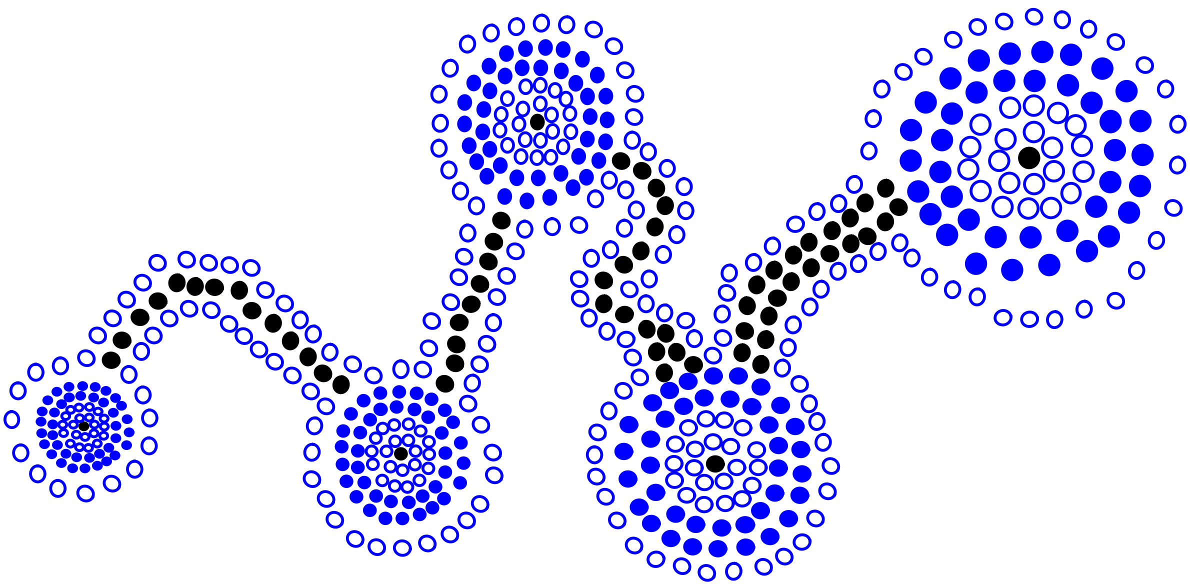
SORTING

- Review of Sorting
- Merge Sort
- Sets



Sorting Algorithms

- Selection Sort uses a priority queue P implemented with an unsorted sequence:
  - **Phase 1**: the insertion of an item into P takes $\boldsymbol{O}(1)$ time; overall $\boldsymbol{O}(n)$
  - **Phase 2**: removing an item takes time proportional to the number of elements in P $\boldsymbol{O}(n)$: overall $\boldsymbol{O}(n^2)$
  - Time Complexity: $\boldsymbol{O}(n^2)$
- Insertion Sort is performed on a priority queue P which is a sorted sequence:
  - **Phase 1**: the first insertItem takes $\boldsymbol{O}(1)$, the second $\boldsymbol{O}(2)$, until the last insertItem takes $\boldsymbol{O}(n)$: overall $\boldsymbol{O}(n^2)$
  - **Phase 2**: removing an item takes $\boldsymbol{O}(1)$ time; overall $\boldsymbol{O}(n)$.
  - Time Complexity: $\boldsymbol{O}(n^2)$
- Heap Sort uses a priority queue K which is a heap.
  - insertItem and removeMinElement each take $\boldsymbol{O}(\log k)$, k being the number of elements in the heap at a given time.
  - **Phase 1**: n elements inserted: $\boldsymbol{O}(n\log n)$ time
  - **Phase 2**: n elements removed: $\boldsymbol{O}(n \log n)$ time.
  - Time Complexity: $\boldsymbol{O}(n\log n)$

Divide-and-Conquer

- *Divide and Conquer* is more than just a military strategy, it is also a method of algorithm design that has created such efficient algorithms as Merge Sort.
- In terms or algorithms, this method has three distinct steps:
  - **Divide**: If the input size is too large to deal with in a straightforward manner, divide the data into two or more disjoint subsets.
  - **Recurse**: Use divide and conquer to solve the subproblems associated with the data subsets.
  - **Conquer**: Take the solutions to the subproblems and "merge" these solutions into a solution for the original problem.

Merge-Sort

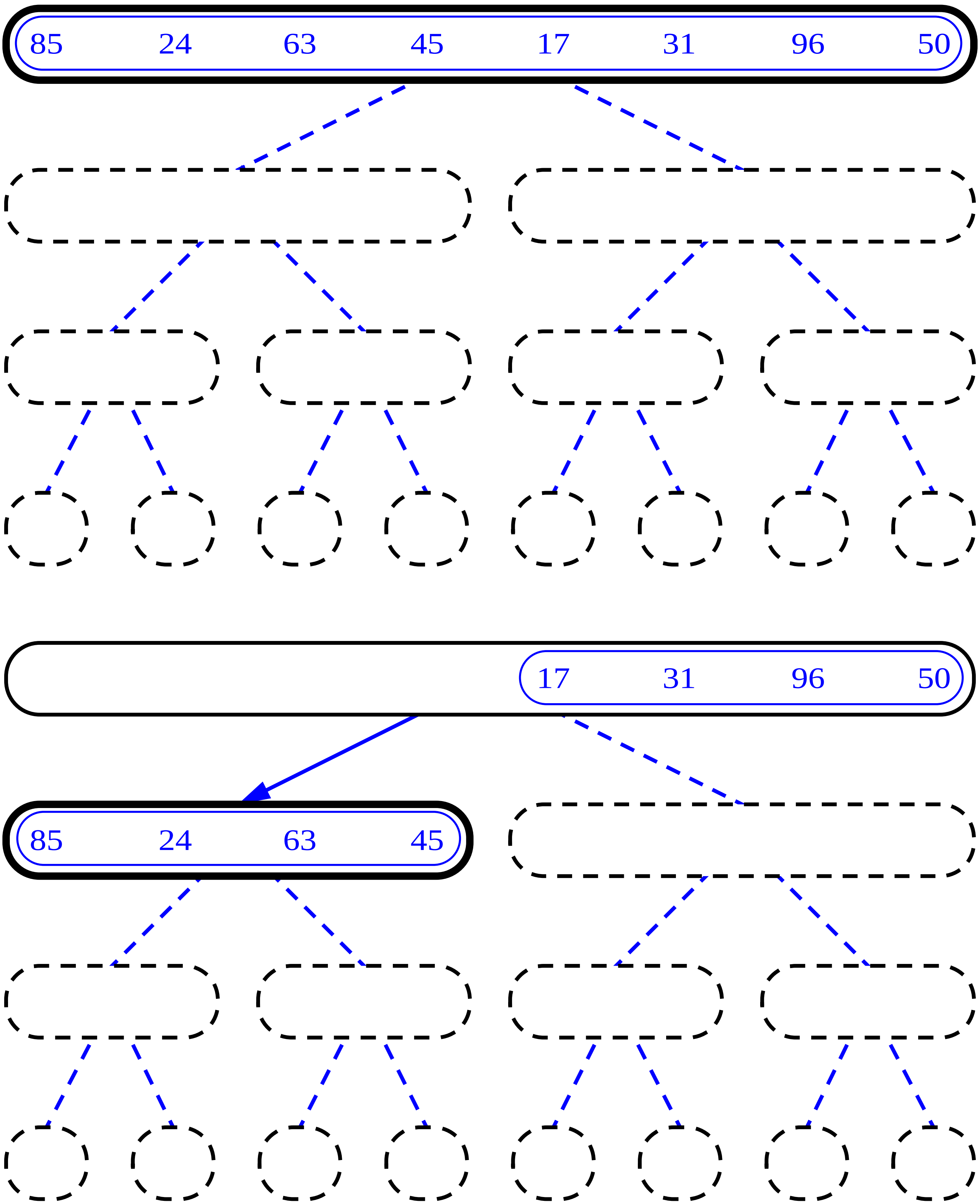
- Algorithm:

  - **Divide**: If S has at leas two elements (nothing needs to be done if S has zero or one elements), remove all the elements from S and put them into two sequences, S_1 and S_2, each containing about half of the elements of S. (i.e. S_1 contains the first $\lceil n/2 \rceil$ elements and S_2 contains the remaining $\lfloor n/2 \rfloor$ elements.
  - **Recurse**: Recursive sort sequences S_1 and S_2.
  - **Conquer**: Put back the elements into S by merging the sorted sequences S_1 and S_2 into a unique sorted sequence.

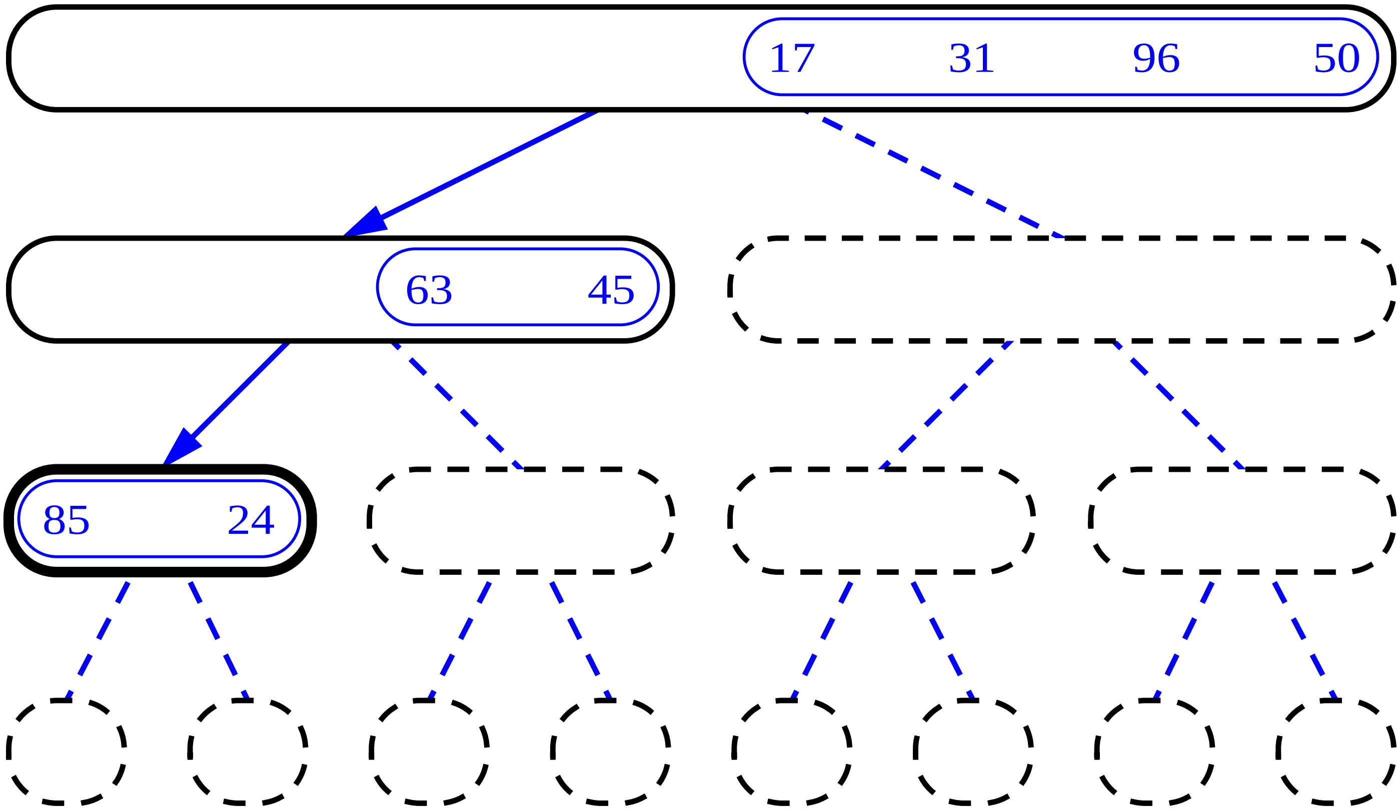
- Merge Sort Tree:

