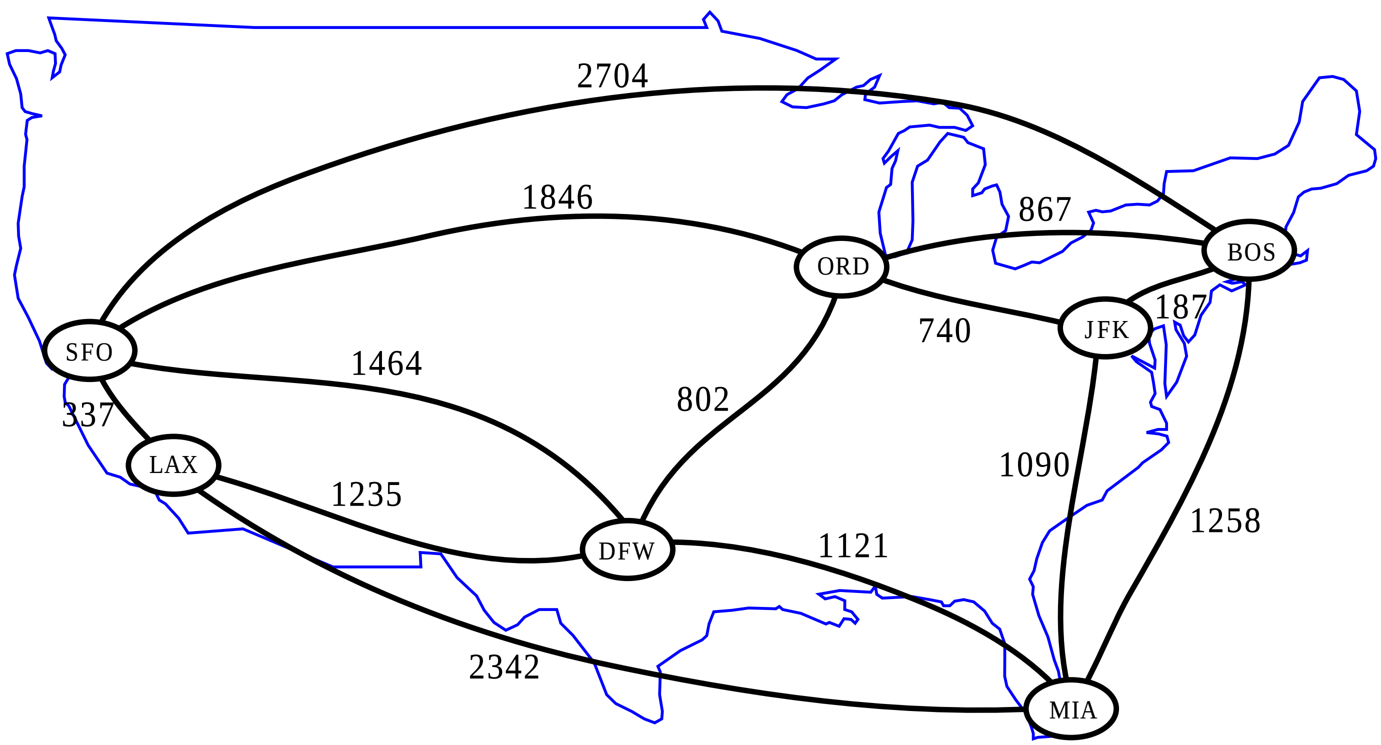
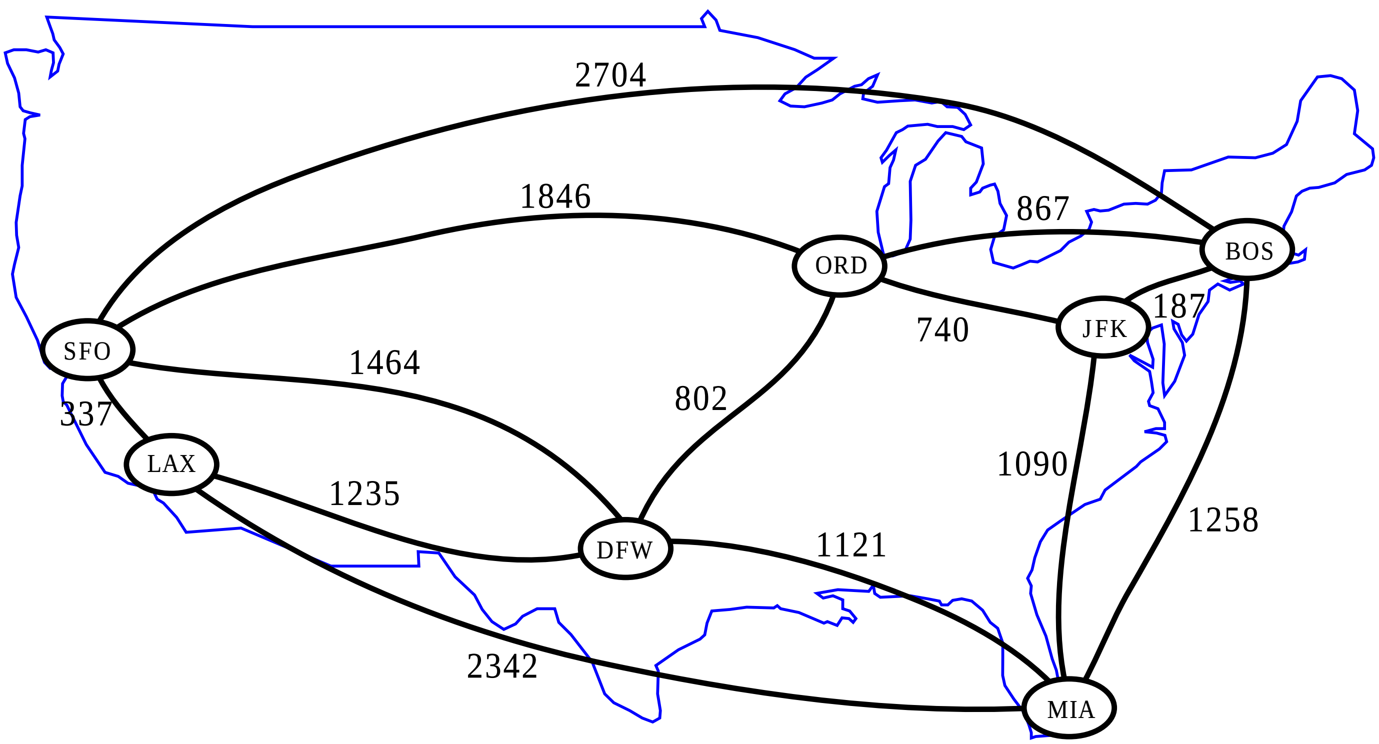
SHORTEST PATHS

