Flow algorithm. Drum roll, please! [Pause for dramatic drum roll music]

  $$O(\,F\,(n + m)\,)$$

  where F is the maximum flow value, n is the number of vertices, and m is the number of edges

- The problem with this algorithm, however, is that it is strongly dependent on the maximum flow value F. For example, if $F=2^n$ the algorithm may take exponential time.

- Then, along came Edmonds & Karp...

# Maximum Flow: Improvement

- Theorem: [Edmonds & Karp, 1972]

  By using BFS, Breadth First Search, a maximum flow can be computed in time...

$$O((n + m)\ n\ m) = O(n^5)$$

- **Note:**
  - Edmonds & Karp algorithm runs in time $O(n^5)$ regardless of the maximum flow value.
  - The worst case usually does not happen in practice.

# CUTS

- What is a *cut* ?

  a)a skin-piercing laceration

  b)sharp lateral movement

  c)opposite of paste

  d)getting dropped from the team

  e)frontsies on the lunch line

  f)common CS16 attendence phenomenon

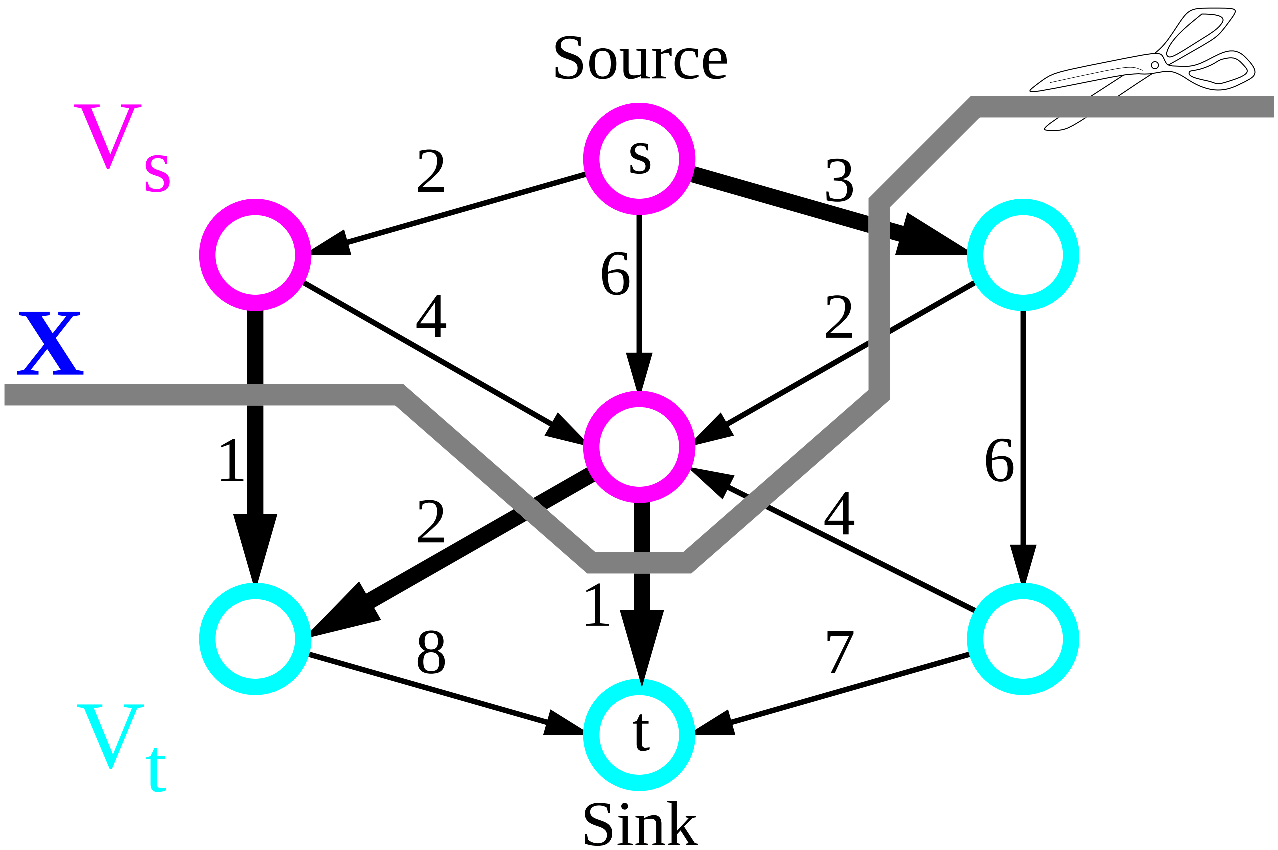
  g)line of muscular definition

  h)**C**omputer **U**ndergraduate **T**orture (e.g., CS16, Roberto Tamassia, dbx, etc.)

