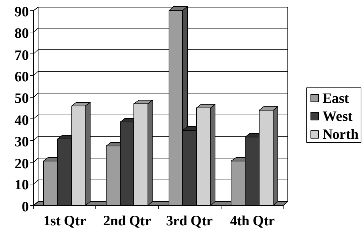
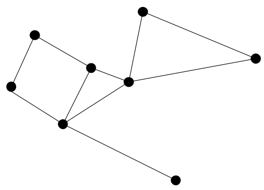
Graphs

Rosen, Chapter 8



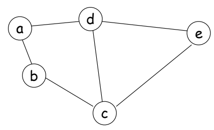
NOT ONE OF THESE!



One of these!

A Simple Graph

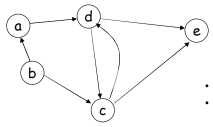
- G = (V,E)
  - V is set of vertices
  - E is set of edges



- V = {a,b,c,d,e}
- E = {(a,b),(a,d),(b,c),(c,d),(c,e),(d,e)}

A Directed Graph

- G = (V,E)
  - V is set of vertices
  - E is set of directed edges
    - directed pairs



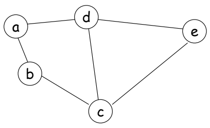
- V = {a,b,c,d,e}
- E = {(a,d),(b,a),(b,c),(c,d),(c,e),(d,c),(d,e)}

Applications

- computer networks
- telecomm networks
- scheduling (precedence graphs)
- transportation problems
- relationships
- chemical structures
- chemical reactions
- pert networks
- services (sewage, cable, …)
- WWW
- …

Terminology

- Vertex x is *adjacent* to vertex y if (x,y) is in E
  - c is adjacent to b, d, and e
- The degree of a vertex x is the number of edges incident on x
  - deg(d) = 3
  - note: degree aka valency
- The graph has a *degree sequence*
  - in this case 3,3,2,2,2



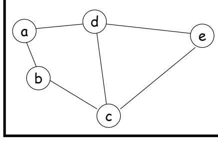
Handshaking Theorem (simple graph)

G = (V,E)

$$2e = \sum_{v \in V} \deg(v)$$

For an undirected graph G with e edges,
the sum of the degrees is 2e

Why?
- An edge (u,v) adds 1 to the degree of vertex u and vertex v
- Therefore edge(u,v) adds 2 to the sum of the degrees of G
- Consequently the sum of the degrees of the vertices is 2e



- 2e = deg(a) + deg(b) + deg(c) + deg(d) + deg(e)
- = 2 + 2 + 3 + 3 + 2
- = 12

Challenge: Draw a graph with degree sequence 2,2,2,1

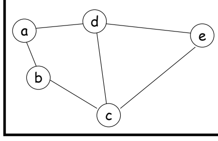
Handshaking Theorem (a consequence, for simple graphs)

There is an even number of vertices of odd degree

$$2e = \sum_{v \in V} \deg(v)$$

$$2e = \sum_{u \in OddDegVertices} \deg(u) + \sum_{v \in EvenDegVertices} \deg(w)$$

$$2k = \sum_{u \in OddDegVertices} \deg(u)$$



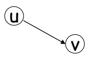
deg(d) = 3 and deg(c) = 3

Is there an algorithm for drawing a graph with a given degree sequence?

Yes, the Havel-Hakimi algorithm

Directed Graphs

- (u,v) is a directed edge
- u is the initial vertex
- v is the terminal or end vertex



- the in-degree of a vertex
  - number of edges with v as terminal vertex

$\deg^+(v)$

- the out-degree of a vertex
  - number of edges with v as initial vertex

$\deg^-(v)$

Directed Graphs

- (u,v) is a directed edge
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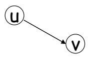
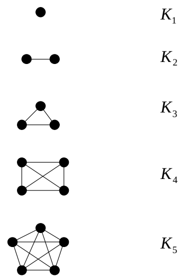


$$\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = |E|$$

Each directed edge (v,w) adds 1 to the out-degree of one vertex and adds 1 to the in-degree of another