- Weighted Digraphs
- Shortest paths



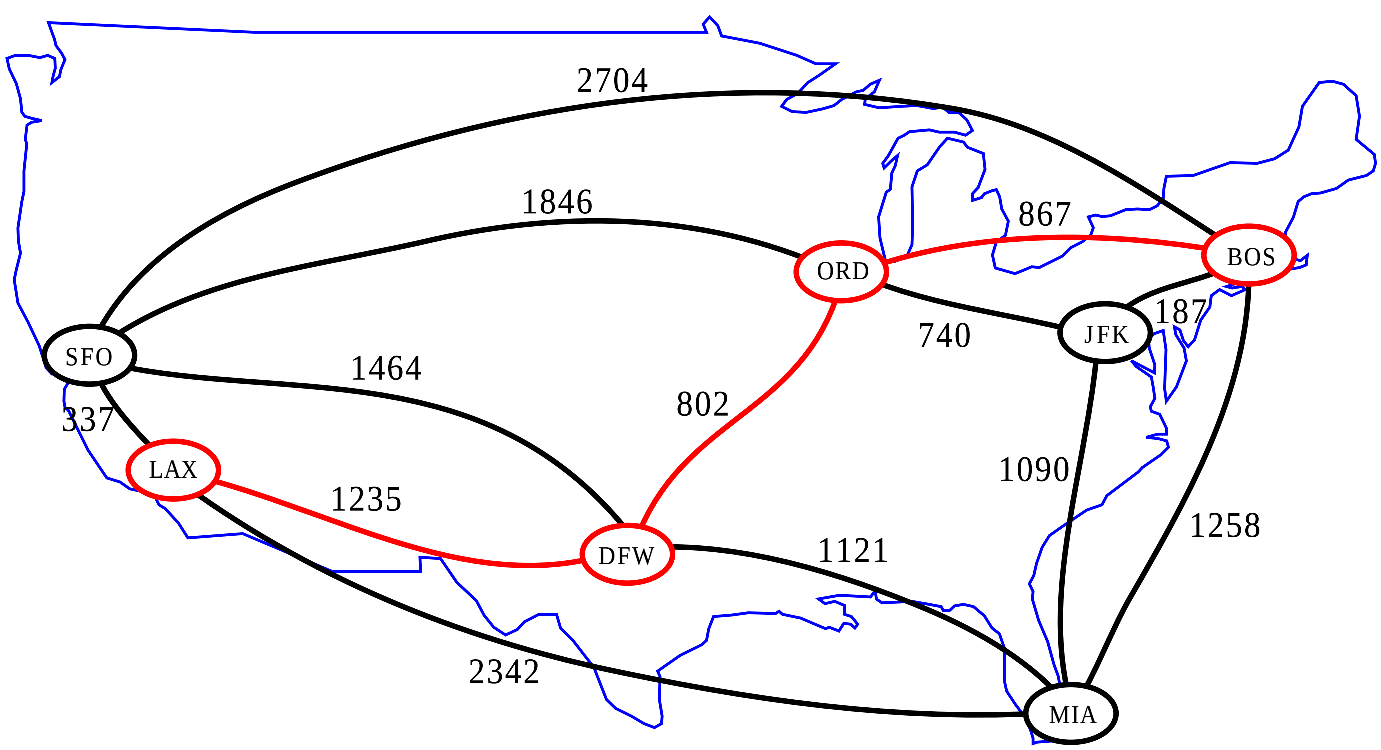
Weighted Graphs

- **weights** on the edges of a graph represent distances, costs, etc.
- An example of an undirected weighted graph:



Shortest Path

- BFS finds paths with the minimum number of edges from the start vertex
- Hencs, BFS finds shortest paths assuming that each edge has the same weight
- In many applications, e.g., transportation networks, the edges of a graph have different weights.
- How can we find paths of minimum total weight?
- Example - Boston to Los Angeles:



Dijkstra's Algorithm

- Dijkstra's algorithm finds shortest paths from a start vertex *s* to all the other vertices in a graph with
  - **undirected edges**
  - **nonnegative edge weights**
- Dijkstra's algorithm uses a **greedy method** (sometimes greed works and is good ...)
- the algorithm computes for each vertex *v* the **distance** of v from the start vertex s, that is, the weight of a shortest path between s and v.
- the algorithm keeps track of the set of vertices for which the distance has been computed, called the **cloud** C
- the algorithm uses a **label** D[v] to store an approximation of the distance between s and v
- when a vertex v is added to the cloud, its label D[v] is equal to the actual distance between s and v
- initially, the cloud C contains s, and we set
  - $D[s] = 0$
  - $D[v] = \infty$ for $v \neq s$

Expanding the Cloud

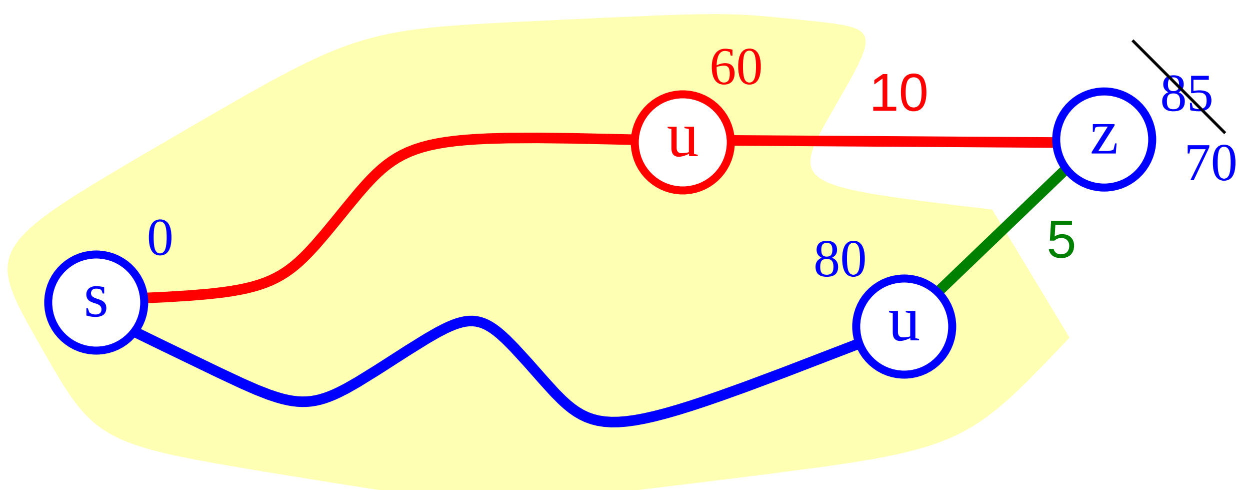
- meaning of D[z]: length of shortest path from s to z that uses only intermediate vertices in the cloud
- after a new vertex u is added to the cloud, we need to check whether u is a better routing vertex to reach z
- let u be a vertex not in the cloud that has smallest label D[u]
  - we add u to the cloud C
  - we update the labels of the adjacent vertices of u as follows

    for each vertex z adjacent to u **do**

      if z is not in the cloud C **then**

        if D[u] + weight(u,z) < D[z] **then**

          D[z] = D[u] + weight(u,z)
- the above step is called a **relaxation** of edge (u,z)