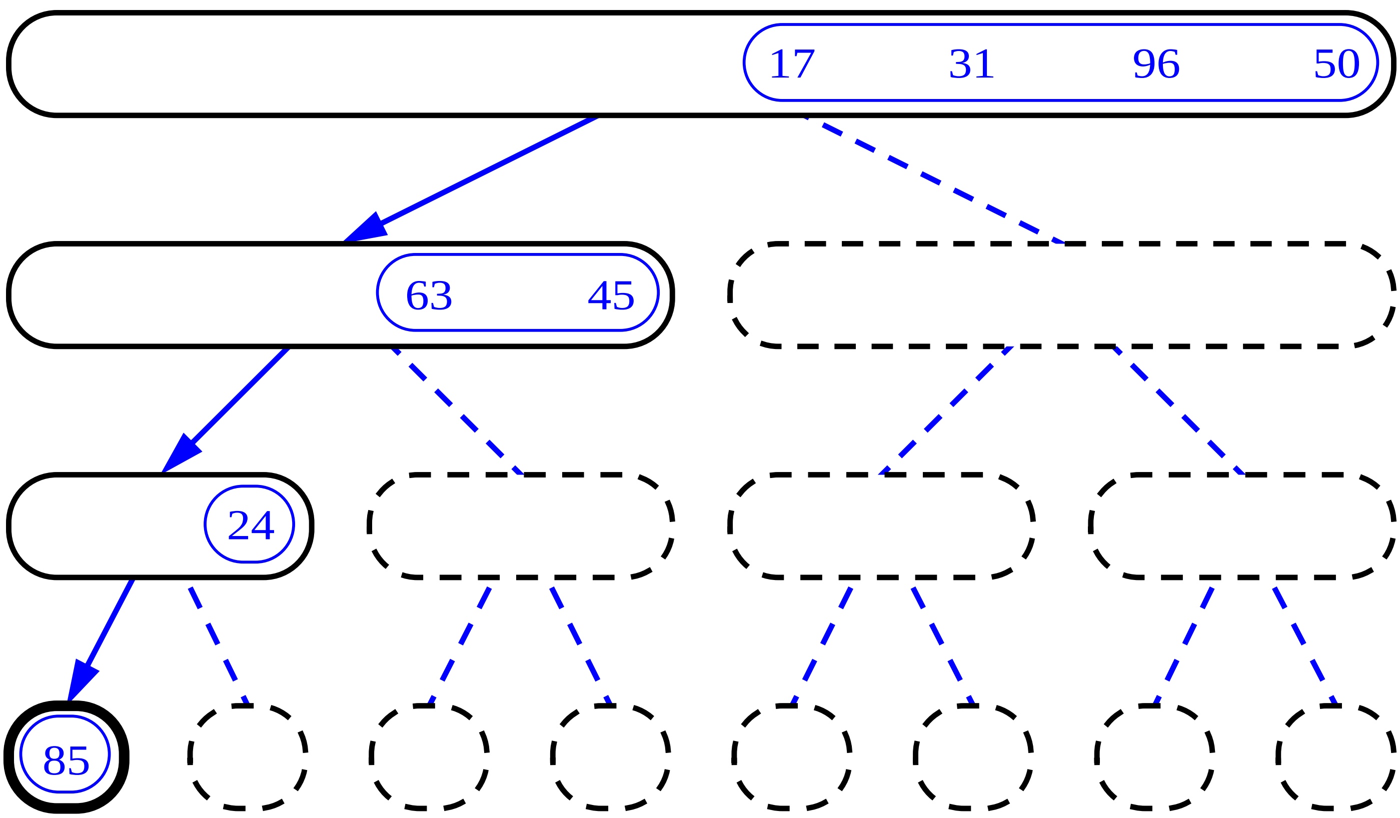
  - Take a binary tree T
  - Each node of T represents a recursive call of the merge sort algorithm.
  - We assocoate with each node v of T a the set of input passed to the invocation v represents.
  - The external nodes are associated with individual elements of S, upon which no recursion is called.

Merge-Sort

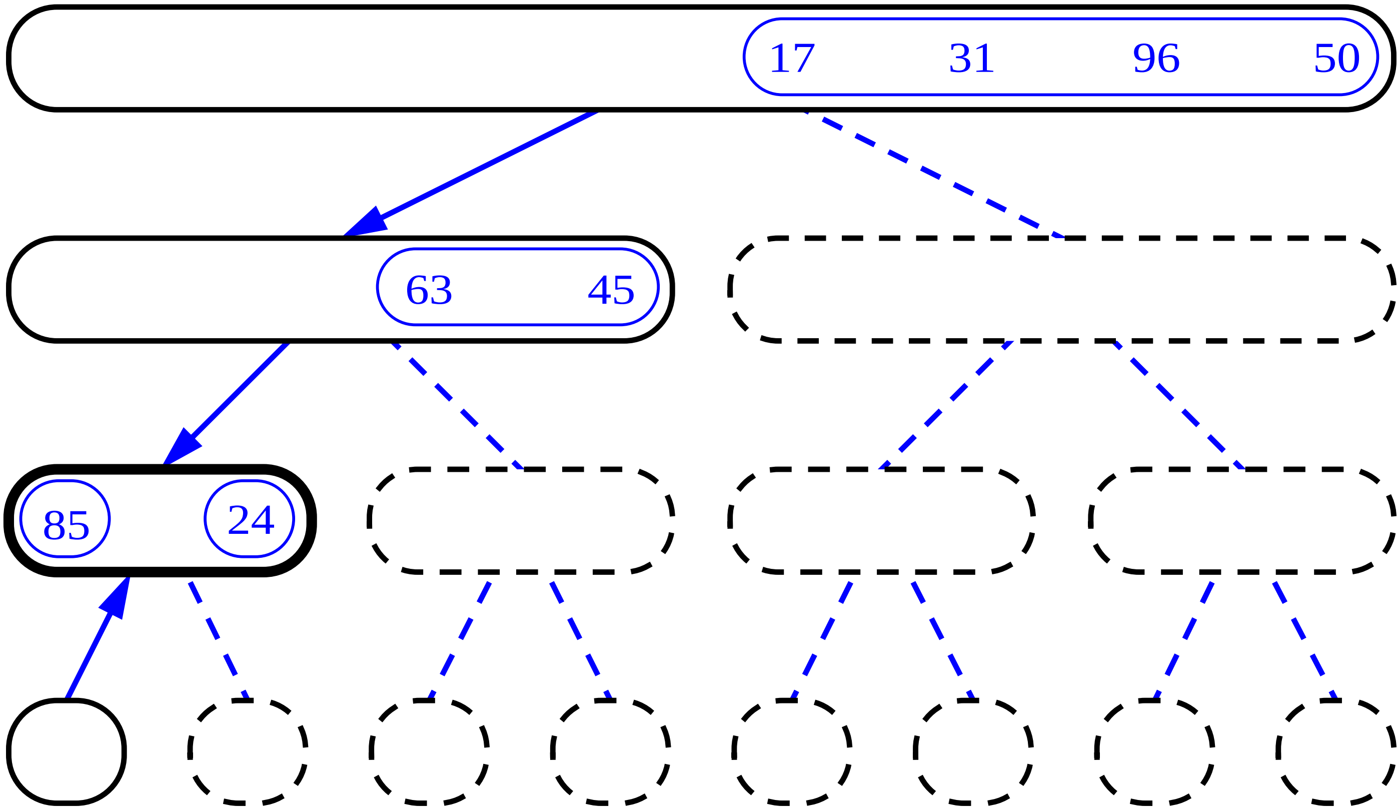


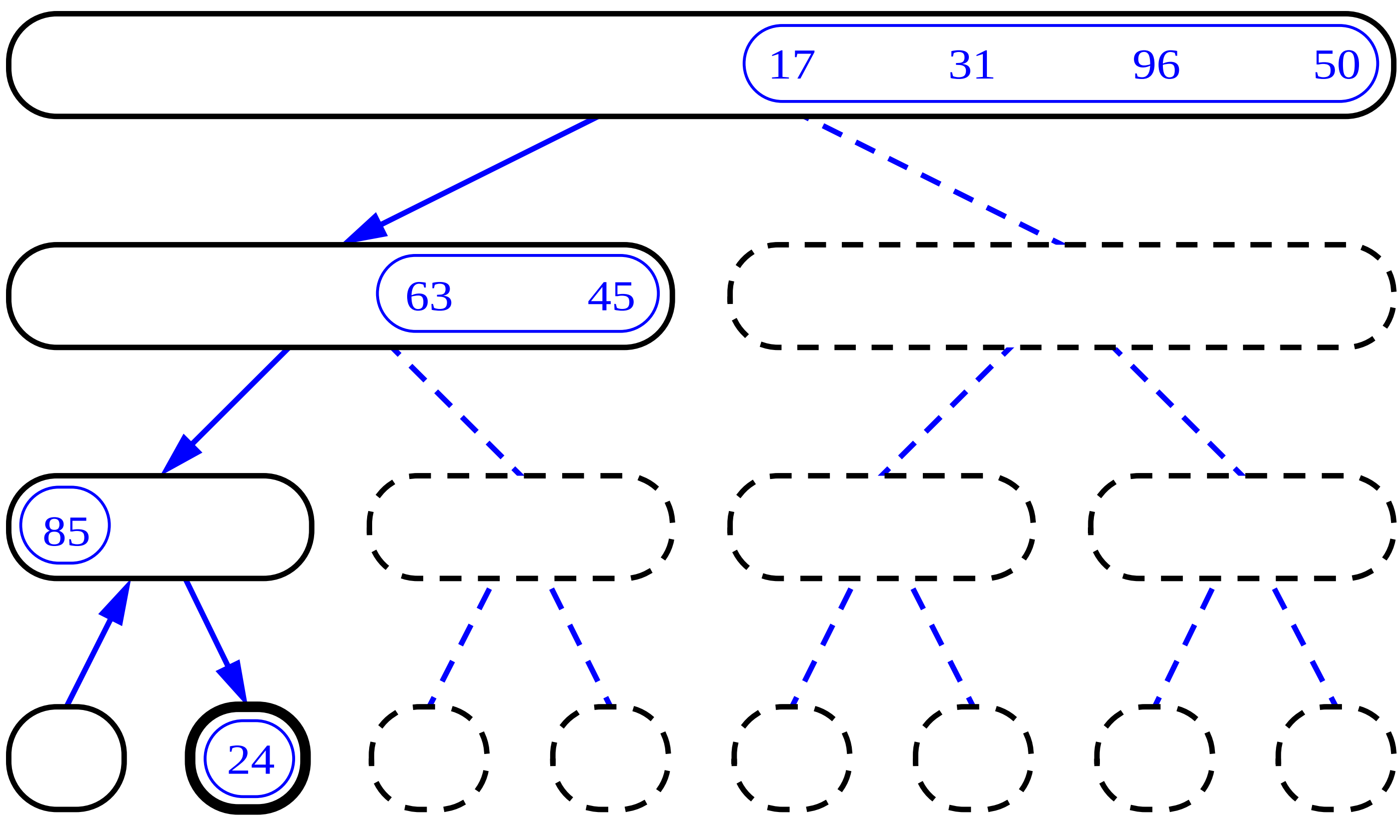
Merge-Sort(cont.)