(Some) Special Graphs



$$G = (V, E)$$
$$n = |V|$$
$$|E| = \frac{n(n-1)}{2}$$

Cliques

See Rosen 548 & 549

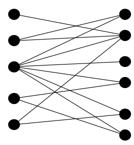
- Cycles
- Wheels
- n-Cubes

Bipartite Graphs

Vertex set can be divided into 2 disjoint sets

$$V = V_1 \cup V_2$$

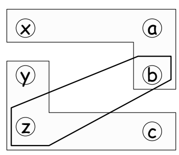
$$(v,w) \in E \rightarrow (v \in V_1 \wedge w \in V_2) \oplus (v \in V_2 \wedge w \in V_1)$$



Other Kinds of Graphs (that we won't cover, but you should know about)

- multigraphs
  - may have multiple edges between a pair of vertices
  - in telecomms, these might be redundant links, or extra capacity
  - Rosen 539
- pseudographs
  - a multigraphs, but edges (v,v) are allowed
  - Rosen 539
- hypergraph
  - hyperedges, involving more than a pair of vertices

A Hypergraph (one I prepared earlier)



The hypergraph might represent the following

- x = a + b
- c = y - z
- z ≥ b

New Graphs from Old?

We can have a subgraph

$$G = (V, E) \quad H = (W, F) \quad W \subseteq V \quad F \subseteq E$$

We can have a union of graphs

$$G_1 = (V_1, E_1) \quad G_2 = (V_2, E_2) \quad G_3 = G_1 \cup G_2 \quad G_3 = (V_1 \cup V_2, E_1 \cup E_2)$$

Representing a Graph (Rosen 7.3, pages 456 to 463)

Adjacency Matrix: a 0/1 matrix A

$$(i, j) \in E \leftrightarrow a_{i,j} = 1$$

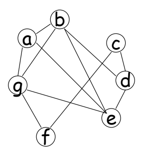
NOTE: A is symmetric for simple graphs!

$$(i, j) \in E \leftrightarrow a_{i,j} = 1 = a_{j,i}$$

NOTE: simple graphs do not have loops (v,v)

$$\forall i (a_{i,i} = 0)$$

Representing a Graph (Rosen 8.3)



$A =$

	a	b	c	d	e	f	g
a	0	1	0	0	1	0	1
b	1	0	0	1	1	0	1
c	0	0	0	1	0	1	0
d	0	1	1	0	1	0	0
e	1	1	0	1	0	0	1
f	0	0	1	0	0	0	1
g	1	1	0	0	1	1	0

A^2 What's that then?

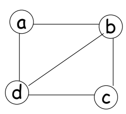
Isomorphism (Rosen 560 to 563)

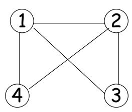
Are two graphs G1 and G2 of equal form?
- That is, could I rename the vertices of G1 such that the graph becomes G2
- Is there a bijection f from vertices in V1 to vertices in V2 such that
  - if (a,b) is in E1 then (f(a),f(b)) is in E2

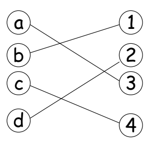
So far, best algorithm is exponential in the worst case

There are necessary conditions
- V1 and V2 must be same cardinality
- E1 and E2 must be same cardinality
- degree sequences must be equal
  - what's that then?

Are these graphs isomorphic?



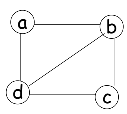


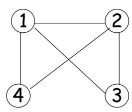


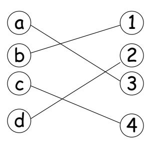
How many possible bijections are there?

Is this the worst case performance?

Are these graphs isomorphic?





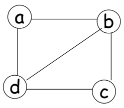


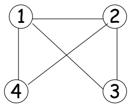
How many bijections?
1234,1243,1324,1342,1423,1432
2134,2143, …
…
4123,4132,… ,4321

4! = 4.3.2.1 = 24

Are these graphs isomorphic?

But not all 4! need be considered





What might the search process look like that constructs the bijection?