Pseudocode

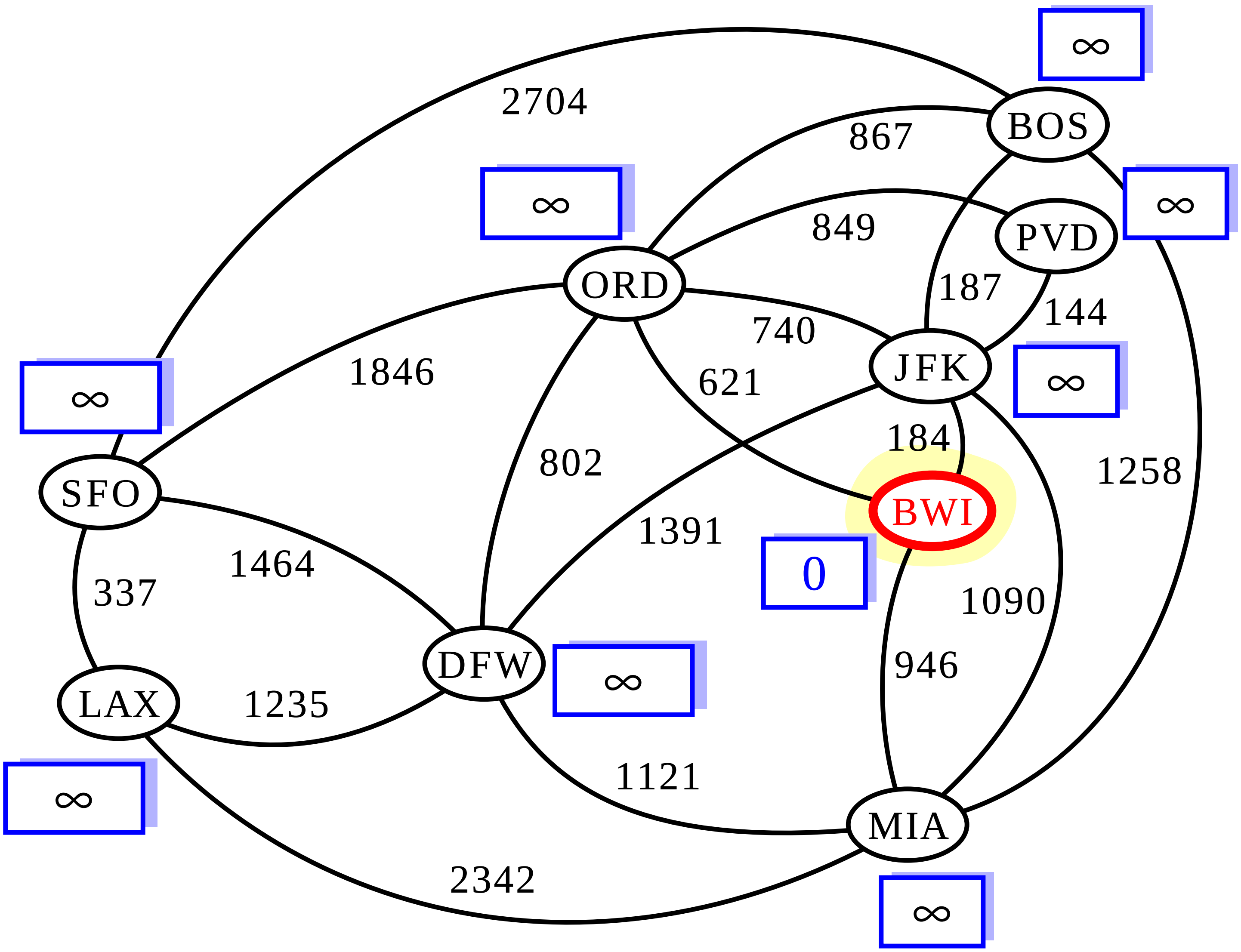
- we use a priority queue Q to store the vertices not in the cloud, where $D[v]$ the key of a vertex v in Q

Algorithm ShortestPath(G, v):
  Input: A weighted graph G and a distinguished vertex v of G.
  Output: A label $D[u]$, for each vertex that u of G, such that $D[u]$ is the length of a shortest path from v to u in G.

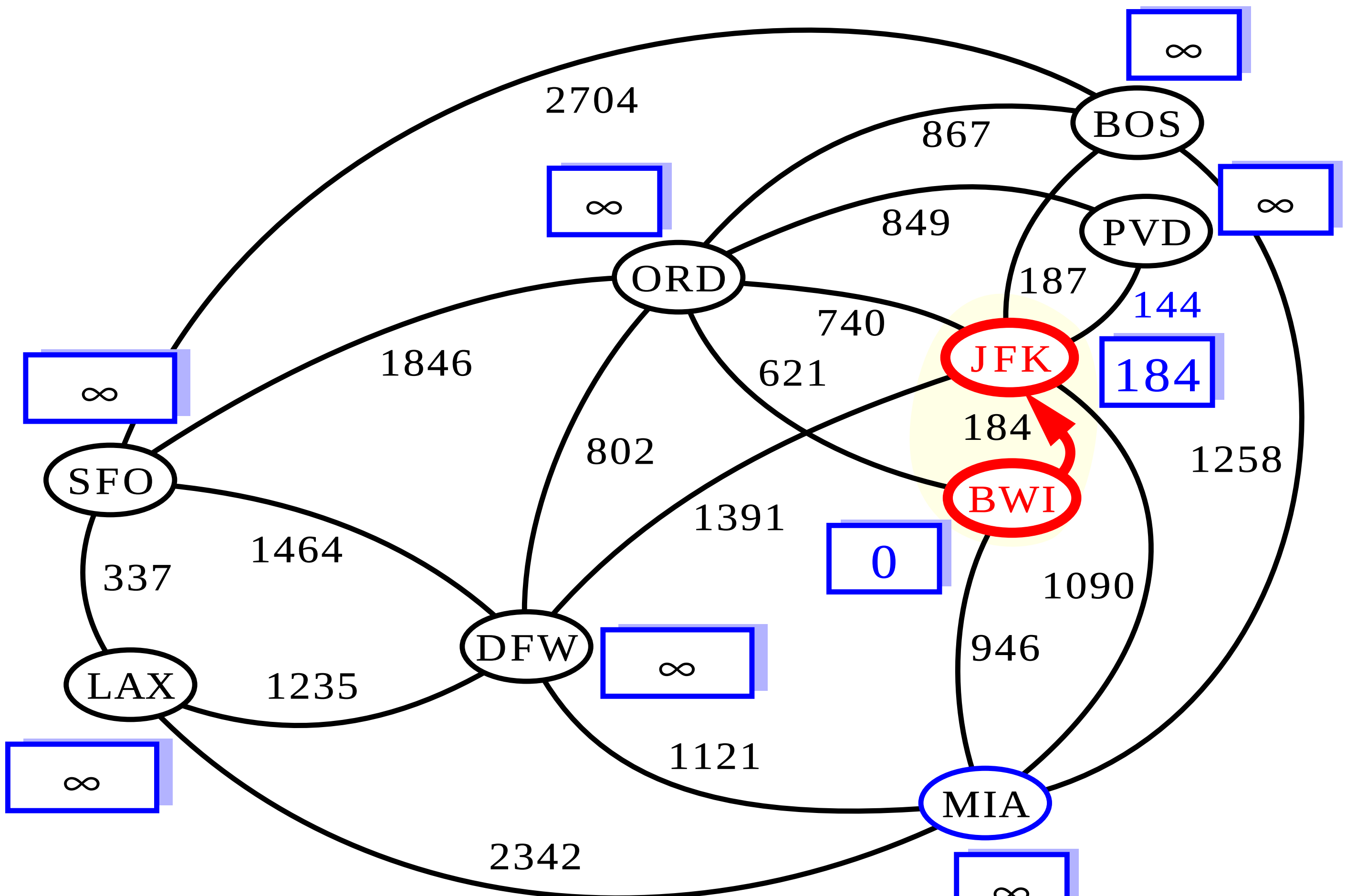
  initialize $D[v] \leftarrow 0$ and $D[u] \leftarrow +\infty$ for each vertex $v \neq u$
  let Q be a priority queue that contains all of the vertices of G using the D lables as keys.
  while $Q \neq \emptyset$ do
    {pull u into the cloud C}
    $u \leftarrow Q$.removeMinElement()
    for each vertex z adjacent to u such that z is in Q do
      {perform the relaxation operation on edge (u, z) }
      if $D[u] + w((u, z)) < D[z]$ then
        $D[z]$ ¨ $D[u] + w((u, z))$
        change the key value of z in Q to $D[z]$
  return the label $D[u]$ of each vertex u.

