Mathematical Induction

Mathematical Induction

- Mathematical Induction is a technique for proving theorems
- Theorems are typically of the form
  - P(n) is true for all positive integers n
    - where P(n) is a propositional function

$$\text{RTP}: \forall n P(n) \text{ where universe of discourse is } Z^+$$

Mathematical Induction

Proof by Mathematical Induction that P(n) is true for every positive integer n consists of two steps

- 1. Basis Step.
  - Show that P(1) is true
- 2. Inductive Step.
  - Prove P(n) $\rightarrow$ P(n+1) is true for all positive n

The statement P(n) is called the ***inductive hypothesis***

Expressed as a rule of inference,
Mathematical Induction can be stated as

$$[P(1) \wedge \forall n(P(n) \rightarrow P(n+1))] \rightarrow \forall n P(n)$$

The Principle of Mathematical Induction

- *show that P(n) is true when n = 1, i.e. P(1) is true*
- *show that P(n) $\rightarrow$ P(n+1) for all n > 0*
  - *show that P(n+1) cannot be false when P(n) is true*
  - *do this as follows*
    - *assume P(n) true*
    - *under this hypothesis that P(n) is true show that P(n+1) is also true*

$\therefore$ *Since P(1) is true and the implication P(n) $\rightarrow$ P(n+1) is true for all positive integers, the principle of mathematical induction shows that P(n) is true for all positive integers*

An Example

Prove, using Mathematical Induction, that the sum of the first n odd integers is n^2

$1+3+5+\ldots+2n-1=n^2$

Let P(n) denote the proposition that the sum of the first n odd integers is n^2

- ***Basis Step*** P(1) , the sum of the first odd integer, is 1^2
  - this is true, since $1^2 = 1$
- ***Inductive Step*** show that P(n) $\rightarrow$ P(n+1) for all n > 0
  - assume P(n) is true for a positive integer n
    - i.e. 1 + 3 + 5 + 7 + 9 + 11 + … + (2n -1) = n^2
    - *note: 2n - 1 is the nth odd number!*
  - show P(n+1) is true assuming P(n) is true
    - assuming P(n) true, P(n+1) is then
      1 + 3 + 5 + … + 2n-1 + (2(n + 1) - 1)
      where (2(n + 1) - 1) is the n+1th odd number
      = n^2 + 2n + 1
      = $(n+1)^2$
- "Since P(1) is true and the implication P(n) $\rightarrow$ P(n+1) is true for all +ve n, the principle of MathInd shows that P(n) is true for all +ve integers"

A Bad Example

What's wrong with this?

$1+3+5+\ldots+2n-1=n^2$

Let P(n) denote the proposition that the sum of the first n odd integers is n^2

- ***Basis Step*** P(1) = 1^2 = 1. Therefore P(1) is true.
- ***Inductive Step*** show that P(n) $\rightarrow$ P(n+1) for all n > 0
  - assume P(n) is true for a positive integer n
    - i.e. P(n) = 1 + 3 + 5 + 7 + 9 + 11 + … + (2n -1) = n^2
  - show P(n+1) is true assuming P(n) is true
    - assuming P(n) true, P(n+1) is then
      P(n+1) = 1 + 3 + 5 + … + 2n-1 + (2(n + 1) - 1)
      = n^2 + 2n + 1
      = $(n+1)^2$
- "Since P(1) is true and the implication P(n) $\rightarrow$ P(n+1) is true for all +ve n, the principle of MathInd shows that P(n) is true for all +ve integers"

What is P(n) ?

An Example

$$\forall n(n \geq 1 \rightarrow 3 \mid n^3 - n)$$

Let P(n) denote the proposition **$3|n^3 - n$**

- ***Basis Step***
  - P(1) is true, since $1^3 - 1 = 0$, and that is divisible by 3
- ***Inductive Step***
  - assume P(n) is true i.e. $3|(n^3 - n)$
  - show P(n + 1) is true assuming P(n) is true, i.e $3|((n + 1)^3 - (n + 1))$

$(n + 1)^3 - (n + 1) = n^3 + 3n^2 + 3n + 1 - (n + 1)$
$= n^3 + 3n^2 + 2n$
$= (n^3 - n) + 3n^2 + 3n$ Note trick
$= (n^3 - n) + 3(n^2 + n)$

$3|(n^3 - n)$ due to our assumption P(n) and …
$3|3(n^2 + n)$ because it is of the form 3k
we know if a|b and a|c then a|(b+c) We proved this
consequently $3|((n^3 - n) + 3(n^2 + n))$

- *By the principle of mathematical induction $n^3 - n$ is divisible by 3 when n is positive*

$$\forall n \exists p \exists q(n \geq 12 \wedge p \geq 0 \wedge q \geq 0 \rightarrow n = 4p + 5q)$$

Let P(n) denote the proposition ***"If n is greater than 11 then n can be expressed as 4p + 5q, where p and q are positive"***

- ***Basis Step***
  - P(12) is true, since 12 = 4x3 + 5x0
    NOTE: we started at 12, since P(n) is defined only for n > 11
- ***Inductive Step***
  - assume P(n) is true and prove P(n+1)
  - We use a "proof by cases"
  - if p > 0 and q > 0 then decrement p and increment q to get 1 more
  - if p > 0 and q = 0 then decrement p and increment q to get 1 more
  - if p = 0 and q > 0
    then q must be at least 3, because n is at least 11
    therefore 5q is at least 15
    decrement q by 3 and increment p by 4 to get 1 more
  - this covers all the cases where we want to increment n by 1
- Therefore, by the principle of mathematical induction we have proved that any integer greater than 11 can be expressed as 4p + 5q

$$\forall n \exists p \exists q(n \geq 12 \wedge p \geq 0 \wedge q \geq 0 \rightarrow n = 4p + 5q)$$

Therefore, by the principle of mathematical induction we have proved that any integer greater than 11 can be expressed as 4p + 5q

If a country's cheapest postage cost is 12 pence then 4 penny and 5 penny stamps will allow us to send any parcel/letter within the country

Could you imagine a similar type of problem with delivering change from a vending machine?