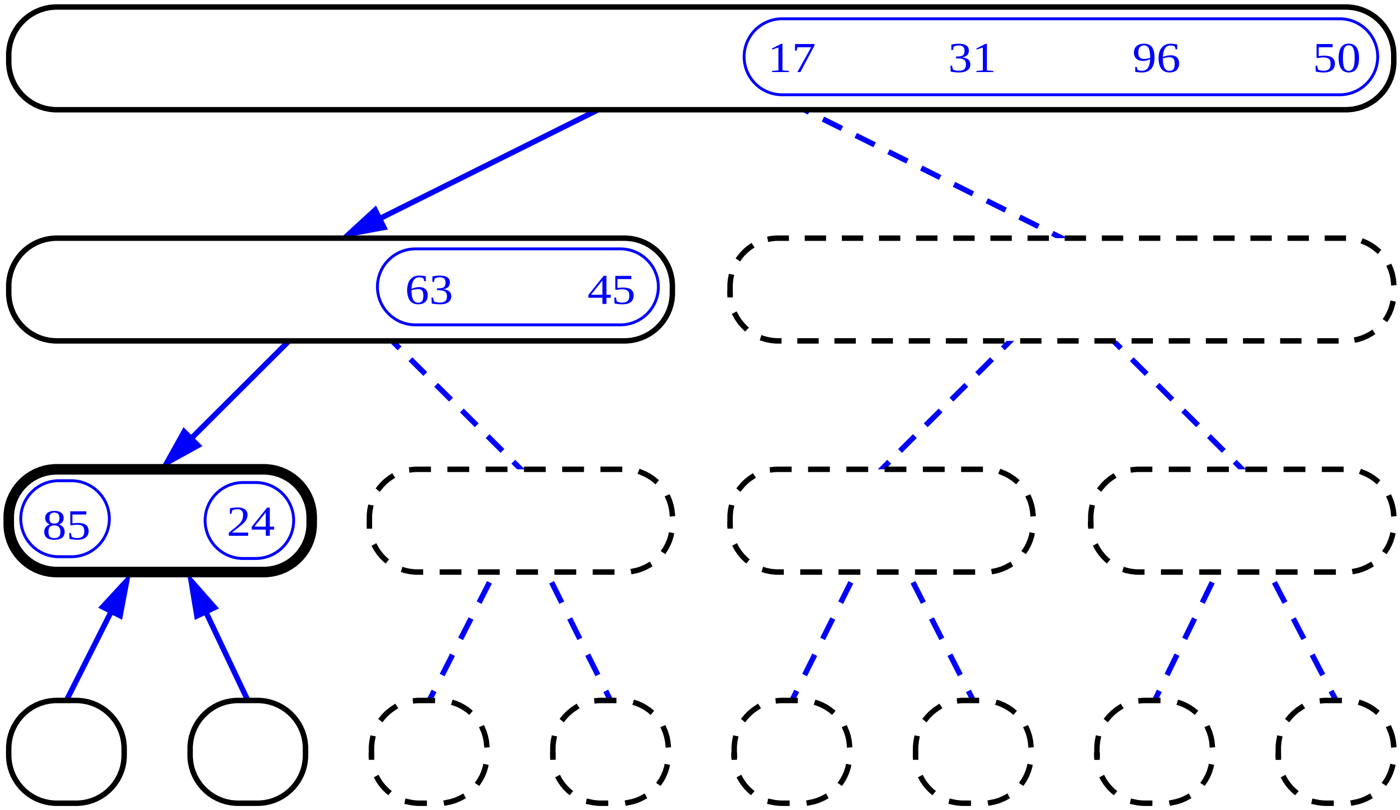


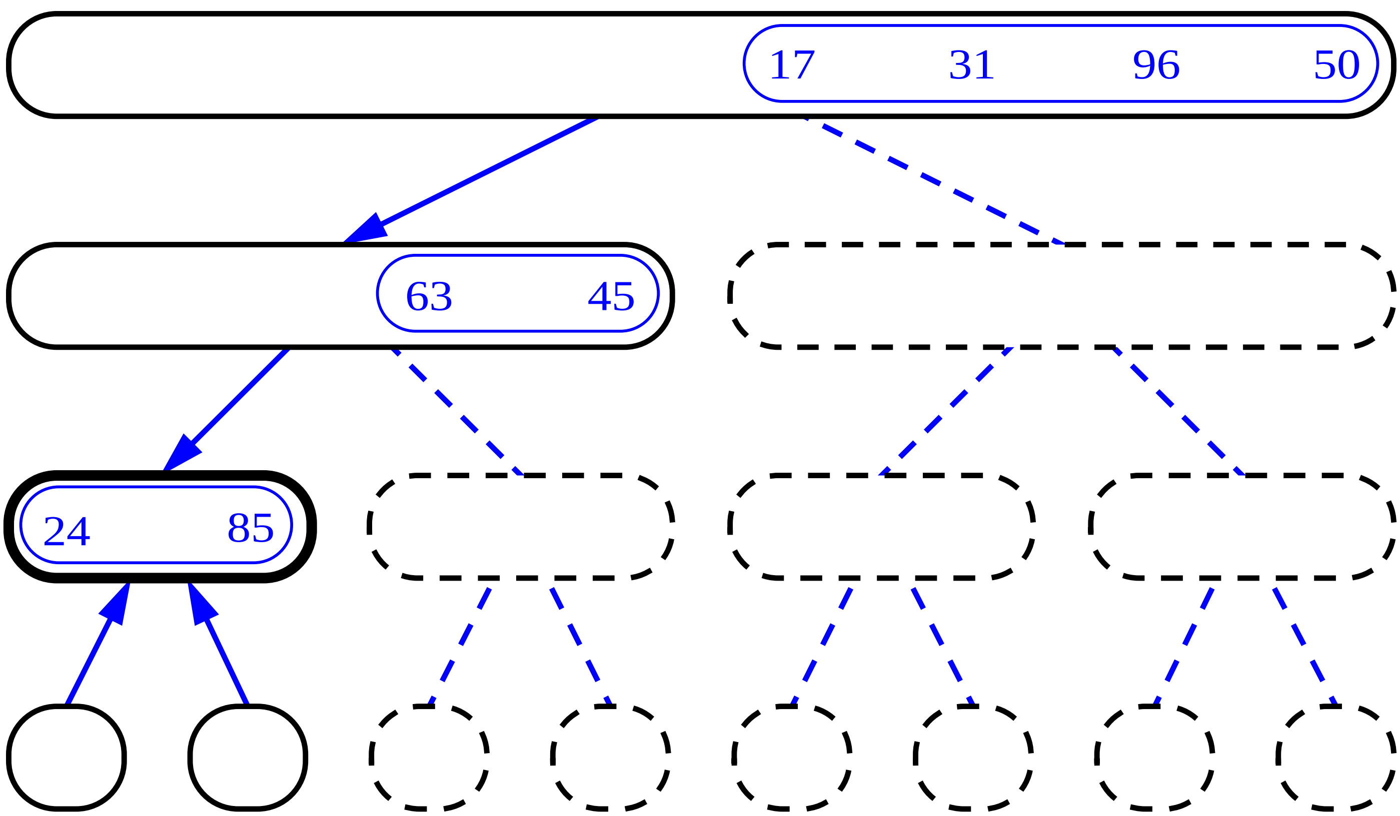
Merge-Sort (cont.)



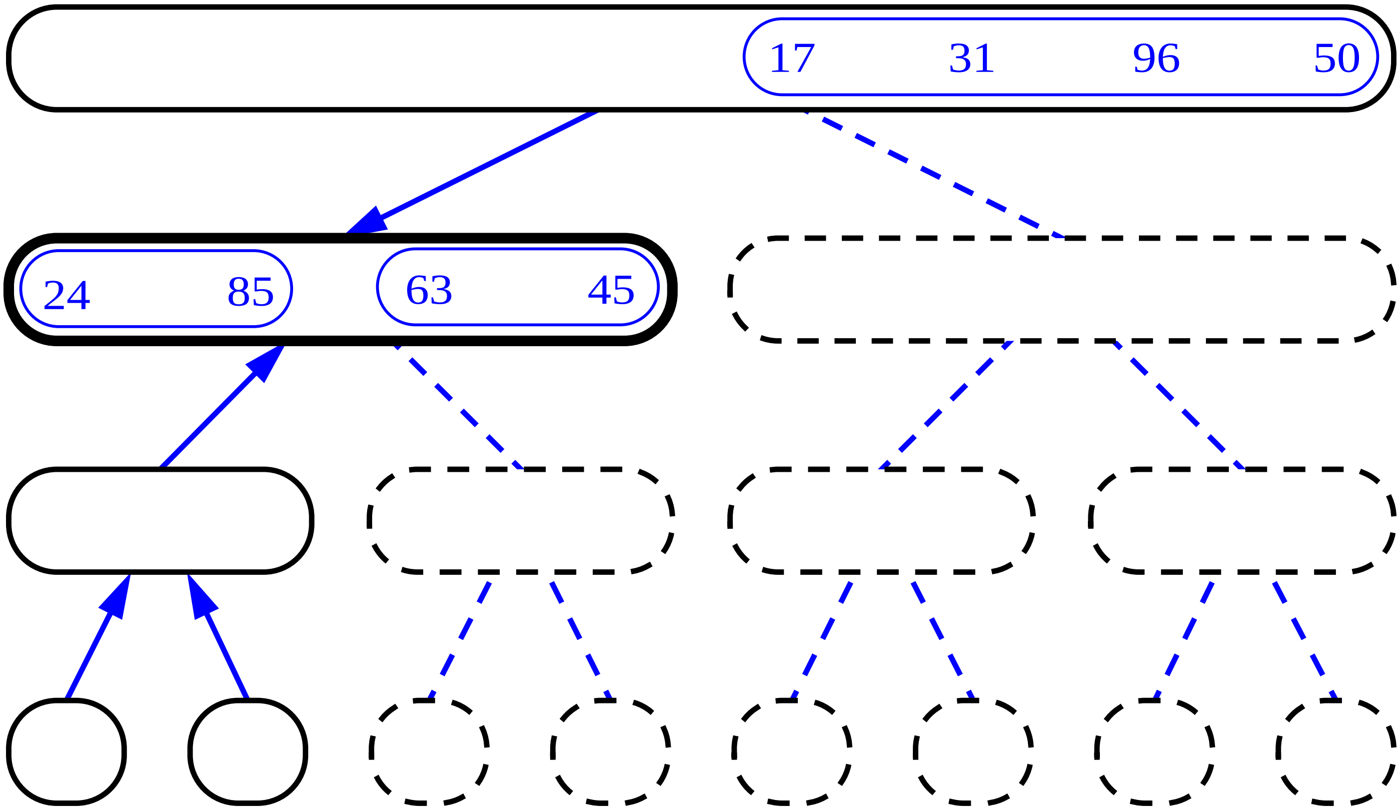


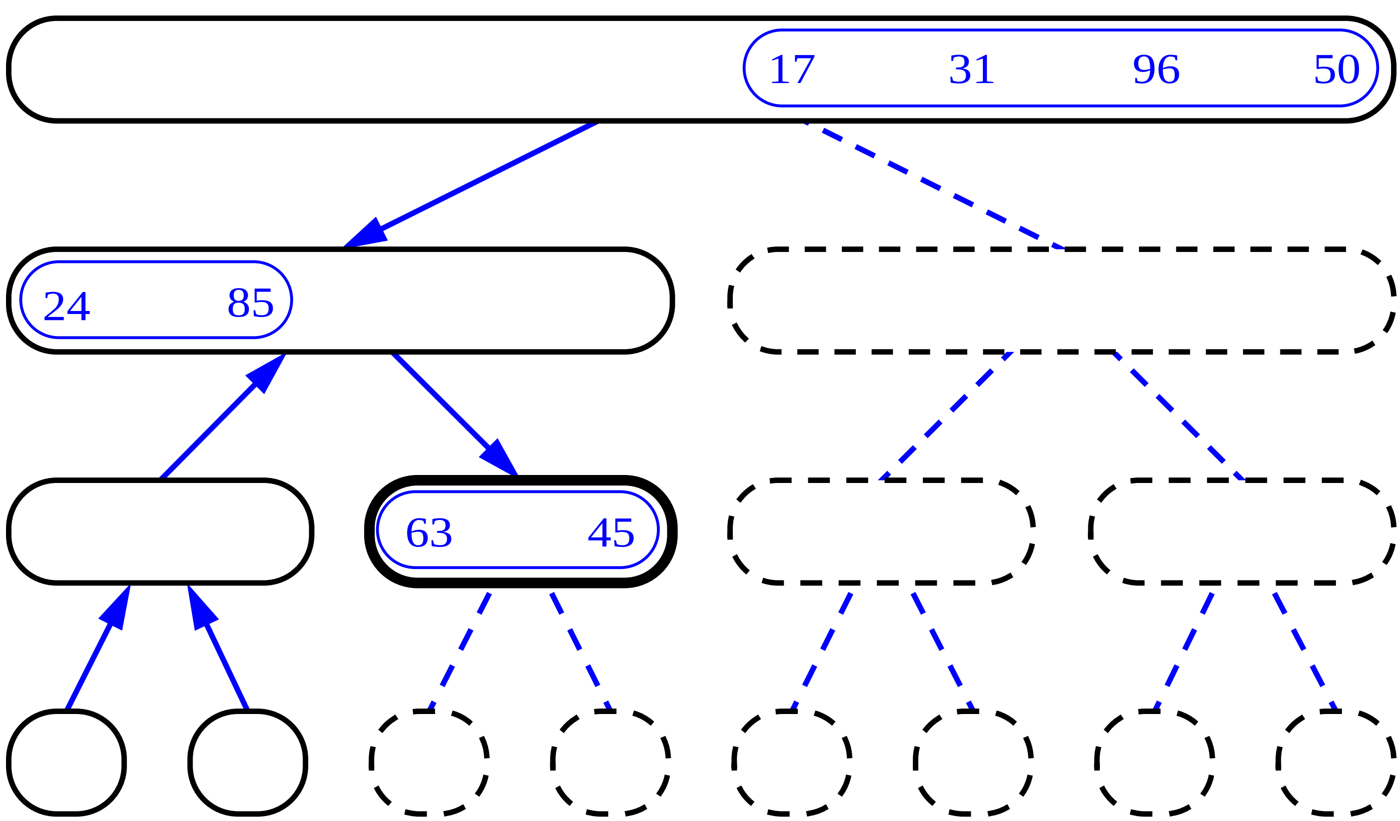
Merge-Sort (cont.)