Example

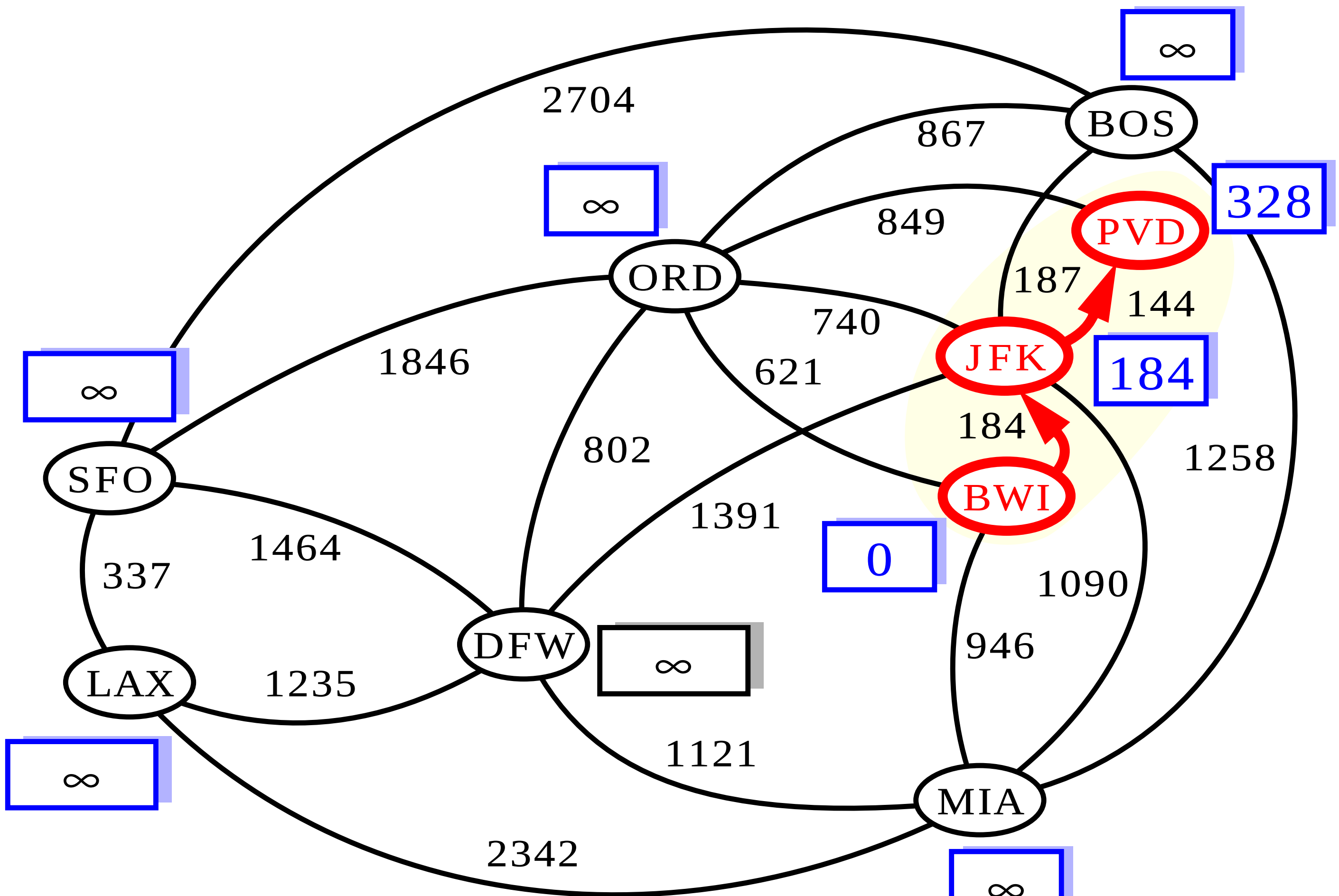
- shortest paths starting from BWI



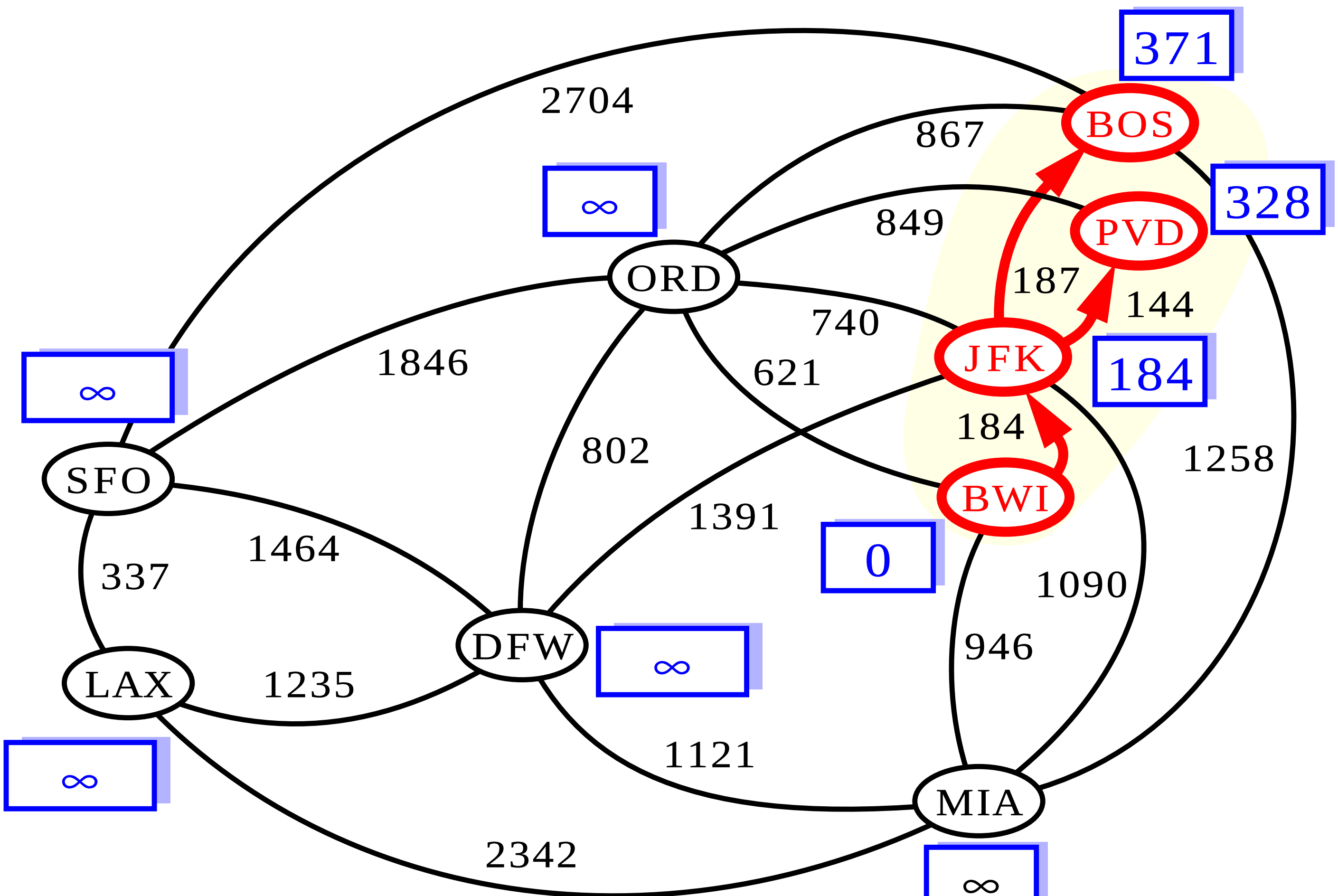