Connectivity

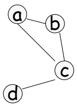
A Path of length n from v to u, is a sequence of edges that take us from u to v by traversing n edges.

A path is *simple* if no edge is repeated

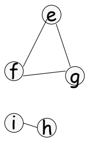
A *circuit* is a path that starts and ends on the same vertex

An undirected graph is connected if there is a path between every pair of distinct vertices

Connectivity



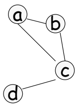
$$G_1 = (\{a,b,c,d\}, \{(a,b),(a,c),(b,c),(c,d)\})$$



$$G_2 = (\{e,f,g,h,i\}, \{(e,f),(e,g),(f,g),(i,h)\})$$

This graph has 2 components

Connectivity



c is a cut vertex
(d,c) is a cut edge

$$G = (\{a,b,c,d\}, \{(a,b),(a,c),(b,c),(c,d)\})$$

A *cut vertex* v, is a vertex such that if we remove v, and all of the edges incident on v, the graph becomes disconnected

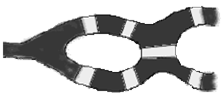
We also have a *cut edge*, whose removal disconnects the graph

Euler Path (the Konigsberg Bridge problem)

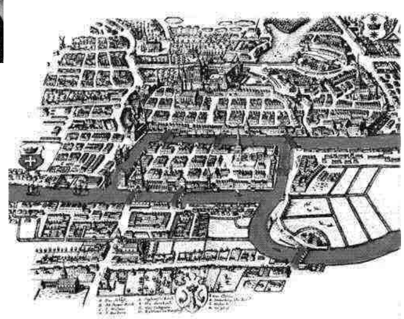
Rosen 8.5

Is it possible to start somewhere, cross all the bridges once only, and return to our starting place?

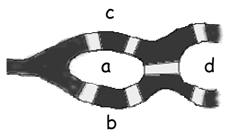
Leonhard Euler 1707-1783)

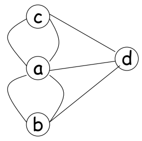


Is there a simple circuit in the given multigraph that contains every edge?



Euler Path (the Konigsberg Bridge problem)





Is there a simple circuit in the given multigraph that contains every edge?

An Euler circuit in a graph *G* is a simple circuit containing every edge of *G*.

An Euler path in a graph *G* is a simple path containing every edge of *G*.

Euler Circuit & Path

Necessary & Sufficient conditions

- every vertex must be of even degree
  - if you enter a vertex across a new edge
  - you must leave it across a new edge

A connected multigraph has an Euler circuit if and only if all vertices have even degree

The proof is in 2 parts (the biconditional)
The proof is in the book, pages 579 - 580

Hamilton Paths & Circuits

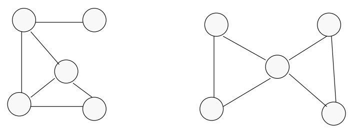
Given a graph, is there a circuit that passes through each vertex once and once only?

Given a graph, is there a path that passes through each vertex once and once only?

Easy or hard?

Due to Sir William Rowan Hamilton (1805 to 1865)

Hamilton Paths & Circuits



Is there an HC?

HC is an instance of TSP!

Connected?

Is the following graph connected?

$$G = (\{a,b,c,d,e,f,g\},\{(a,b),(b,c),(b,d),(c,d),(g,e),(e,f),(f,g)\})$$

Draw the graph

What kind of algorithm could we use to test if connected?

Connected?

$$G = (\{a,b,c,d,e,f,g\},\{(a,b),(b,c),(b,d),(c,d),(g,e),(e,f),(f,g)\})$$

- (0) assume all vertices have an attribute visited(v)
- (1) have a stack S, and put on it any vertex v
- (2) remove a vertex v from the stack S
- (3) mark v as visited
- (4) let X be the set of vertices adjacent to v
- (5) for w in X do
  - (5.1) if w is unvisited, add w to the top of the stack S
- (6) if S is not empty go to (2)
- (7) the vertices that are marked as visited are connected

A demo?

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