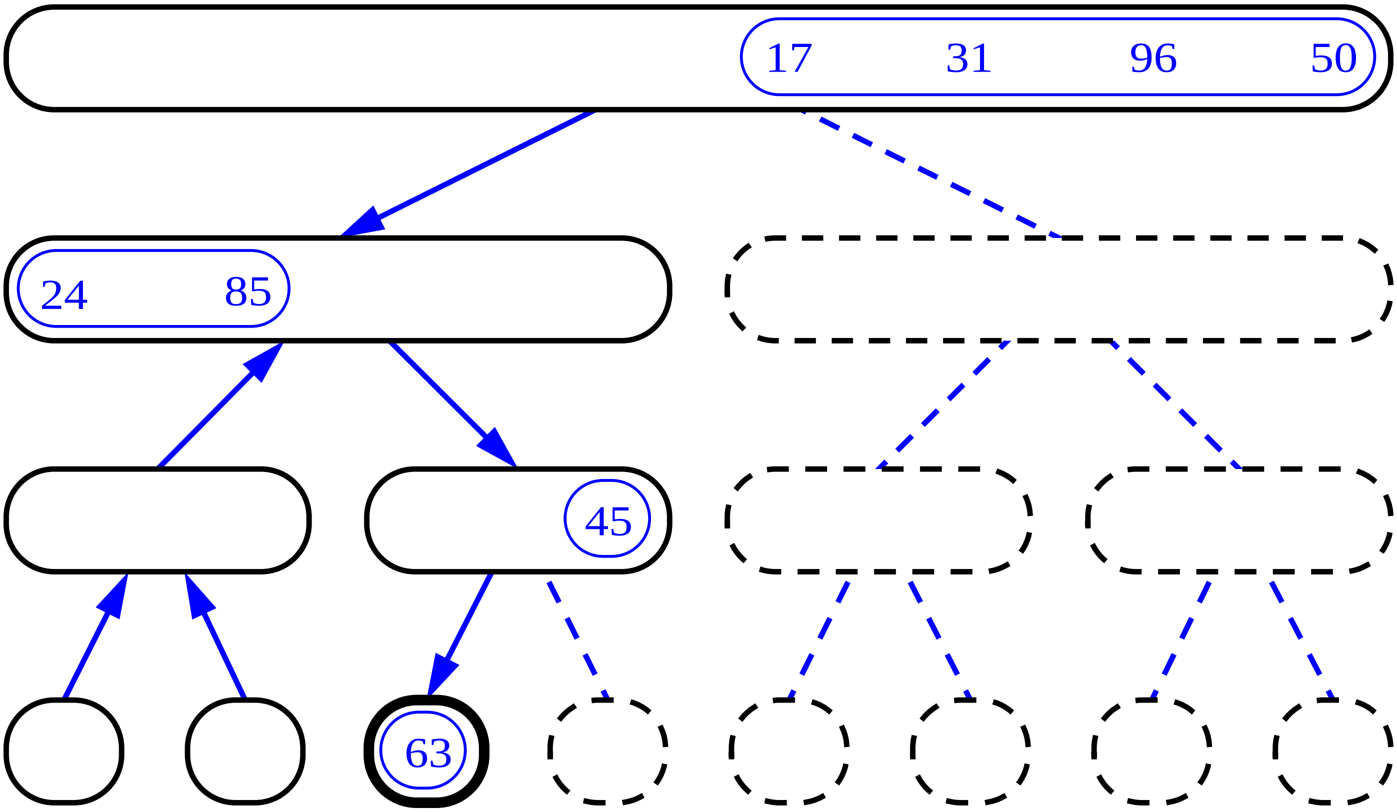


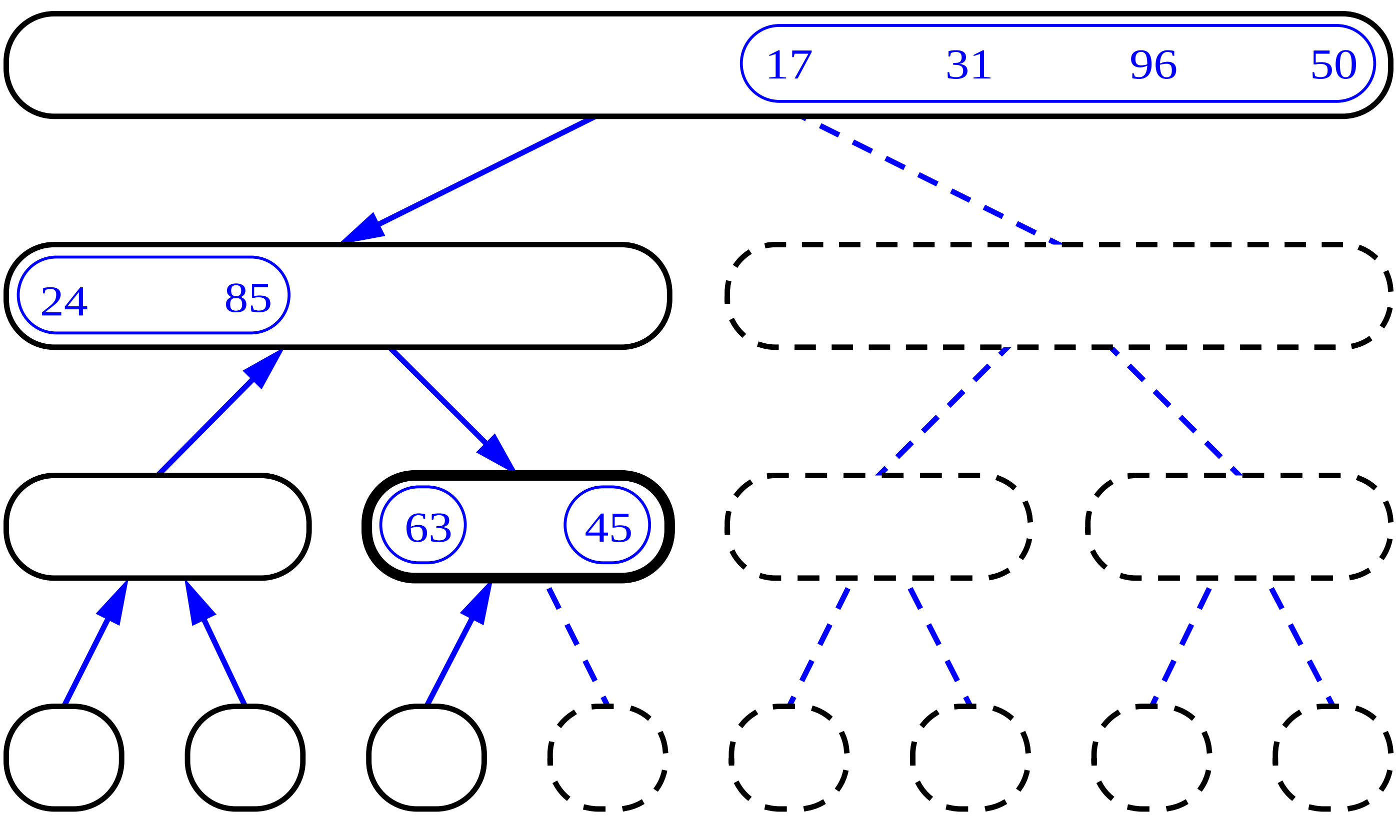
Merge-Sort (cont.)



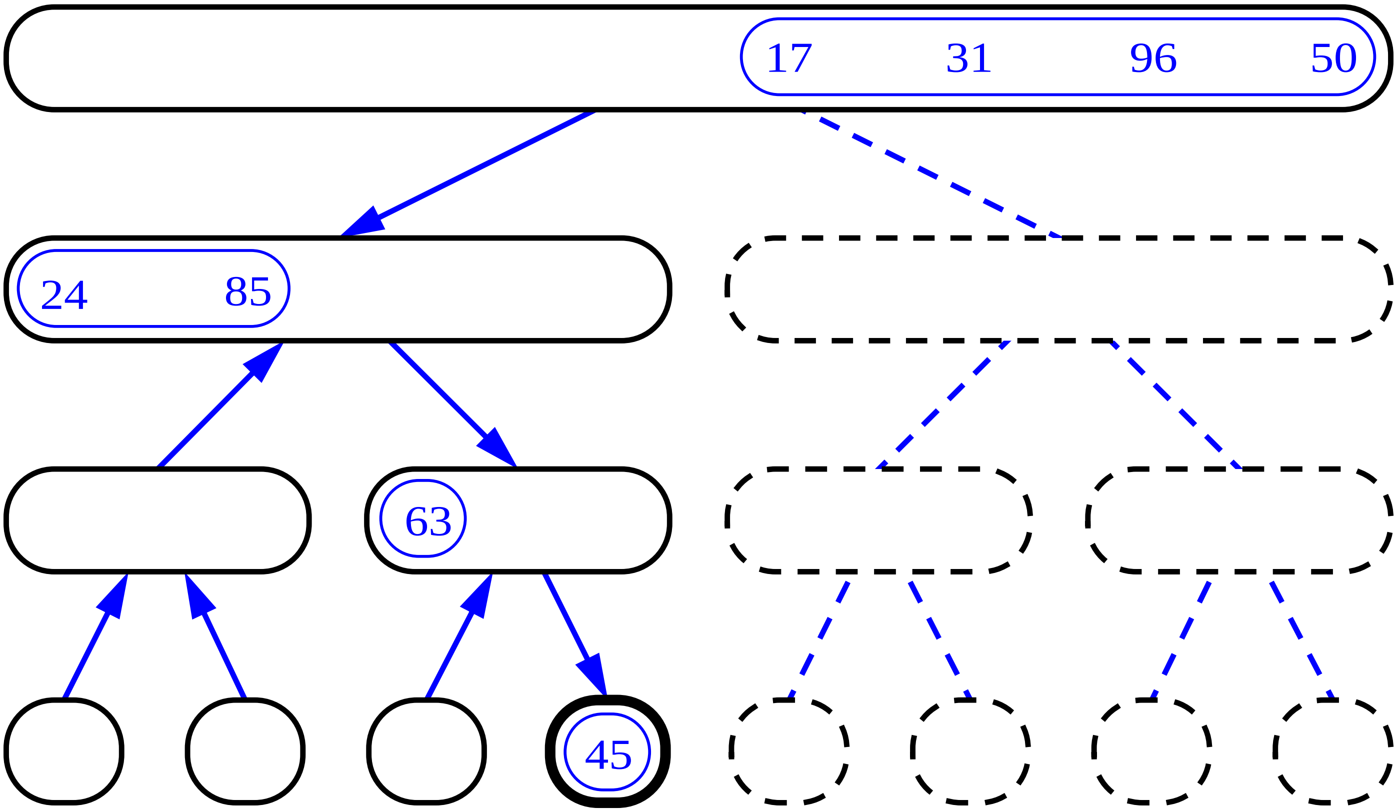


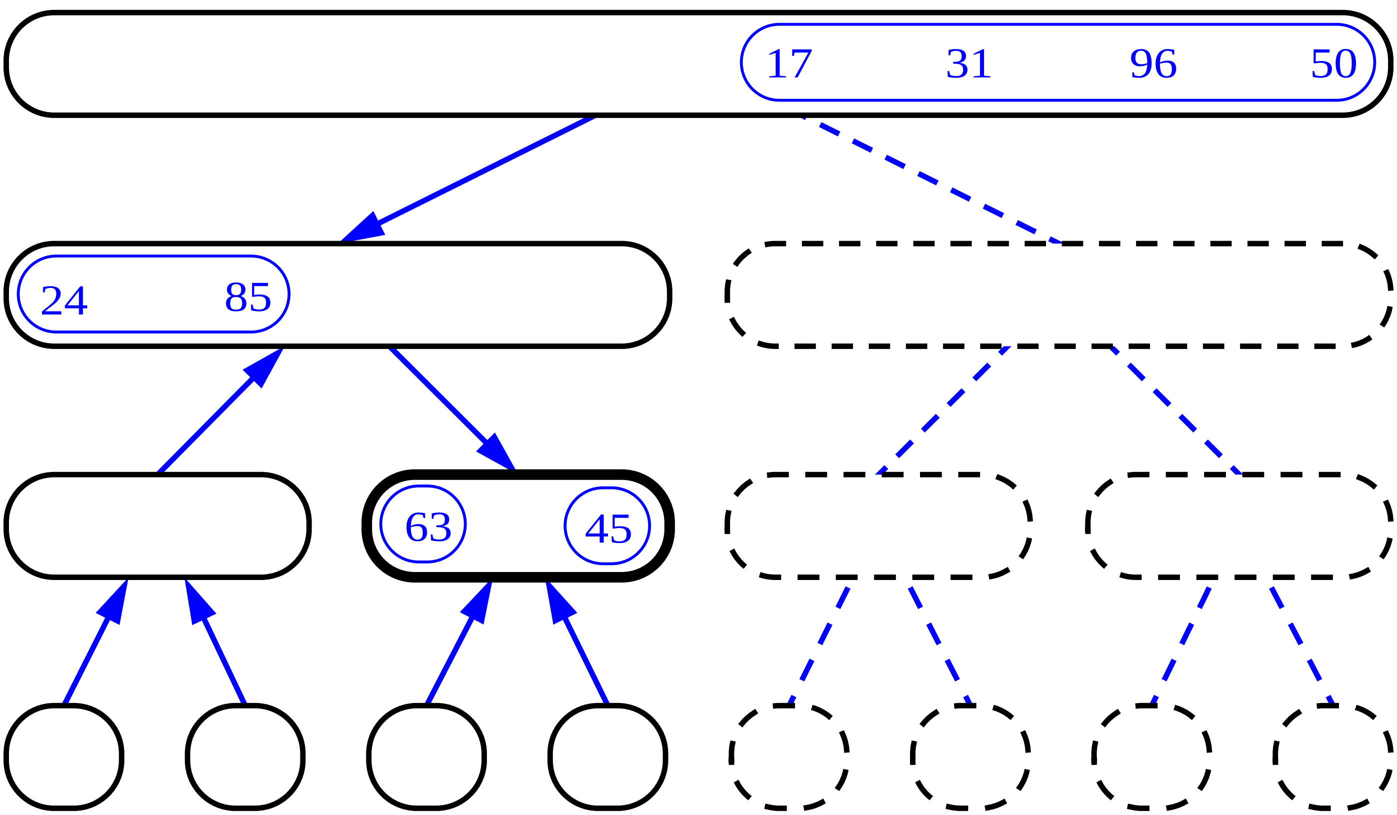
Merge-Sort (cont.)