• JFK is the nearest...



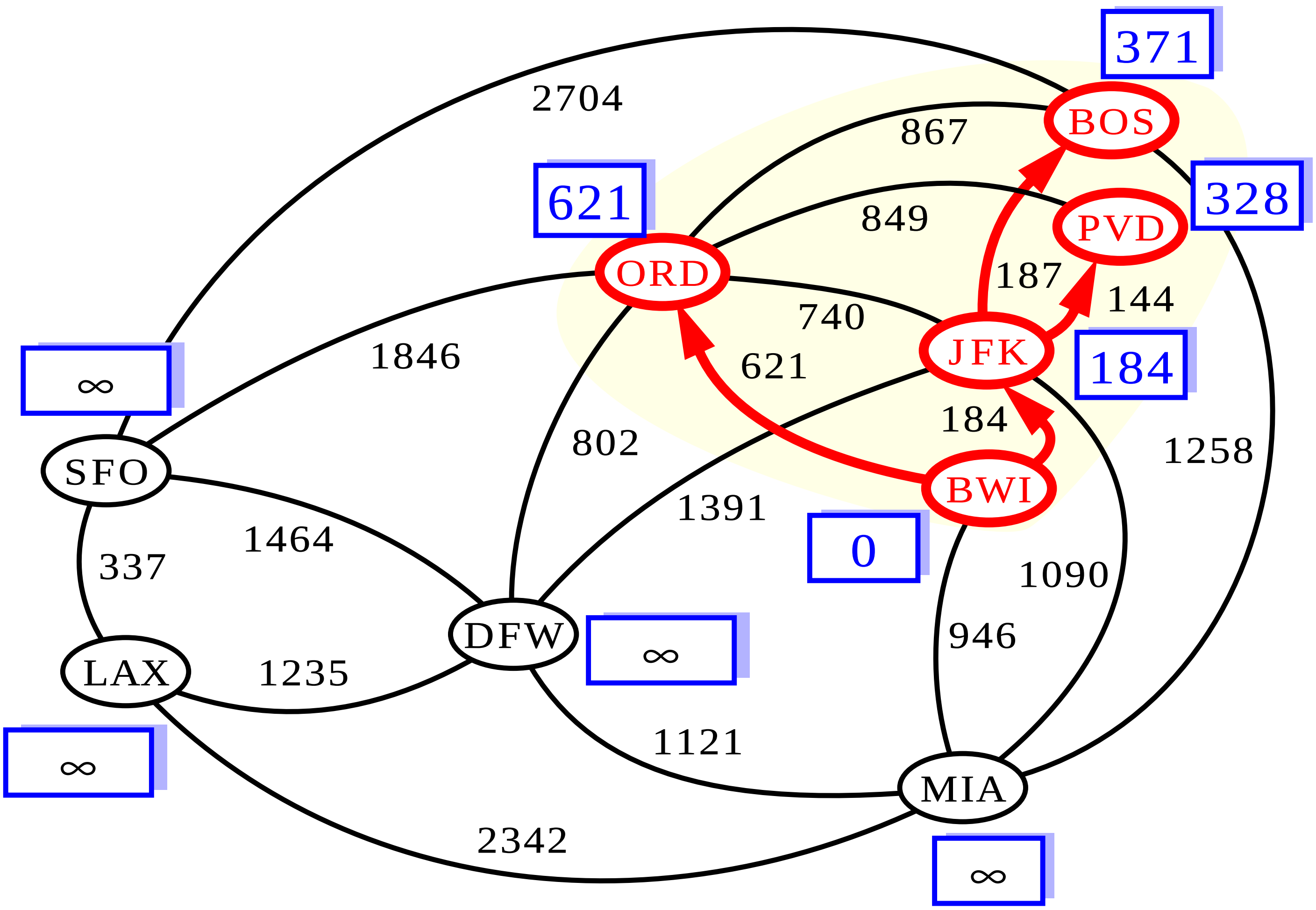
- followed by sunny PVD.