Your Example

$$\sum_{i=1}^{n} i = n(n+1)/2$$

Let P(n) denote the proposition ***"The sum of the first n positive integers, Sum(n), is n(n+1)/2"***

- ***Basis Step***
- ***Inductive Step***
- *Since P(1) is true and the implication P(n) → P(n+1) is true for all positive integers, the principle of mathematical induction shows that P(n) is true for all positive integers*

Your Example

$$\sum_{i=1}^{n} i = n(n+1)/2$$

Let P(n) denote the proposition
"The sum of the first n positive integers is n(n+1)/2"

- ***Basis Step***
  - P(1) is true, since 1(1+1)/2 = 1
- ***Inductive Step***
  - assume P(n) is true, i.e. 1 + 2 + … + n = n(n+1)/2
    show P(n+1) is true assuming P(n) is true, i.e.
    1 + 2 + 3 + … + n + (n+1) = (n + 1)(n + 2)/2
  - using the inductive hypothesis P(n) it follows that

$1 + 2 + 3 + \ldots + n + (n+1) = n(n + 1)/2 + (n + 1)$
$= (n^2 + n + 2n + 2)/2$
$= (n^2 + 3n + 2)/2$
$= (n + 1)(n + 2)/2$

  - Therefore P(n+1) follows from P(n)
- This completes the inductive proof

Our Example

$$|P(S)| = 2^{|S|}$$

Let P(n) denote the proposition
"A set S has 2^n subsets, where n is the cardinality of S"

Basis Step

Inductive Step

Since P(0) is true and the implication P(n) → P(n+1) is true for all positive integers, the principle of mathematical induction shows that P(n) is true for all positive integers

Our Example $|P(S)| = 2^{|S|}$

Let P(n) denote the proposition
"A set S has 2^n subsets, where n is the cardinality of S"

- ***Basis Step***
  - P(0) is true, since the only subset of the empty set is the empty set {}
- ***Inductive Step***
  - assume P(n) is true
  - show P(n + 1) is true assuming P(n) is true
    - Create a new set T = S $\cup$ {e}, where e is not already in S
    - of the 2^n subsets of S we can have each of those subsets
      - *with* element e or
      - *without* element e
    - therefore we have twice as many subsets of T as of S
    - therefore the number of subsets of T is $2(2^n) = 2^{(n+1)}$
- This shows that P(n+1) is true when P(n) is true, and completes the inductive step. Hence, it follows that a set of size n has 2^n subsets

Example

$$\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$$

Example $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$

Basis Step
P(0) is true, since $ar^0 = (ar^{0+1} - a)/(r - 1) = a(r - 1)/(r - 1) = a$

Inductive Step
assume P(n) true and P(n + 1) follows, therefore

$$ar^0 + ar^1 + \ldots + ar^n + ar^{n+1}$$
$$= \frac{a(r^{n+1} - 1)}{r - 1} + ar^{n+1}$$
$$= [ar^{n+1} - a + ar^{n+1}(r - 1)]/(r - 1)$$
$$= [ar^{n+1} - a + ar^{n+2} - ar^{n+1}]/(r - 1)$$
$$= \frac{a(r^{n+2} - 1)}{r - 1}$$

Since P(0) is true and the implication P(n) $\rightarrow$ P(n+1) is true for all positive integers, the principle of mathematical induction shows that P(n) is true for all positive integers

Mathematical Induction

Used frequently in CS when analysing the complexity of an algorithm or section of code.

When we have a loop, such as

for i from 1 to n do

Bubble Sort

Bubble sort?

```
for i := 1 to n do
   for j := 1 to n-i do
     if s[j] > s[j+1]        // *
     then swap(s,j,j+1)      // *
```

- Assume we have an array s to be sorted into non-decreasing order
- how many times is the section * executed for a given size of array

Bubble Sort

What is bubble sort?

```
for i := 1 to n do
   for j := 1 to n-i do
     if s[j] > s[j+1]        // *
     then swap(s,j,j+1)      // *
```

Assume we have an array s to be sorted into non-decreasing order

1	2	3	4	5	6	7	← index
9	3	8	1	7	4	2	← content

n = 7
i = 1   (first time)
j = 1 to 6
first compares s[1] with s[2], and swaps s[1] and s[2]
then compares s[2] with s[3], and swaps s[2] with s[3]
…
finally compares s[6] with s[7]

n = 7
i = 2   (second time)
j = 1 to 5
first compares s[1] with s[2],
…
finally compares s[5] with s[6]

n = 7
i = 3   (third time)
j = 1 to 4
first compares s[1] with s[2],
…
finally compares s[4] with s[5]

Bubble Sort

What 's bubble sort's complexity ?

```
for i := 1 to n do
   for j := 1 to n-i do
     if s[j] > s[j+1]        // *
      then swap(s,j,j+1)     // *
```

- inner loop executes * n-i times, for each i
- outer loop executes inner loop n times

$$\sum_{i=1}^{n} n-i = n-1+n-2+n-3+...+n-n$$
$$=1+2+...+(n-2)+(n-1)$$
$$=\sum_{1}^{n-1} i$$
$$=\frac{n(n-1)}{2}$$
$$=O(n^2)$$

Bubblesort demo in java?
Find the jdk?

sortDemo

Reading

Please read the following before the next tutorial

- Recursive definitions and structural induction
  - recursively defined functions
  - recursively defines sets and structures