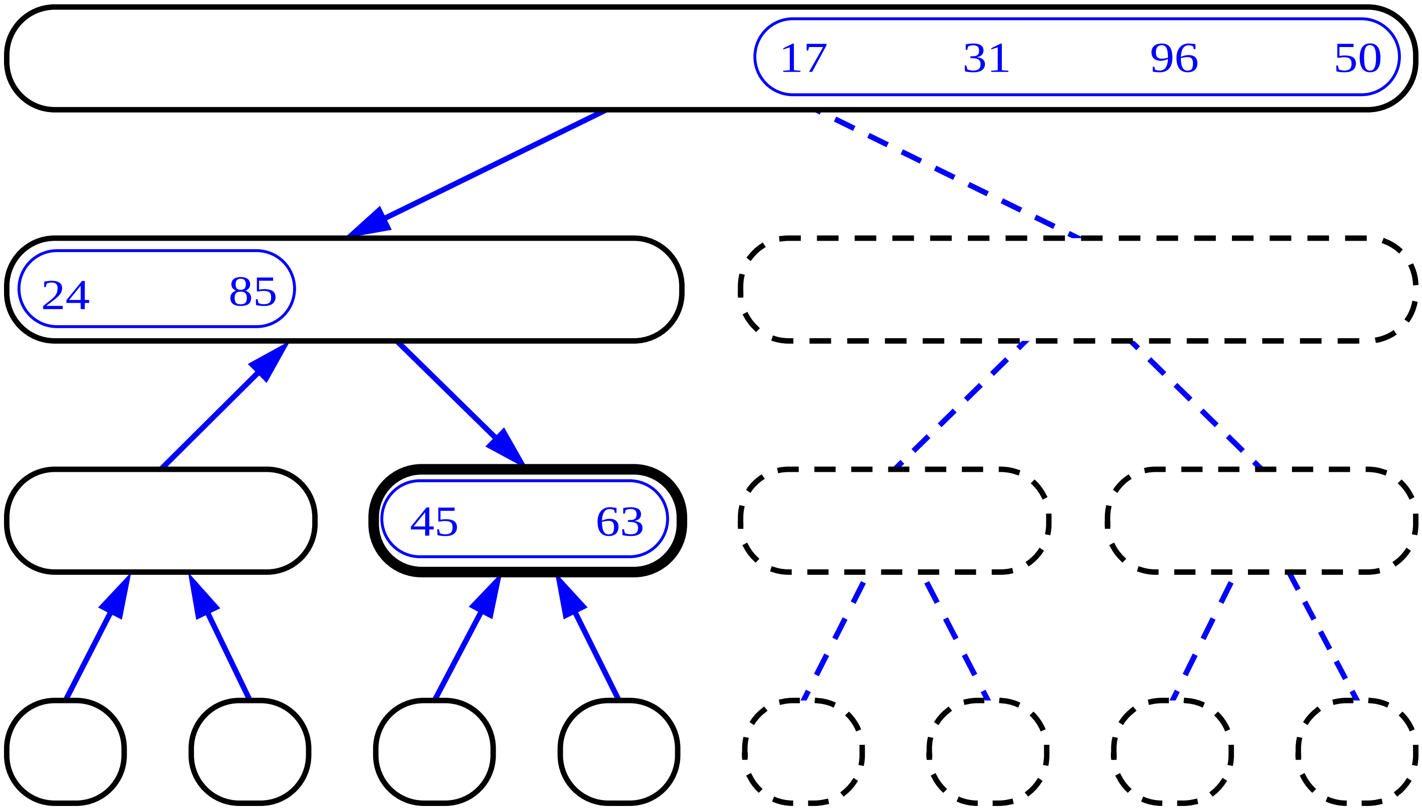


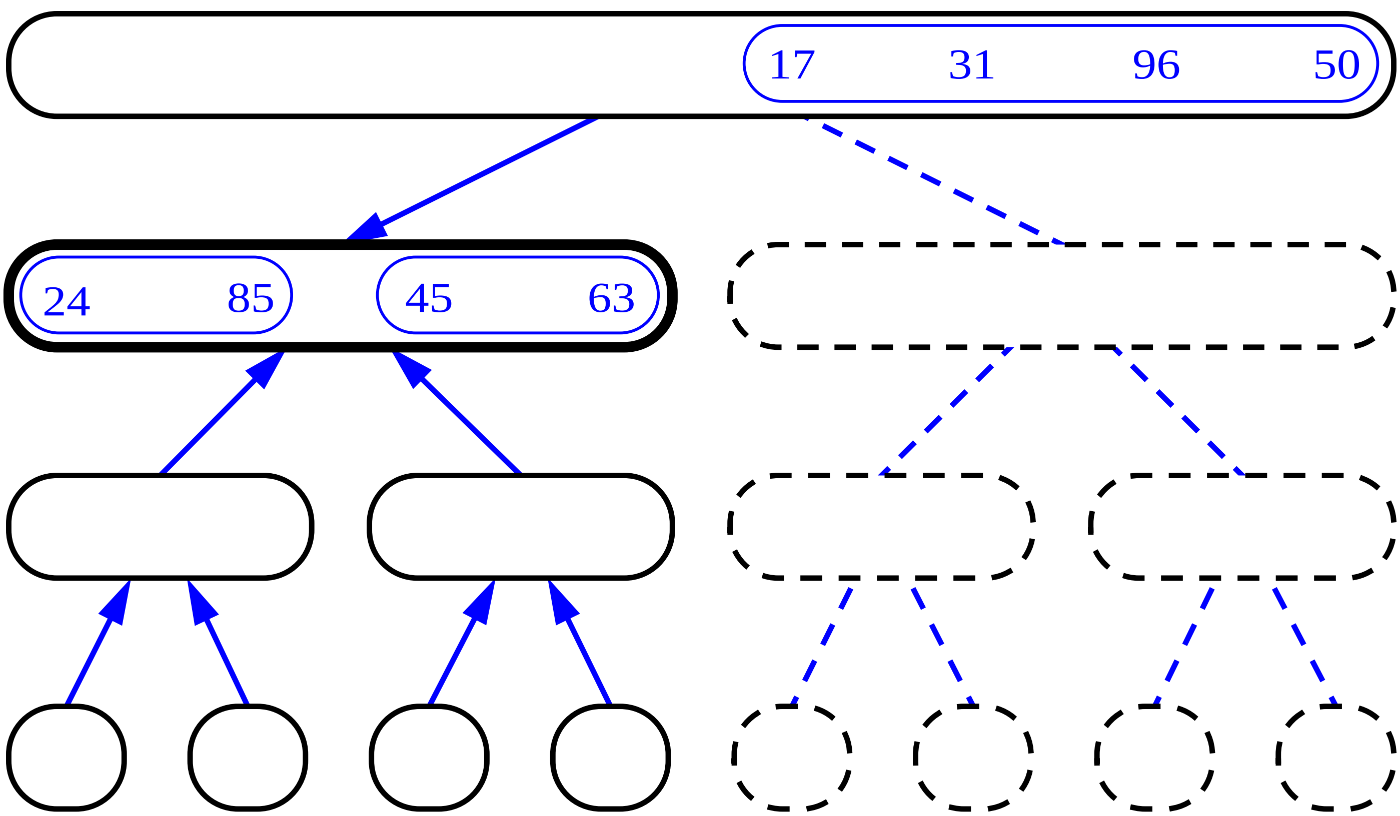
Merge-Sort (cont.)



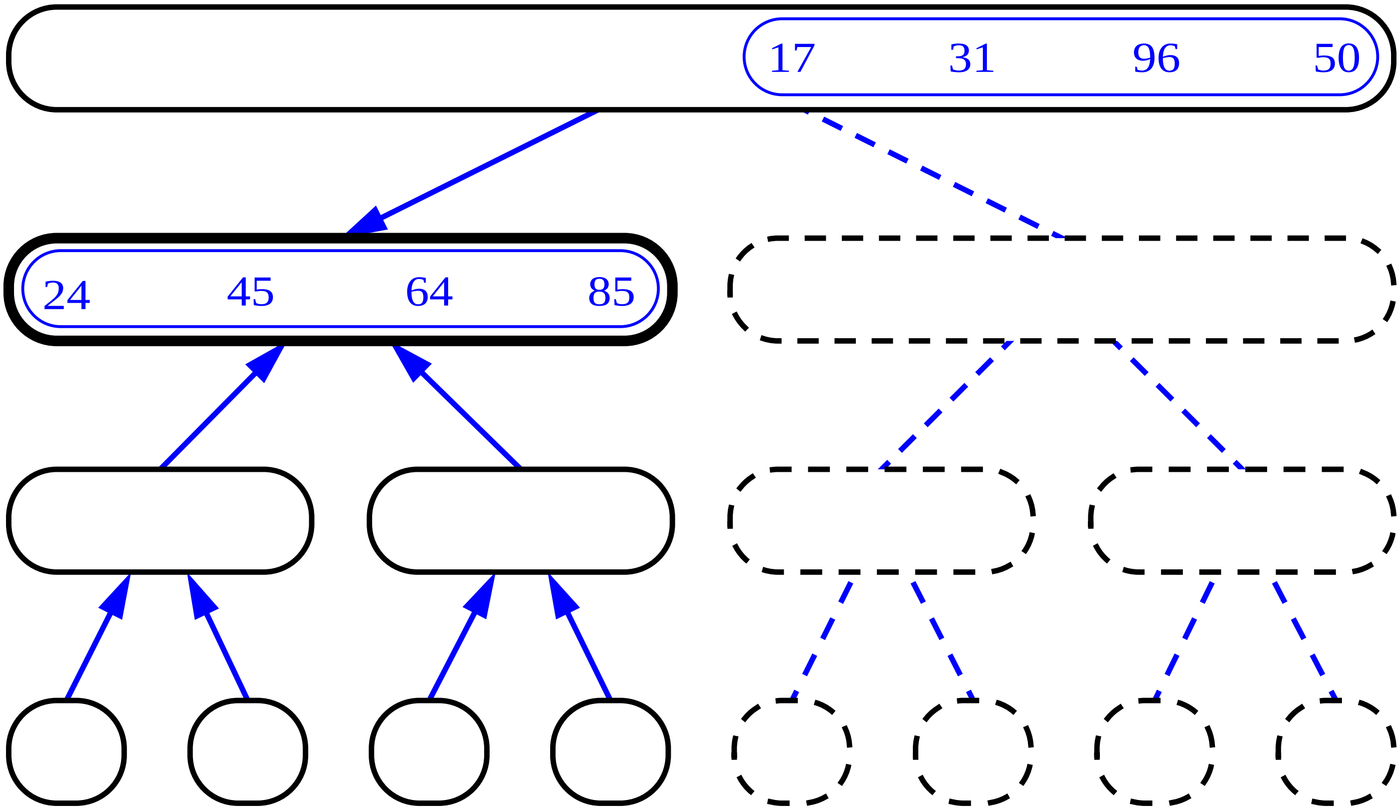


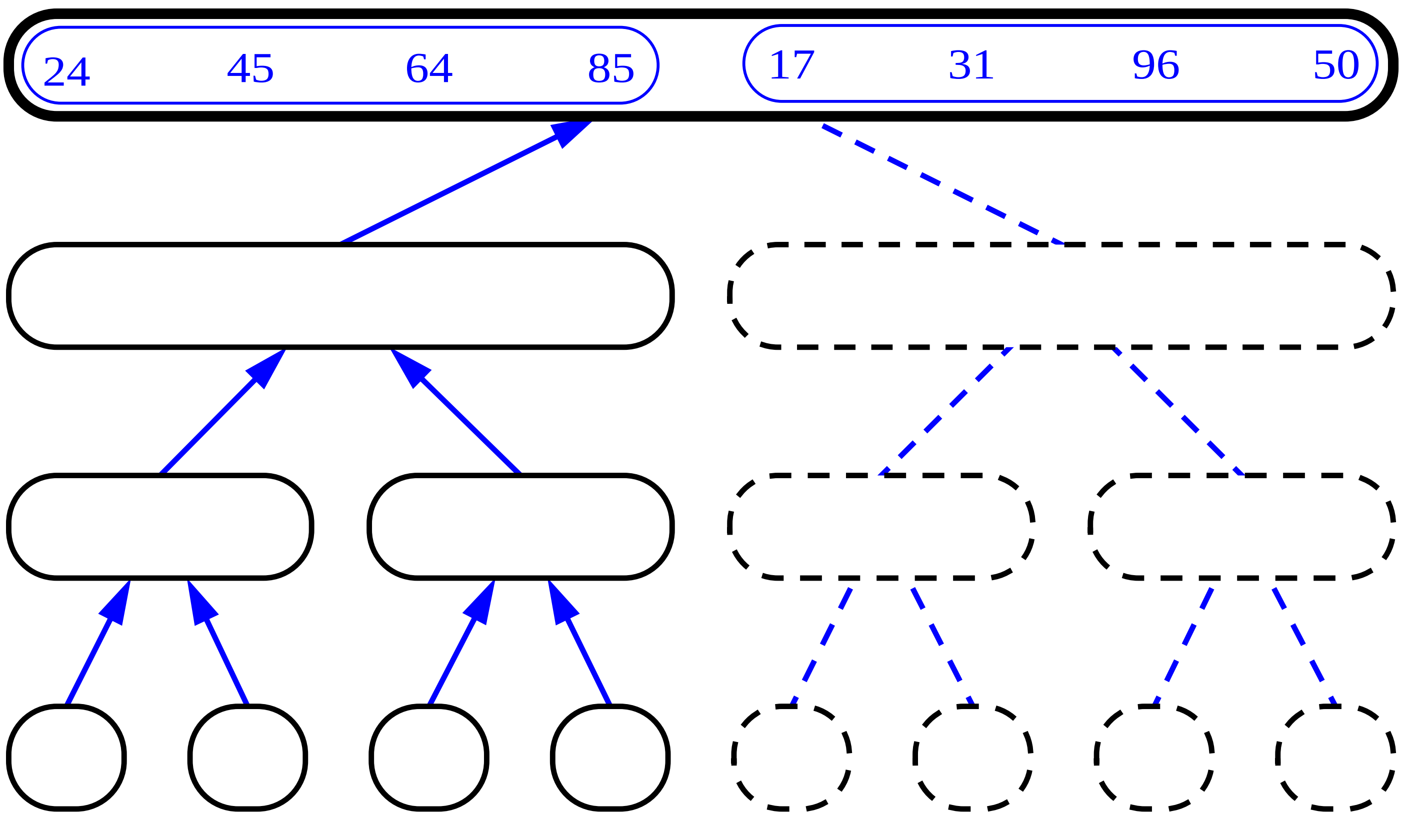
Merge-Sort(cont.)