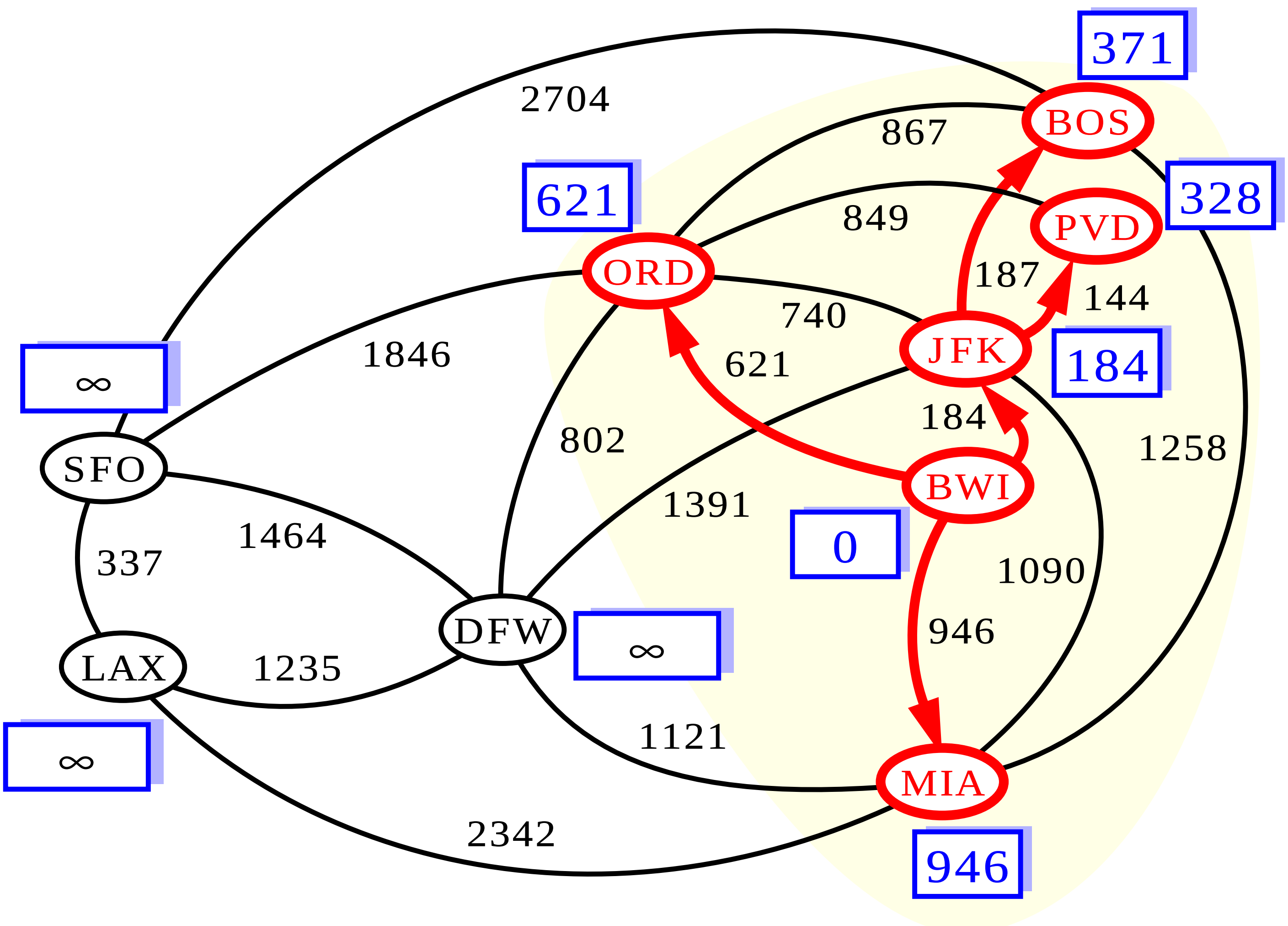
- BOS is just a little further.



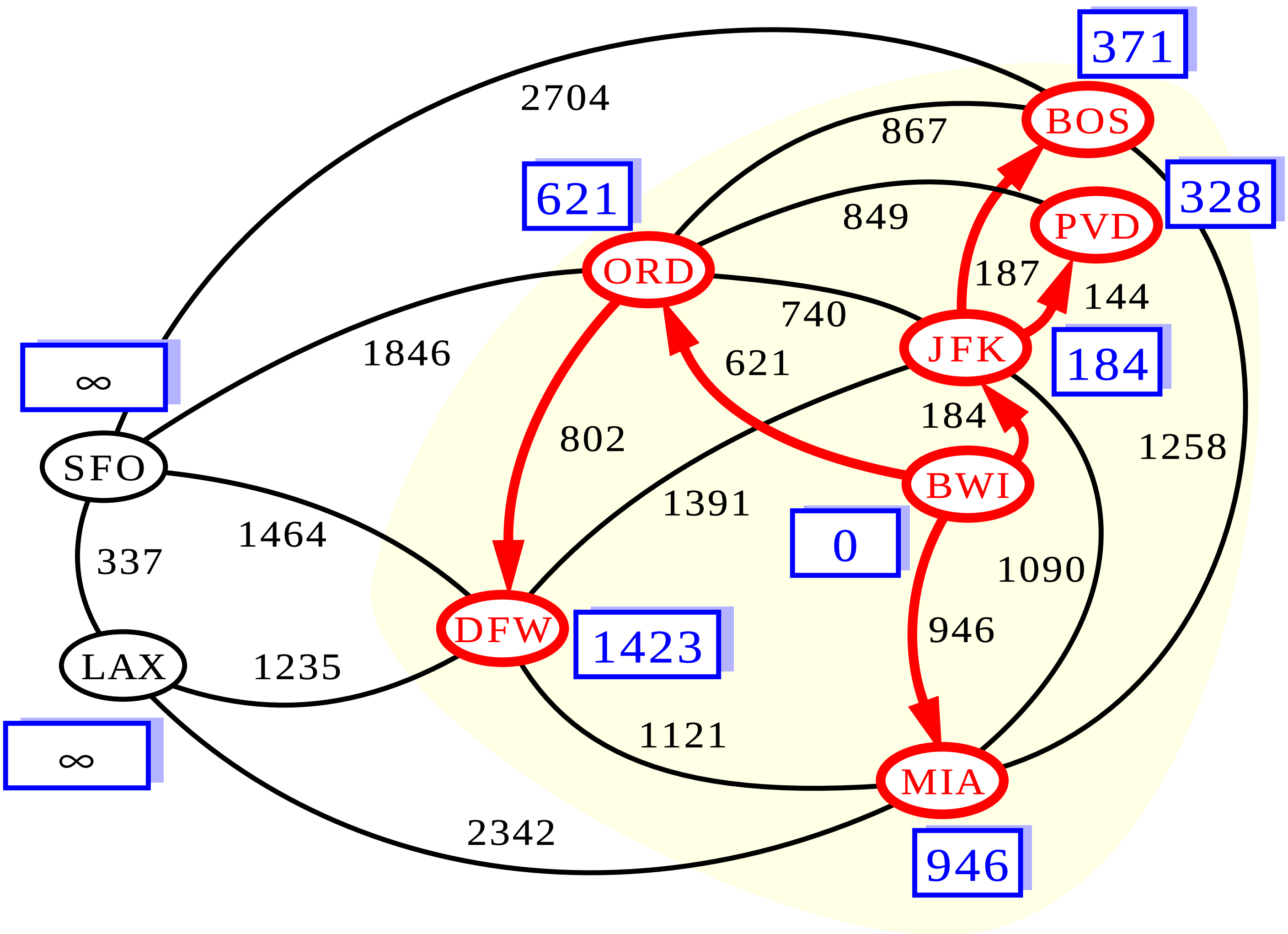
- ORD: Chicago is my kind of town.