  i)a partition of the vertices $X=(V_s,V_t)$, with $s \in V_s$ and $t \in V_t$

- The answer is:

# What Is A Cut?




- Capacity of a cut X = ($V_s$,$V_t$):
  - $c(X) = \sum_{\substack{v \in V_s \\ w \in V_t}} \text{capacity}(v,w) = (1+2+1+3) = 7$

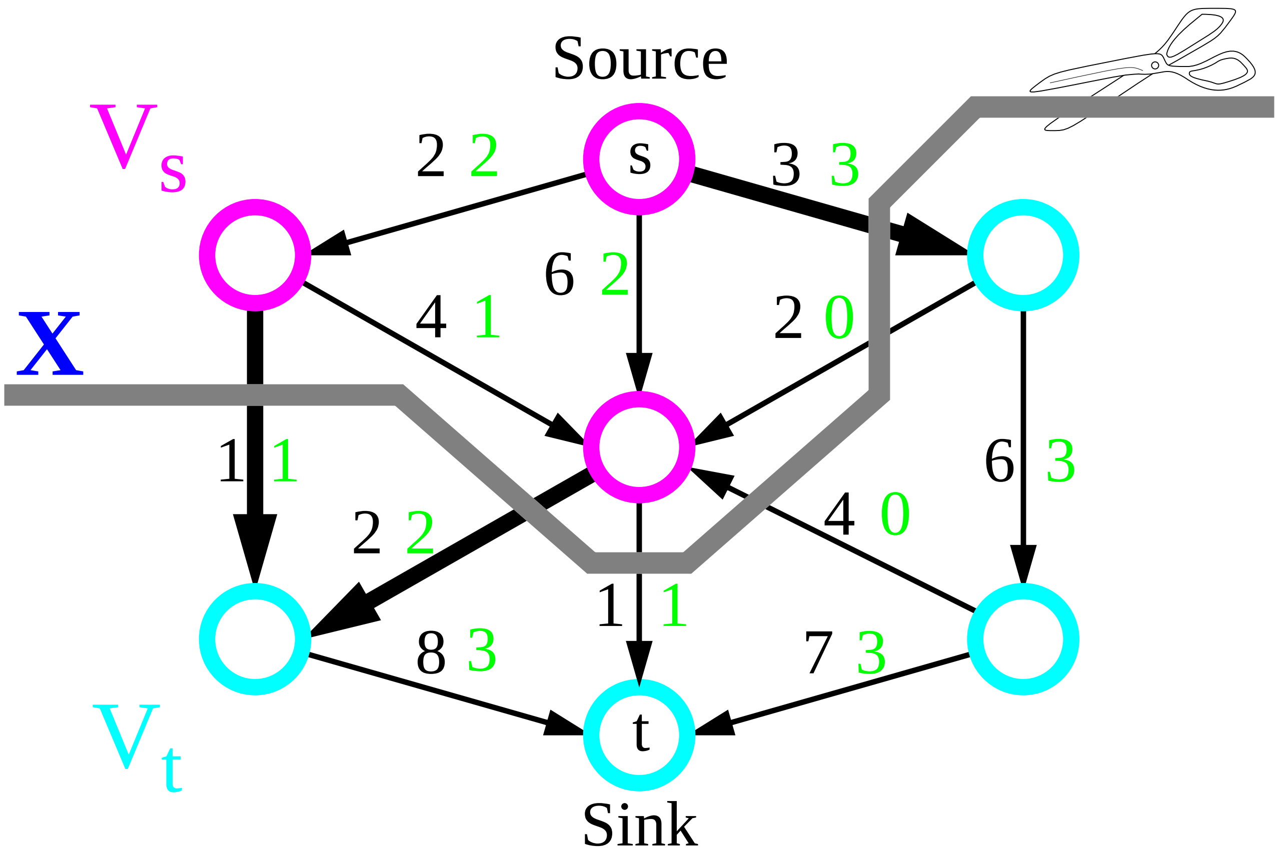
- The cut partition (X in this case) must pass through the entire network, and can not pass through a vertex.