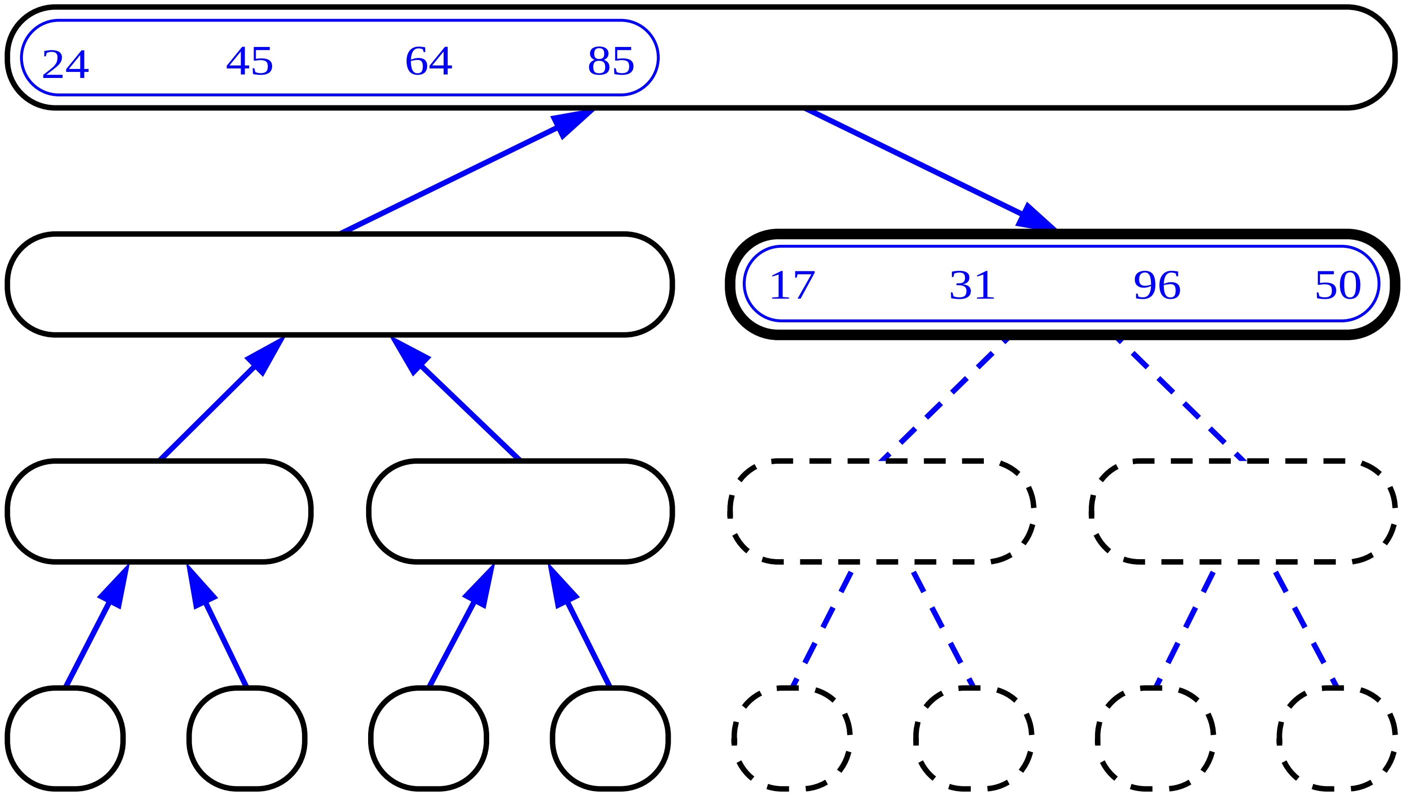


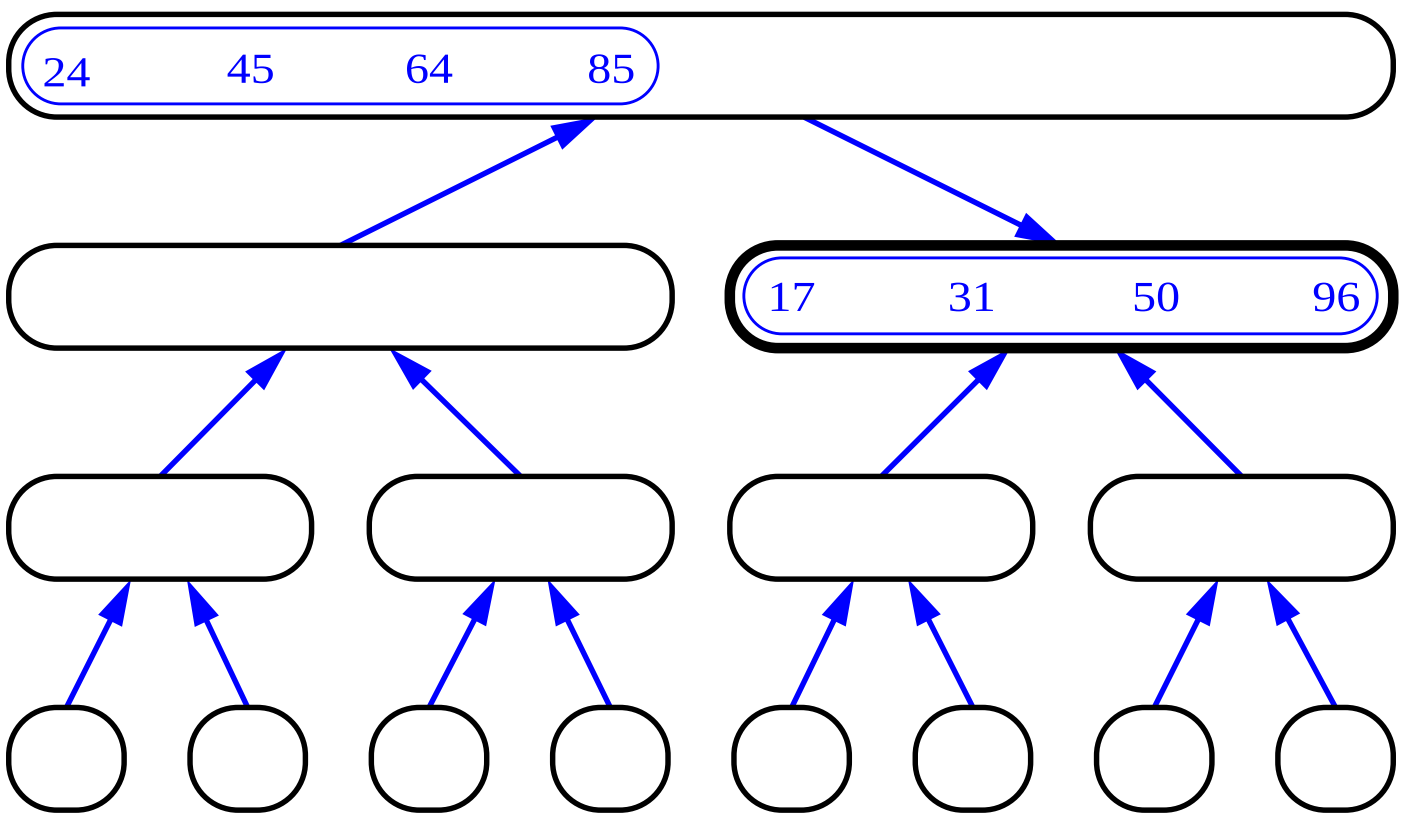
Merge-Sort (cont.)



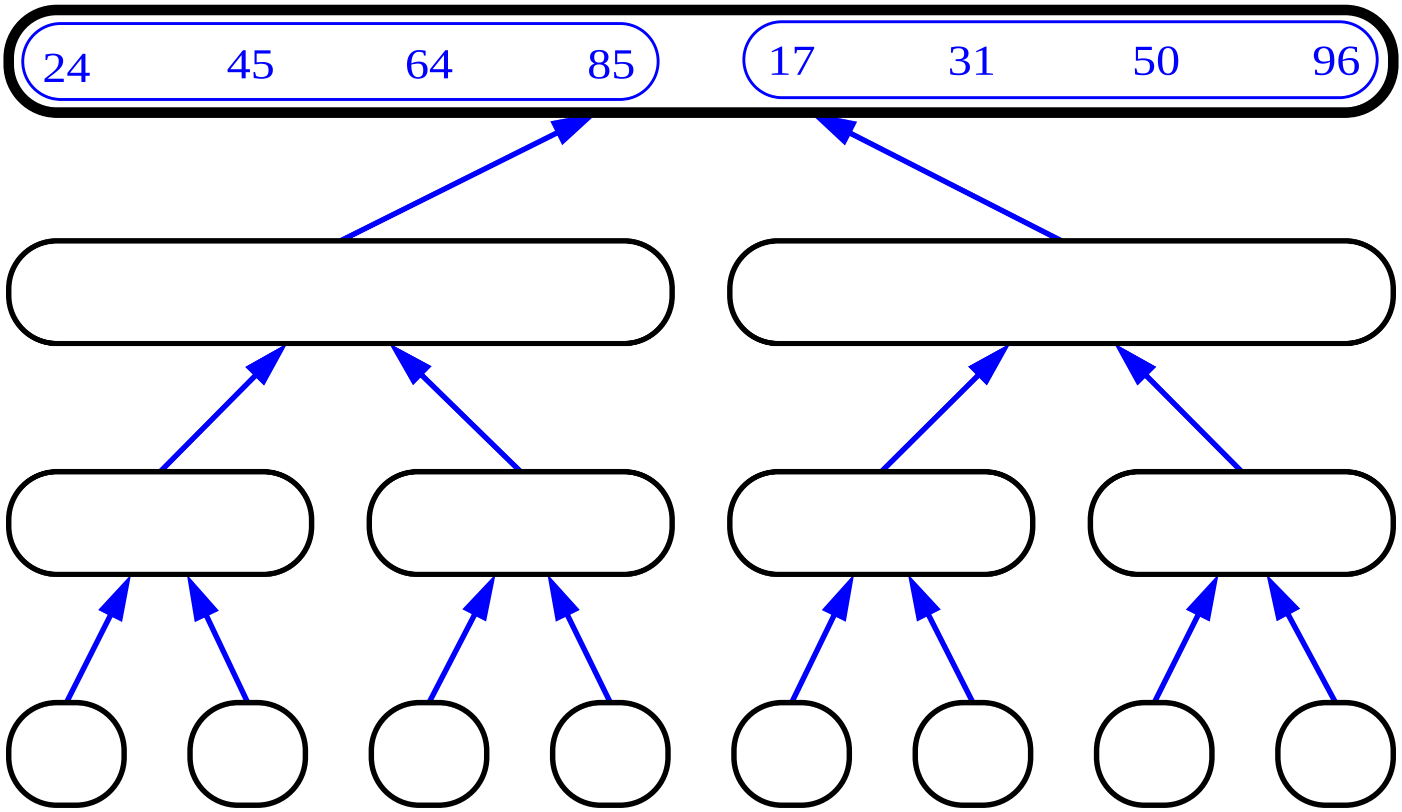


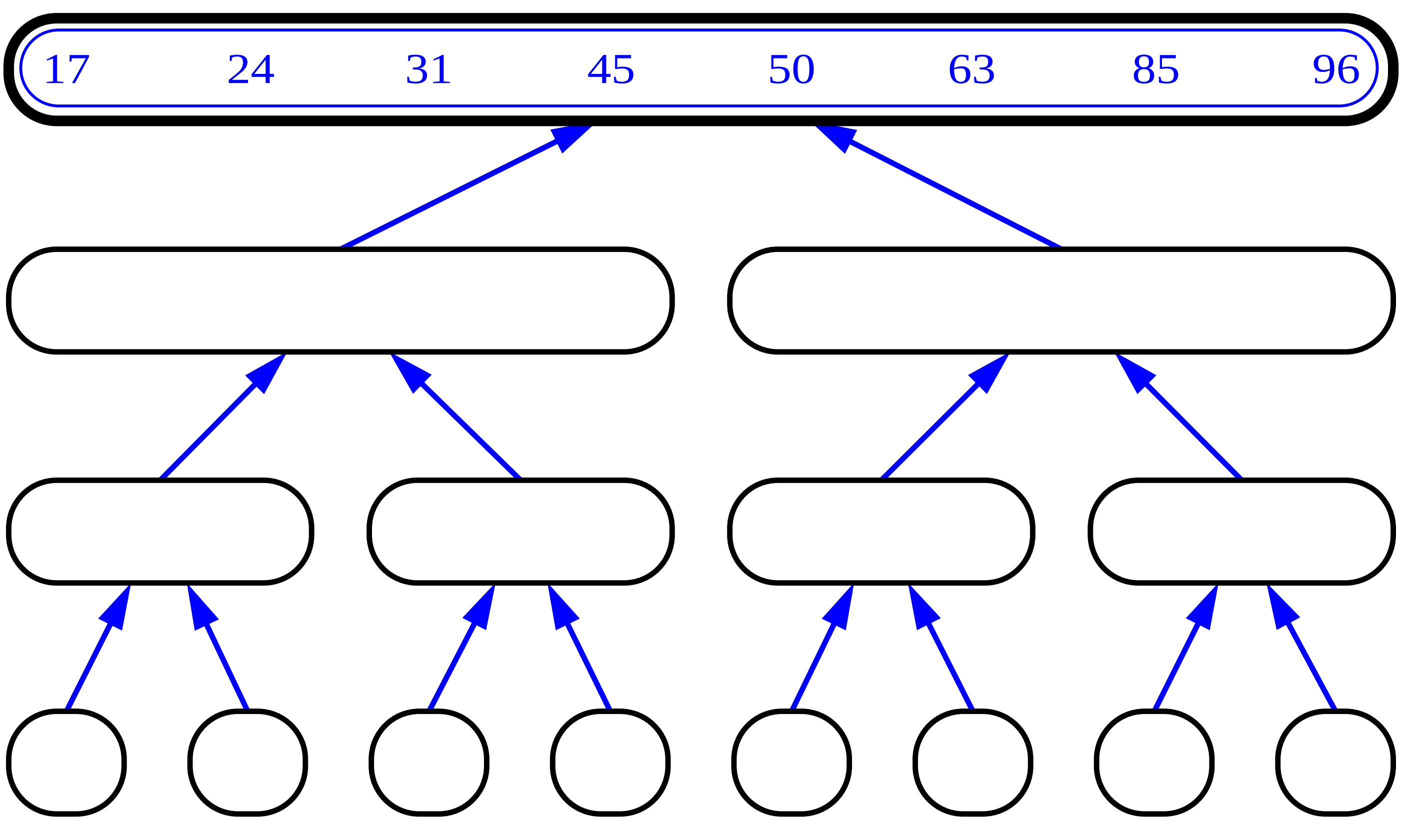
Merge-Sort (cont.)