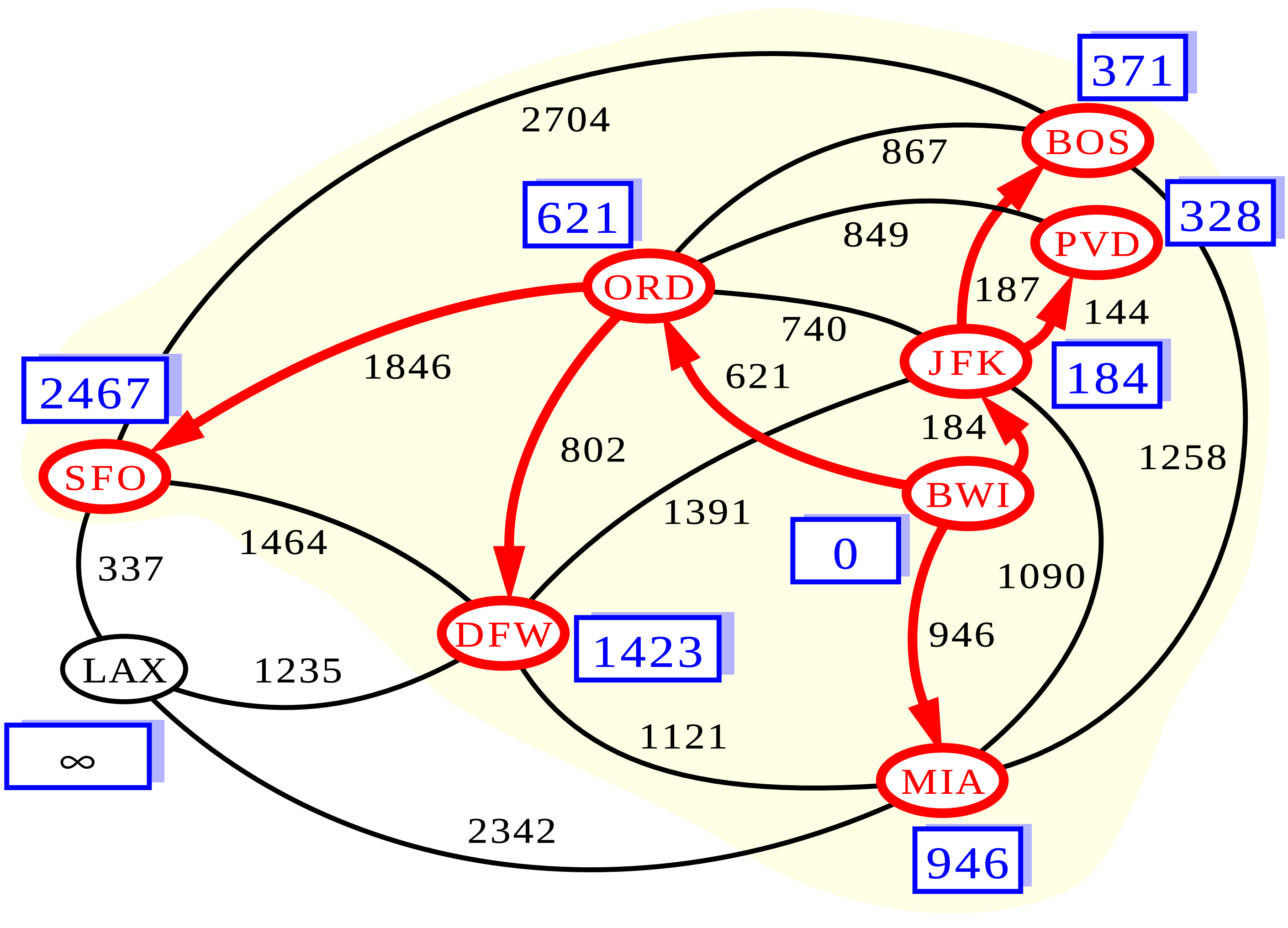
- MIA, just after Spring Break.



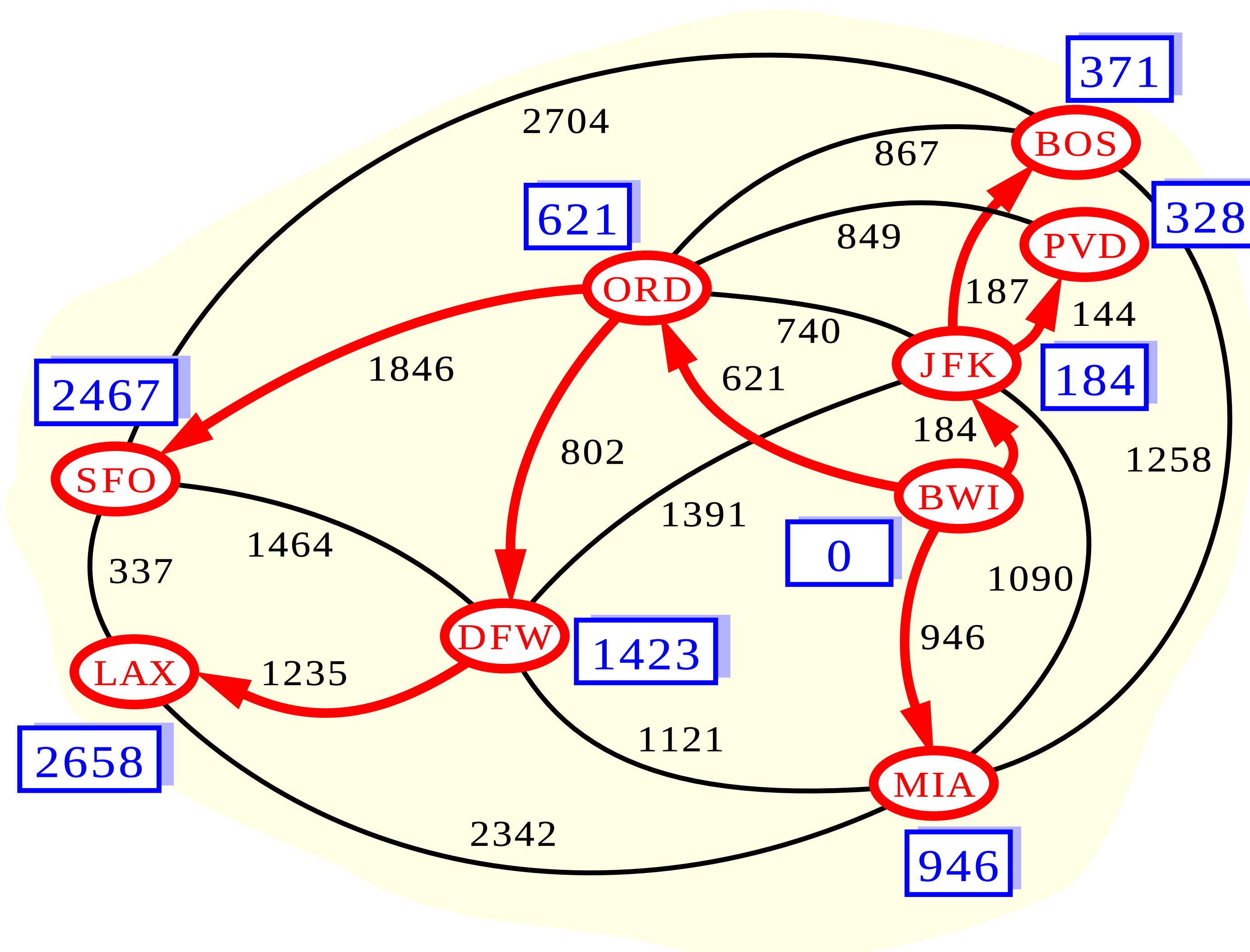
- DFW is huge like Texas.