# Maximum Flow vs. Minimum Cut

(value of maximum flow)

=

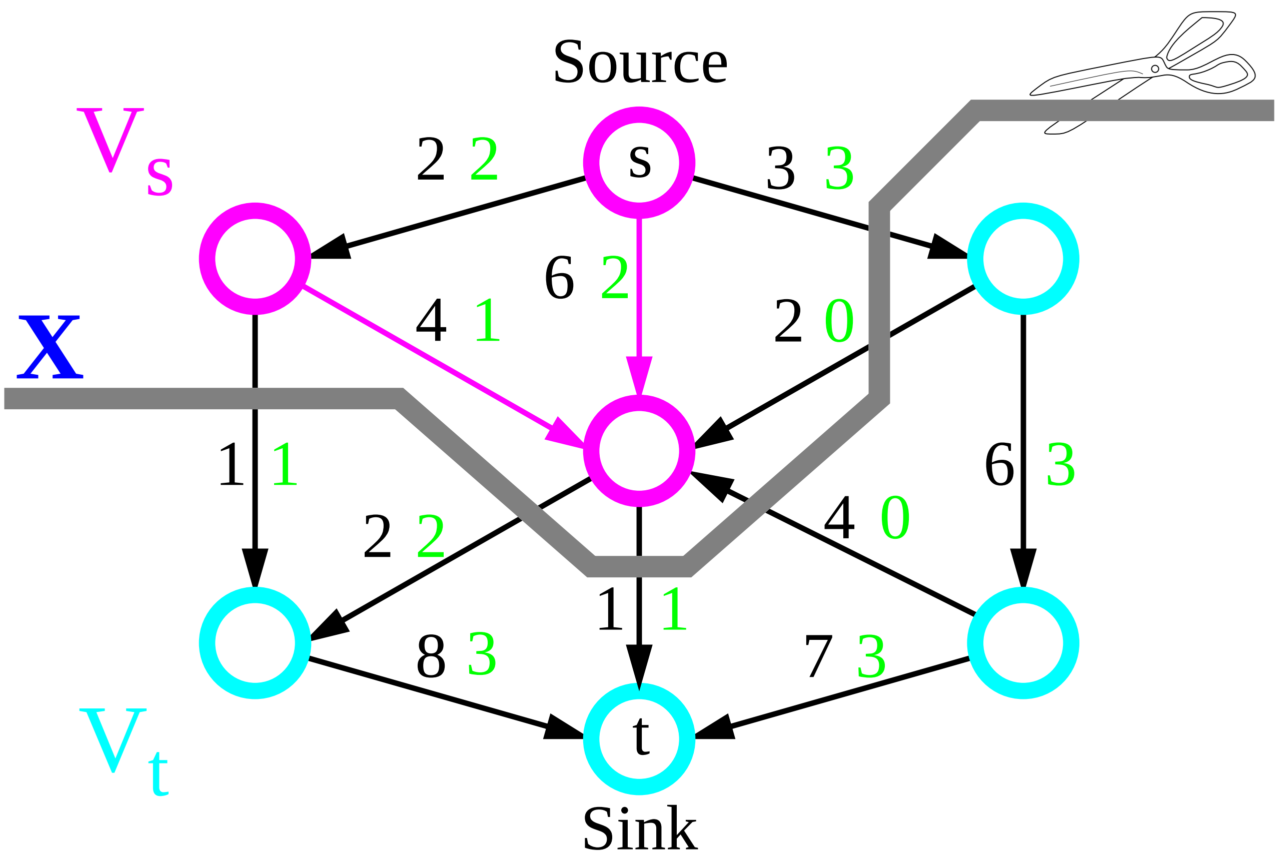
(capacity of minimum cut)




- Value of maximum flow: 7 flow units
- Capacity of minimum cut: 7 flow units

# Finding a Minimum Cut

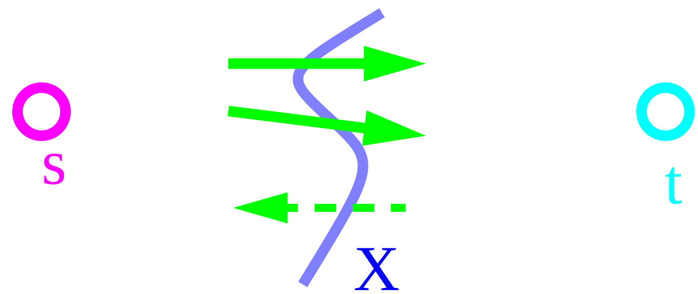
- Let $V_s$ be the set of vertices reached by augmenting paths from the source s, $V_t$ the set of remaining vertices, and place the cut partition X accordingly.




- Hence, a minor modification of the Ford & Fulkerson algorithm gives a minimum cut
- **Value of maximum flow: 7 flow units**
- **Capacity of minimum cut: 7 flow units**

# Why is that a Minimum Cut?

- Let f be a flow of value |f| and X a cut ofcapacity |X|. Then,  |f| <=|X|.




- Hence, if we find a flow  f* of value |f*| and a cut X* of capacity |X*|=|f*|, then  f* must be the maximum flow and X* must be the minimum cut.

- We have seen that from the flow obtained by the Ford and Fulkerson algorithm we can construct a cut with capacity equal to the flow value. Therefore,
  - we have given an alternative proof that the Ford and Fulkerson algorithm yields a maximum flow
  - we have shown how to construct a minimum cut