Merge-Sort (cont.)





Merging Two Sequences

- Pseudo-code for merging two sorted sequences into a unique sorted sequence

  Algorithm merge (S1, S2, S):

  Input: Sequence *S1* and *S2* (on whose elements a total order relation is defined) sorted in nondecreas ing order, and an empty sequence *S*.

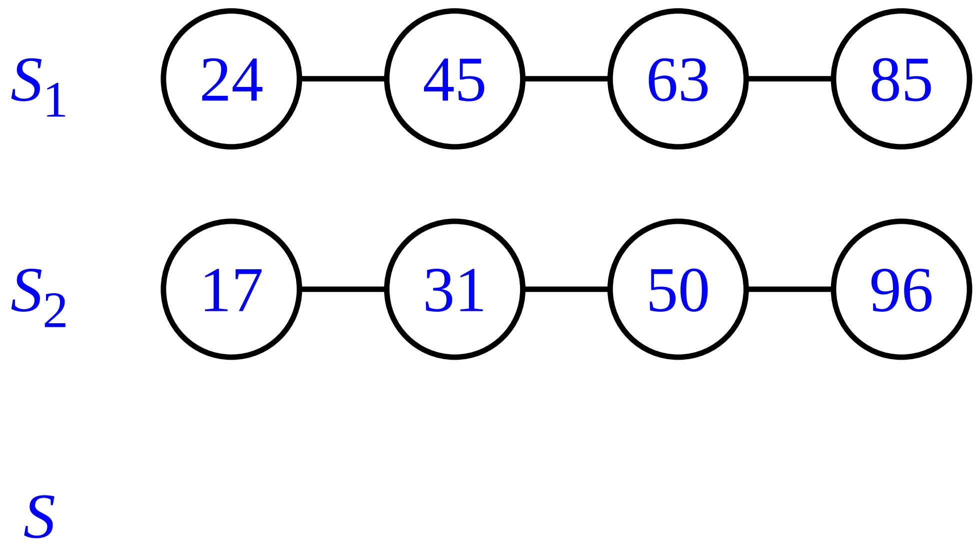
  Ouput: Sequence *S* containing the union of the ele ments from *S1* and *S2* sorted in nondecreasing order; sequence *S1* and *S2* become empty at the end of the execution

```
while S1 is not empty and S2 is not empty do
   if S1.first().element() ≤ S2.first().element() then
      {move the first element of S1 at the end of S}
      S.insertLast(S1.remove(S1.first()))
   else
      { move the first element of S2 at the end of S}
      S.insertLast(S2.remove(S2.first()))
while S1 is not empty do
   S.insertLast(S1.remove(S1.first()))
   {move the remaining elements of S2 to S}
while S2 is not empty do
   S.insertLast(S2.remove(S2.first()))
```

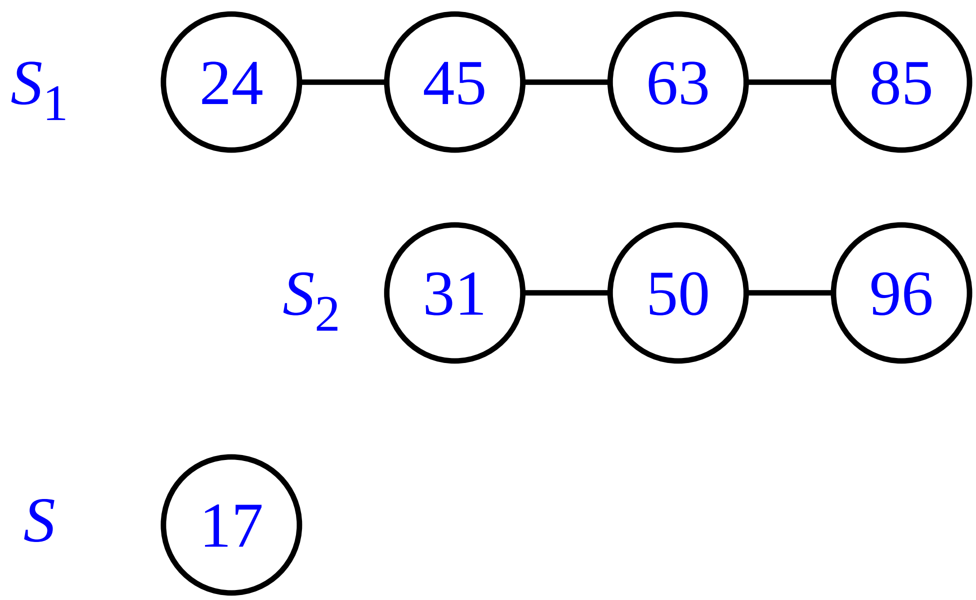
Merging Two Sequences (cont.)

- Some pictures:

a)

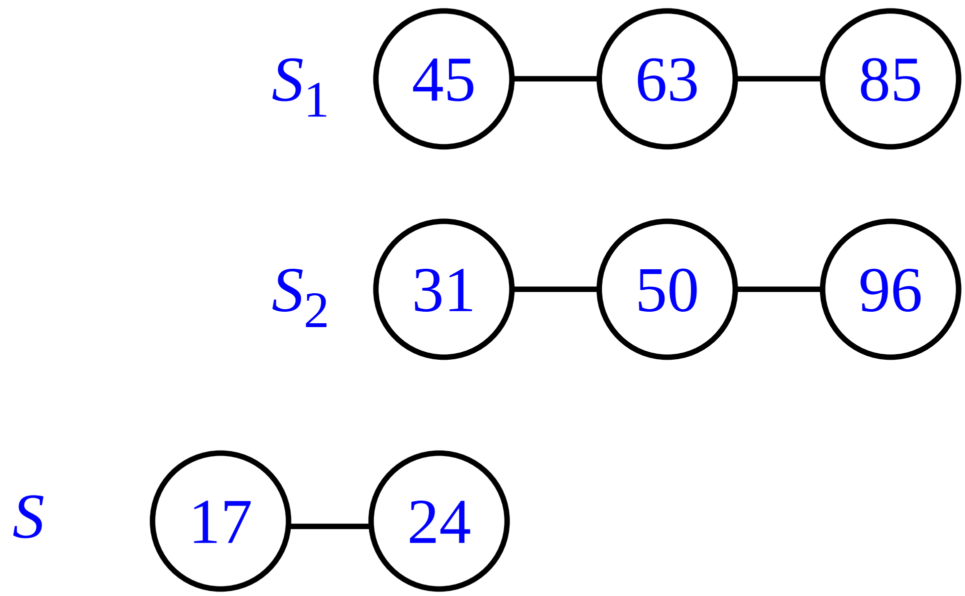


b)

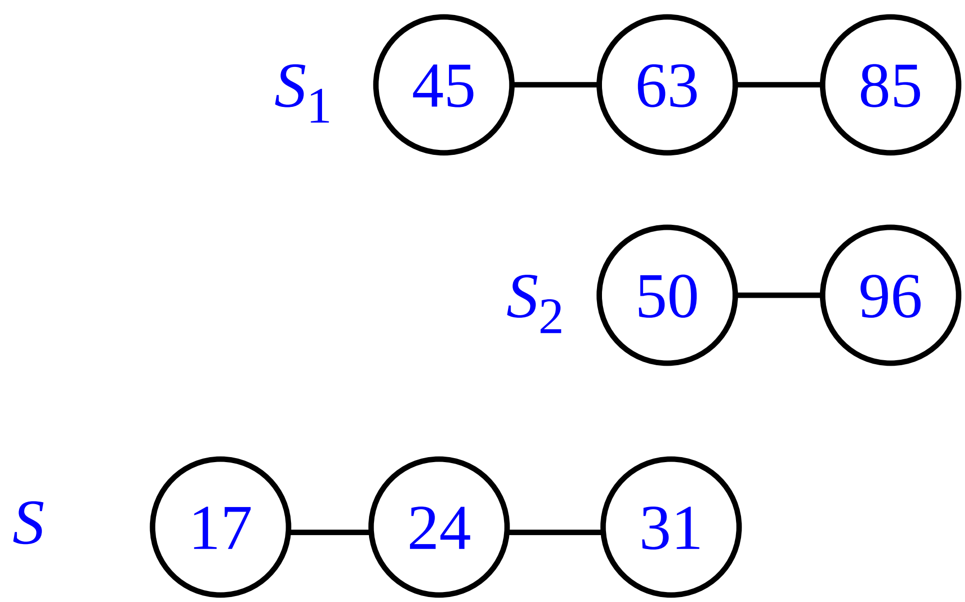


Merging Two Sequences (cont.)

c)

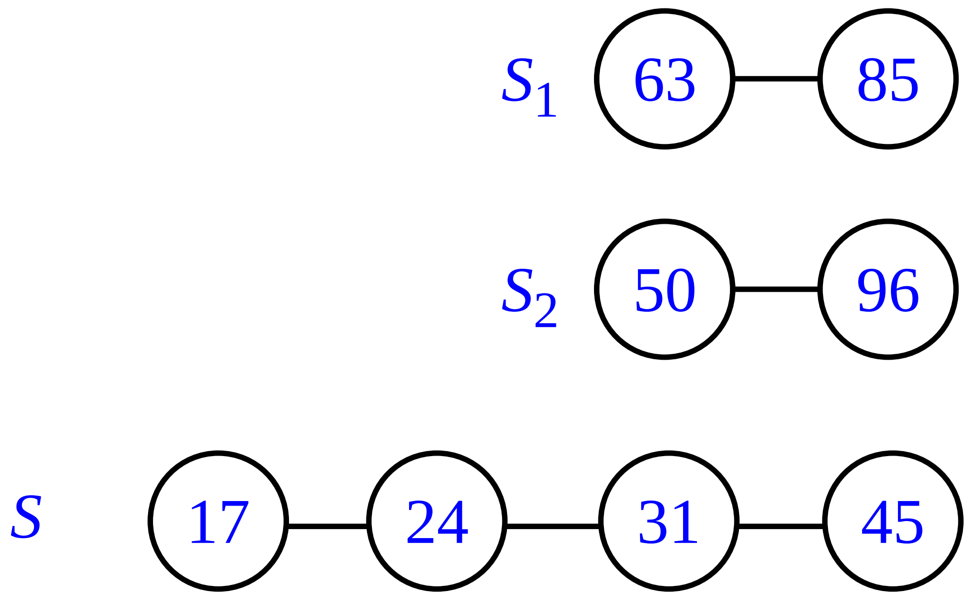


d)

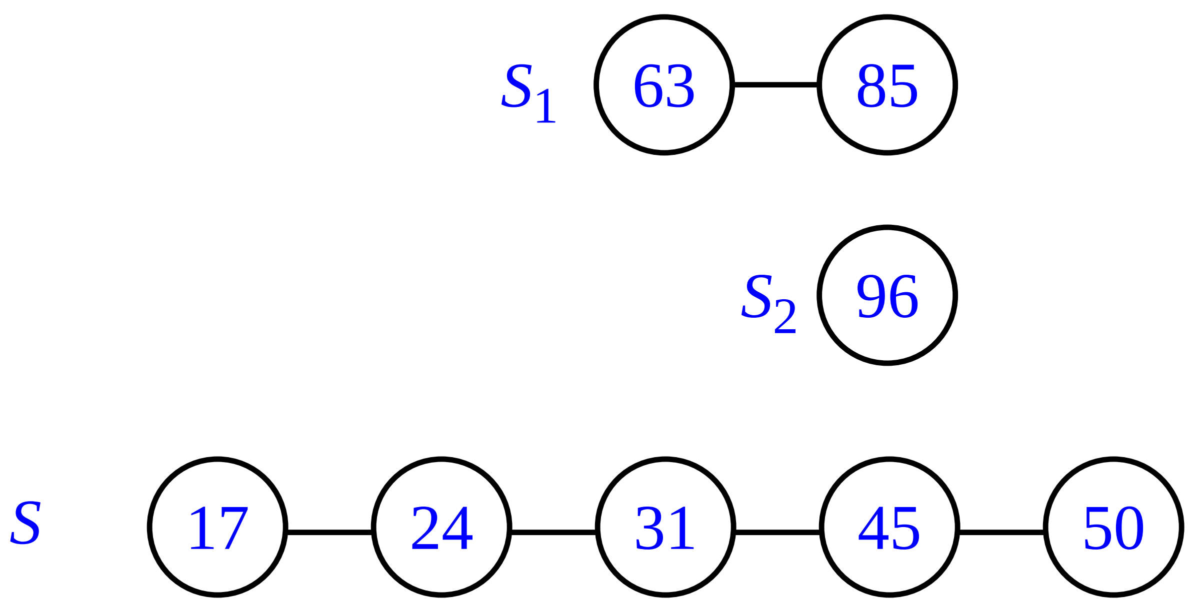


Merging Two Sequences (cont.)

e)