- SFO: the 49'ers will take the prize next year.



- LAX is the last stop on the journey.



Running Time

- Let's assume that we represent G with an adjacency list. We can then step through all the vertices adjacent to u in time proportional to their number (i.e. O(j) where j in the number of vertices adjacent to u)
- The priority queue Q:
  - A Heap: Implementing Q with a heap allows for efficient extraction of vertices with the smallest D label(O(logN)). If Q is implented with locators, key updates can be performed in O(logN) time. The total run time is O((n+m)logn) where n is the number of vertices in G and m in the number of edges. In terms of n, worst case time is (On2log)
  - Unsorted Sequence: O(n) when we extract minimum elements, but fast key updates (O(1)). There are only n-1 extractions and m relaxations. The running time is O(n2+m)
- In terms of worst case time, heap is good for small data sets and sequence for larger.
- For each vertex, its neighbors are pulled into the cloud in random order. There are only O(logn) updates to the key of a vertex. Under this

Running Time (cont)

assumption, the run time of the head is O(nlogn+m), which is always O(n2) the heap implementation is thus preferable for all but degenerate cases.

Java Implementation

- we use a priority queue Q supporting locator-based methods in the implementation of Dijkstra's shortest path algorithm
- when we insert a vertex u into Q, we associate with u the locator returned by insert (e.g., via a dictionary)

```
Locator u_loc = Q.insert(new Integer(u_dist), u);
setLocator(u, u_loc);
```

- in the relaxation of an edge (u,z), the update of the distance of z is performed with operation replaceKey

```
for (Enumeration u_edges = graph.incidentEdges(u);
                        u_edges.hasMoreElements(); ) {
   Edge e = (Edge) u_edges.nextElement();
   Vertex z = graph.opposite(u,e);
   Locator z_loc = getLocator(z);
   if (z_loc.isContained()) { // test whether z is in Q
     int e_weight = weight(e);
     int z_dist = value(z_loc);
     if ( u_dist + e_weight < z_dist )
    Q.replaceKey(z_loc, new Integer(u_dist e_weight));
   }
}
```

Java Implementation (contd.)

```
public abstract class Dijkstra {
  private static final int INFINITE = Integer.MAX_VALUE;
  protected InspectableGraph graph;
   // priority queue used by the algorithm
    protected PriorityQueue Q;
    public Object execute(InspectableGraph g, Vertex
         start) {
    graph = g;
    dijkstraVisit(start);
    return distances();
  }
  // initialization
  abstract void init();
  // create an empty priority queue
  abstract PriorityQueue initPQ(Comparator comp);
  // return the weight of edge e
  abstract int weight(Edge e);
  // attach to u its locator loc in Q
  abstract void setLocator(Vertex u, Locator loc);
  // return the locator attached to u
  abstract Locator getLocator(Vertex u);
```

Java Implementation(cont)

```
// attach to u its distance dist
  abstract void setDistance(Vertex u, int dist);
// return the vertex distances in a data structure
  abstract Object distances();
  // return as an int the key of a vertex in Q
  private int value(Locator u_loc) {
    return ((Integer) u_loc.key()).intValue();
  }
```

Java Implementation (cont.)

```
protected void dijkstraVisit (Vertex v) {
    // initialize the priority queue Q and store all the
vertices in it
    init();
    Q = initPQ(new IntegerComparator());
    for (Enumeration vertices = graph.vertices();
         vertices.hasMoreElements(); ) {
      Vertex u = (Vertex) vertices.nextElement();
      int u_dist;
      if (u==v)
        u_dist = 0;
      else
          u_dist = INFINITE;
      Locator u_loc = Q.insert(new Integer(u_dist), u);
      setLocator(u, u_loc);
    }
     // grow the cloud, one vertex at a time
    while (! Q.isEmpty()) {
      // remove from Q and insert into cloud a vertex with
minimum distance
      Locator u_loc = Q.min();
```

Java Implementation (cont)

```
    Q.remove(u_loc);
    setDistance(u, u_dist); // the distance of u is final
    // examine all the neighbors of u and update their distances
    for (Enumeration u_edges = graph.incidentEdges(u);
        u_edges.hasMoreElements(); ) {
      Edge e = (Edge) u_edges.nextElement();
      Vertex z = graph.opposite(u,e);
      Locator z_loc = getLocator(z);
      // check if z is not in the cloud, i.e., z is in Q
      if (z_loc.isContained()) {
        // relaxation of edge e = (u,z)
        int e_weight = weight(e);
        int z_dist = value(z_loc);
        if ( u_dist + e_weight < z_dist )
          Q.replaceKey(z_loc, new Integer(u_dist + e_weight));
      }
    }
  }
}
```

Java Implementation (cont)

```
public class MyDijkstra extends Dijkstra {
  protected Hashtable locators = new Hashtable();
  protected Hashtable distances = new Hashtable();
  protected Hashtable weights = new Hashtable();
  public void init() { }
  public PriorityQueue initPQ(Comparator comp) {
    return (PriorityQueue) new
SequenceLocPriorityQueue(comp);
  }
  public int weight(Edge e) {
    return ((Integer) weights.get(e)).intValue();
  }
  public void setWeight(Edge e, int w) {
    weights.put(e, new Integer(w));
  }
  public void setLocator(Vertex u, Locator loc) {
    locators.put(u, loc);
  }
  public Locator getLocator(Vertex u) {
    return (Locator) locators.get(u);
```

Java Implementation (cont.)

```
    }
    public void setDistance(Vertex u, int dist) {
      distances.put(u, new Integer(dist));
    }
    public int distance(Vertex u) {
   return ((Integer) distances.get(u)).intValue();
  }
  public  Object distances() {
   return distances;
  }
}
```