

## Permutations and Combinations

### Perms

- A **permutation** is
  - an ordered arrangement of objects
- An **r-permutation**
  - is an ordered arrangement of size r
- The number of r-permutations of a set of size n is
  - $P(n,r) = n(n-1)(n-2) \dots (n-r)$
  - n ways to choose 1st object
  - n-1 ways to choose 2nd object
  - n-2 ways to choose 3rd object
  - ...
  - n-r+1 ways to choose the rth object
  - $P(n,r) = n!/(n-r)!$

### Perms

### When n = r ?

- An **r-permutation**
  - is an ordered arrangement of size r
  - $P(n,r) = n!/(n-r)!$
- What happens when we have n = r?

$$P(n,n) = n!/(n-n)! = n!/0! = n!/1 = n!$$

### Perms

### Example 3

There are 8 runners in a race. 1st gets a gold medal, 2nd gets a silver medal, and 3rd gets a bronze medal. How many ways are there to award the medals?

- n = 8
- r = 3 (gold, silver, bronze medalists)
- We are selecting 3 winners and then permuting them

$$\begin{aligned} P(8,3) &= 8!/(8-3)! \\ &= (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \\ &= 8 \cdot 7 \cdot 6 \\ &= 336 \text{ possible different outcomes} \end{aligned}$$

An **r-permutation** is an ordered arrangement of size r  $P(n,r) = n!/(n-r)!$

### Perms

### Example 4

A sales person has to visit 7 cities (Aberdeen, Brighton, Coventry, Dundee, Edinburgh, London, Manchester) starting at the head office (Glasgow) and returning to the head office (Glasgow). The sales person wishes to minimise the distance travelled. How many different tours must be considered in order to determine the shortest tour?

- The number of tours is the number of ways we can permute the the seven cities: (Aberdeen, Brighton, Coventry, Dundee, Edinburgh, London, Manchester)
- There are 7! Different tours
  - 7.6.5.4.3.2.1
  - 5040 tours

Draw the tree?

### Combs

- A **combination** is
  - an unordered arrangement of objects
- An **r-combination**
  - is an unordered arrangement of size r
- The number of r-combinations of a set of size n is
  - $C(n,r) = P(n,r)/r! = n!/r!(n-r)!$ 
    - because we over count in  $P(n,r)$  by r!
    - because ordering/permutations are ignored
  - $C(n,r)$  is "n choose r"
  - $C(n,r)$  is "the binomial coefficient"
  - $C(n,r) = n!/r!(n-r)!$

**Combs**

$P(n,r)$  can be considered in terms of  $C(n,r)$

- first generate the  $C(n,r)$  combinations
- each combination is of size  $r$
- now permute each of the combinations
  - this increases the number by  $r!$

**Combs** **Example 8**

We have to assemble a committee, with 4 computer scientists and 3 mathematicians. There are 11 computer scientists and 9 mathematicians. How many ways can I make up the committee?

- Decompose the problem into 2
  - (a) we need 4 from CS, and there are 11 to choose from
    - $C(n,r) = C(11,4) = 11!/4!(11-4)! = 11!/(4!7!)$
  - (b) we need 3 from Maths, and there are 9 to choose from
    - $C(n,r) = C(9,3) = 9!/3!(9-3)! = 9!/(3!6!)$
- We need to do (a) and (b)
  - therefore the product rule applies
  - $(11!/(4!7!)) \cdot (9!/(3!6!)) = 27,720$

$C(n,r) = n!/(r!(n-r)!)$

**Perms & Combs** **Exercise**

1. In how many different orders can five runners finish a race assuming there are no ties?
2. In how many ways can a set of two positive integers less than 100 be chosen?
3. How many subsets with an odd number of elements does a set with 10 elements have?

$P(n,r) = n!/(n-r)!$   $C(n,r) = n!/(r!(n-r)!)$

**Perms & Combs** **Answers**

1. In how many different orders can five runners finish a race assuming there are no ties?
 

$P(5,5) = 5! = 120$
2. In how many ways can a set of two positive integers less than 100 be chosen?
 

*A combination, order insignificant, therefore  $C(n,r)$*   
 $C(99,2) = n!/(r!(n-r)!) = 99!/2!(97!) = 99 \cdot 98/2 = 4851$
3. How many subsets with an odd number of elements does a set with 10 elements have?
 

*There are  $2^n$  subsets of a set, in this case  $2^{10}$*   
*Half of these sets have an even number of elements*  
*Half have an odd number of elements*  
*Therefore  $1024/2 = 512$*

$P(n,r) = n!/(n-r)!$   $C(n,r) = n!/(r!(n-r)!)$

**Perms & Combs** **One more example**

A shop has 10 men and 15 women assistants. How many ways are there to form a committee with 6 members, if it must have more women than men?

Consider the possible make ups of the committee

- (a) 2 men and 4 women =  $C(10,2) \cdot C(15,4)$
- (b) 1 man and 5 women =  $C(10,1) \cdot C(15,5)$
- (c) 0 men and 6 women =  $C(15,6)$

$C(10,2) \cdot C(15,4) + C(10,1) \cdot C(15,5) + C(15,6) = 96,460$

**NOTE: use of product and sum rules**

$C(n,r) = n!/(r!(n-r)!)$

**Perms with Reps**

How many strings of length  $n$  can be formed using the 26 lower case letters of the alphabet?

$26^n$

Every letter can be used repeatedly, so it is like a  $n$  digit number to the base 26

This is an application of the product rule.

Perms with Reps

The number of  $r$ -permutations of a set of  $n$  objects with repetition allowed is  $r^n$  ( $r$  to the power  $n$ )

Combs with Reps

Select 4 pieces of fruit from a bowl that contains 5 apples, 5 pears and 5 oranges.