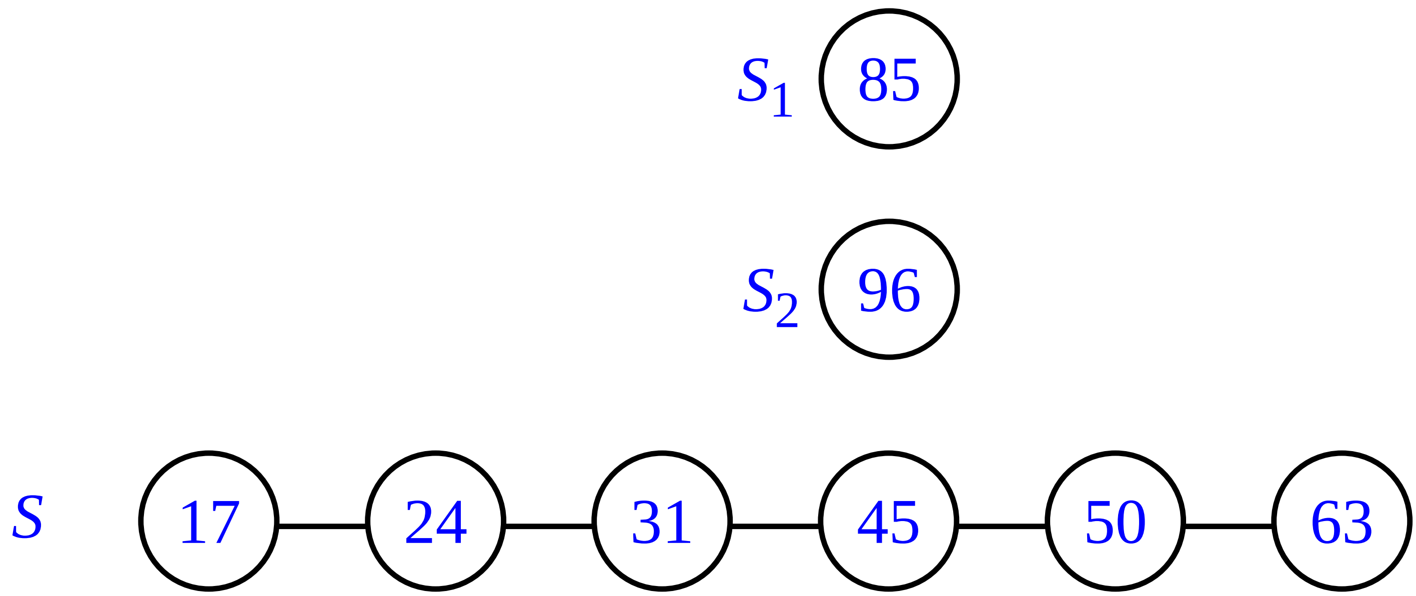


f)

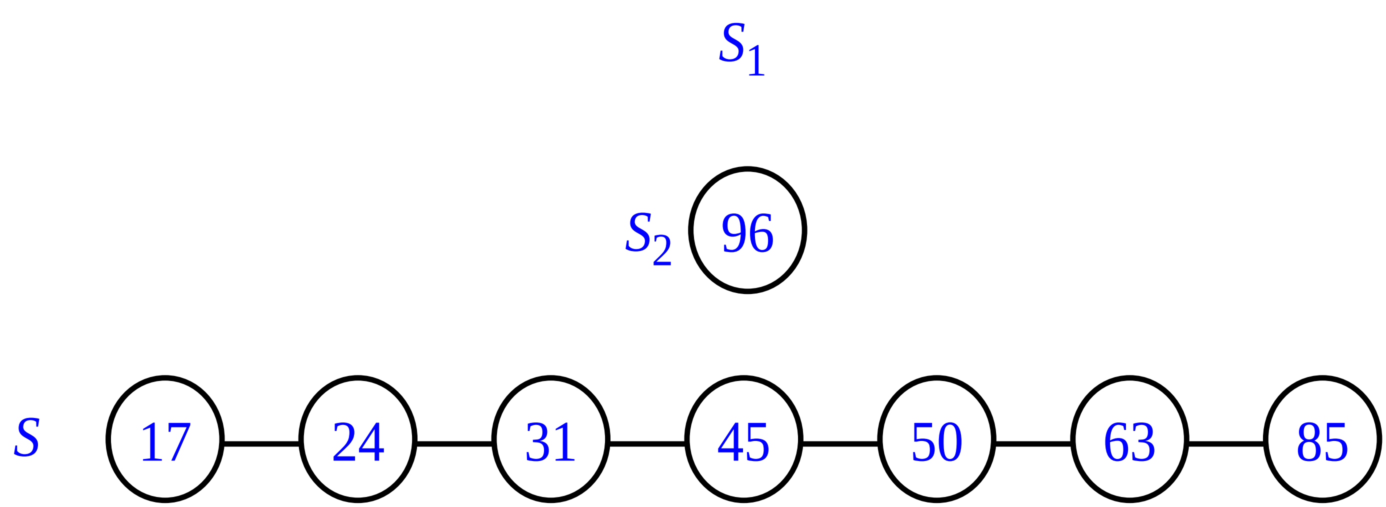


Merging Two Sequences (cont.)

g)

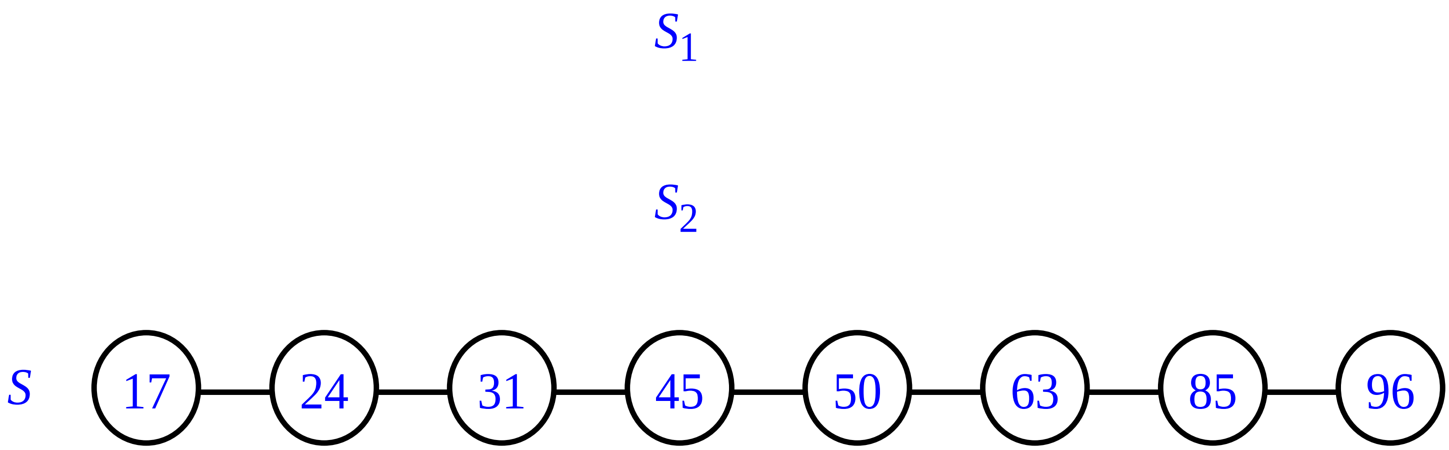


h)



Merging Two Sequences (cont.)

i)



Java Implementation

- Interface SortObject

```
public interface SortObject {
  //sort sequence S in nondecreasing order
  using compartor c
  public void sort (Sequence S, Comparator c);
}
```

Java Implementation (cont.)

```
public class ListMergeSort implements SortObject {
  public void sort(Sequence S, Comparator c) {
     int n = S.size();
    // a sequence with 0 or 1 element is
      already sorted
    if (n < 2) return;
    // divide
    Sequence S1 = (Sequence)S.newContainer();
    for (int i=1; i <= (n+1)/2; i++) {
     S1.insertLast(S.remove(S.first()));
   }
   Sequence S2 = (Sequence)S.newContainer();
   for (int i=1; i <= n/2; i++) {
    S2.insertLast(S.remove(S.first()));
}
    // recur
   sort(S1,c);
   sort(S2,c);
   //conquer
   merge(S1,S2,c,S);
  }
```

Java Implementation (cont.)

```
public void merge(Sequence S1, Sequence S2,
        Comparator c, Sequence S) {
while(!S1.isEmpty() && !S2.isEmpty()) {
    if(c.isLessThanOrEqualTo(S1.first().element(),
        S2.first().element())) {
        S.insertLast(S1.remove(S1.first()));
     }
    else
        S.insertLast(S2.remove(S2.first()));
}
     if(S1.isEmpty()) {
       while(!S2.isEmpty()) {
    S.insertLast(S2.remove(S2.first()));
       }
}
     if(S2.isEmpty()) {
       while(!S1.isEmpty()) {
    S.insertLast(S1.remove(S1.first()));
       }
     }
   }
```

Running Time of Merge-Sort

- **Proposition 1**: The merge-sort tree associated with the execution of a merge-sort on a sequence of n elements has a height of $\lceil \log n \rceil$
- **Proposition 2**: A merge sort algorithm sorts a sequence of size n in $\boldsymbol{O}(n\log n)$ time
- We assume only that the input sequence S and each of the sub-sequences created by each recursive call of the algorithm can access, insert to, and delete from the first and last nodes in $\boldsymbol{O}(1)$ time.
- We call the time spent at node v of merge-sort tree T the running time of the recusive call associated with v, excluding the recursive calls sent to v's children.
- If we let i represent the depth of node v in the merge-sort tree, the time spent at node v is $\boldsymbol{O}(n/2^i)$ since the size of the sequence associated with v is $n/2^i$.
- Observe that T has exactly 2^i nodes at depth i. The total time spent at depth i in the tree is then $\boldsymbol{O}(2^i n/2^i)$, which is $\boldsymbol{O}(n)$. We know the tree has height $\lceil \log n \rceil$
- Therefore, the time complexity is $\boldsymbol{O}(n\log n)$

Set ADT

- A Set is a data structure modeled after the mathematical notation of a set. The fundamaental set operations are *union*, *intersection*, and *subtraction*.
- A brief aside on mathemeatical set notation:
  - $A \cup B = \{ x: x \in A \text{ or } x \in B \}$
  - $A \cap B = \{ x: x \in A \text{ and } x \in B \}$
  - $A - B = \{ x: x \in A \text{ and } x \notin B \}$
- The specific methods for a Set A include the following:

  - size():

    Return the number of elements in set A

    Input: None; **Output**: integer.

  - isEmpty():

    Return if the set A is empty or not.

    Input: None; **Output**: boolean.

  - insertElement(*e*):

    Insert the element *e* into the set A, unless *e* is already in A.

    Input: Object; **Output**: None.

Set ADT (contd.)

- elements():

  Return an enumeration of the elements in set A.

  Input: None; **Output**: Enumeration.

- isMember(*e*):

  Determine if *e* is in A.

  Input: Object; **Output**: Boolean.

- union(B):

  Return $A \cup B$.

  Input: Set; **Output**: Set.

- intersect(B):

  Return $A \cap B$.

  Input: Set; **Output**: Set.

- subtract(B):

  Return $A - B$.

  Input: Set; **Output**: Set.

- isEqual(B):

  Return true if and only if $A = B$.

  Input: Set; **Output**: boolean.

Generic Merging

Algorithm genericMerge(*A*, *B*):
  Input: Sorted sequences *A* and *B*
  Output: Sorted sequence *C*
  let *A'* be a copy of *A* { We won't destroy *A* and *B*}
  let *B'* be a copy of *B*
  while *A'* and *B'* are not empty **do**
    $a \leftarrow A'$.first()
    $b \leftarrow B'$.first()
    if $a<b$ **then**
      firstIsLess(*a*, *C*)
      A'.removeFirst()
    else if $a=b$ **then**
      bothAreEqual(*a*, *b*, *C*)
      A'.removeFirst()
      B'.removeFirst()
      else
        firstIsGreater(*b*, *C*)
        B'.removeFirst()
    while *A'* is not empty **do**
      $a \leftarrow A'$.first()
        firstIsLess(*a*, *C*)
      A'.removeFirst()
    while *B'* is not empty **do**
      $b \leftarrow B'$.first()
      firstIsGreater(*b*, *C*)
      B'.removeFirst()

Set Operations

- We can specialize the generic merge algorithm to perform set operations like union, intersection, and subtraction.
- The generic merge algorithm examines and compare the current elements of *A* and *B*.
- Based upon the outcome of the comparision, it determines if it should copy one or none of the elements *a* and *b* into *C*.
- This decision is based upon the particular operation we are performing, i.e. union, intersection or subtraction.
- For example, if our operation is union, we copy the smaller of *a* and *b* to *C* and if a=b then it copies either one (say *a*).
- We define our copy actions in firstIsLess, bothAreEqual, and firstIsGreater.
- Let's see how this is done ...

Set Operations (cont.)

- For union

```
public class UnionMerger extends Merger {
   protected void firstIsLess(Object a, Object b, Sequence
         C) {
      C.insertLast(a);
   }
   protected void bothAreEqual(Object a, Object b,
         Sequence C) {
      C.insertLast(a);
   }
   protected void firstIsGreater(Object b, Sequence C) {
      C.insertLast(b);
   }
```

- For intersect

```
public class IntersectMerger extends Merger {
   protected void firstIsLess(Object a, Object b, Sequence
C) { } // null method
   protected void bothAreEqual(Object a, Object b,
Sequence C) {
      C.insertLast(a);
}
```

Set Operations (cont.)

```
protected void firstIsGreater(Object b, Sequence C) { }
      // null method
```

- For subtraction

```
public class SubtractMerger extends Merger {
  protected void firstIsLess(Object a, Object b, Sequence
        C) {
     C.insertLast(a);
  }
  protected void bothAreEqual(Object a, Object b,
        Sequence C) { } // null method
  protected void firstIsGreater(Object b, Sequence C) {
}
     // null method
```