4a, 3a + 1p, 3a + 1o, 2a + 2p, 2a + 2o, 2a + 1p + 1o  
 4p, 3p + 1a, 3p + 1o, 2p + 2a, 2p + 2o, 2p + 1a + 1o  
 4o, 3o + 1a, 3o + 1p, 2o + 2a, 2o + 2p, 2o + 1a + 1p

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Combs with Reps

Stars and Bars

Reconsider our problem of selecting 4 pieces of fruit where the four pieces are made up of apples, pears and oranges

- We can think of this as having 3 containers
  - one is for apples
  - one is for pears
  - one is for oranges
- The sum of the contents has to be 4
- We can draw this as "stars and bars"
  - we have 4 stars
    - one for each piece of fruit
  - two bars
    - to give us 3 regions

Combs with Reps

Stars and Bars

Reconsider our problem of selecting 4 pieces of fruit where the four pieces are made up of apples, pears and oranges

- apples|pears|oranges
  - ▶ \*\*\*\*| | 4 apples
  - ▶ \*\*|\*\*| 2 apples, 2 pears
  - ▶ \*\*|\*\*|\*\* 2 apples, 2 oranges
  - ▶ \*\*|\*|\* 2 apples, 1 pear, 1 orange
  - ▶ \*|\*\*|\* 1 apple, 2 pears, 1 orange
  - ▶ \*|\*|\*\* 1 apple, 1 pear, 2 oranges
- and so on

Combs with Reps

Stars and Bars

Reconsider our problem of selecting 4 pieces of fruit where the four pieces are made up of apples, pears and oranges

- Therefore we consider all permutations of
  - 4 stars and 2 bars
  - 6 objects
  - 6!
- But we are over counting
  - the stars are indistinguishable
  - the bars are indistinguishable
- We over count by
  - the 4! permutations of the 4 stars
  - the 2! permutations of the 2 bars
- therefore we should have
  - $6!/(4!2!) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 / (4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1) = 6 \cdot 5 / 2 = 15$

Combs with Reps

Generalisation

There are  $C(n+r-1, r)$   $r$ -combinations from a set of  $n$  elements when repetition of elements is allowed

Revisiting the fruit question, we want the number of 4-combinations (4 pieces of fruit,  $r = 4$ ) from a set with three elements (apples, pears, oranges,  $n = 3$ ) allowing repetitions

?  $n = 3, r = 4$   
 ?  $C(n+r-1, r) = C(6, 4)$   
 ?  $C(6, 4) = 6! / 4!(6-4)! = 6 \cdot 5 / 2 = 15$

**Combs with Reps** **Example**

There are  $C(n+r-1, r)$  r-combinations from a set of n elements when repetition of elements is allowed

*How many positive integer solutions are there to the equation*

$$x + y + z = 11$$

? we are selecting 11 items from a set with three elements  
 ? we want so many x's, so many y's, and so many z's  
 ? such that they add up to 11  
 ? an 11-combination from a set of size 3  
 ?  $n = 3, r = 11$   
 ?  $C(n+r-1, r) = C(13, 11)$   
 ?  $C(13, 11) = 13! / 11!(13 - 11)! = 13! / 11!2! = 13 \cdot 12 / 2$   
 ? 78

**Perms with indistinguishable objects** **Success?**

How many permutations are there of the word "success"

- there are 7 letters
- first guess
  - $7! = 5040$
- but we are over counting (again)
  - we do not distinguish between the 3 "s"
  - we do not distinguish between the 2 "c"
- we over count by  $3! \text{ And } 2!$
- It should be  $7! / 3!2!$ 
  - $7 \cdot 6 \cdot 5 \cdot 4 / 2 = 420$

**Perms with indistinguishable objects** **Theorem**

The number of different permutations of n objects, where  $n_1$  are indistinguishable and  $n_2$  are indistinguishable and ... and  $n_k$  are indistinguishable is

$$n! / (n_1! n_2! \dots n_k!)$$

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

**Exercise**

1. How many positive integer solutions are there to the equation
 
$$A + B + C + D = 17$$
2. How many different ways are there to choose 8 bottles of beer in Bar Brel, when they have Leffe, Chimay, Budvar and Hoeggarden?
3. How many strings with five or more characters can be made from the word SEEREES?
4. What is the probability that I will win the lottery this week, assuming I buy a ticket?

**Answers**

1. How many positive integer solutions are there to the equation
 
$$A + B + C + D = 17$$

**Answer:**  $C(4+17-1, 17) = C(20, 7) = 1140$
2. How many different ways are there to choose 8 bottles of beer in Bar Brel, when they have Leffe, Chimay, Budvar and Hoeggarden?
 

**Answer:**  $C(4+8-1, 8) = C(11, 8) = 11! / 8!(11-8)! = 11 \cdot 10 \cdot 9 / 3! = 165$
3. How many strings with five or more characters can be made from the word SEEREES?
 

**Answer:**  $7! / 2!4! = 7 \cdot 6 \cdot 5 / 2 = 105$

**Win the lottery**

4. What is the probability that I will win the lottery this week, assuming I buy a ticket?
  - This is an r-combination
    - there are 49 numbers
    - we have to choose 6  $P(n, r) = n! / (n-r)!$
    - $C(49, 6) = 49! / 6!(49 - 6)! = 49! / 6!43! = 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 / 720$
    - = 13,983,816
  - My chance is 1 in 14 million

Reading

- Tree Diagrams page 240
- Binomial Coefficients & Theorem, pages 253-257
- Distributing into boxes, page 293
- 6 pages in total

*Permutations and Combinations*