Rules of Inference

Rosen 1.5

Proofs in mathematics are *valid arguments*

An *argument* is a sequence of statements that end in a conclusion

By *valid* we mean the conclusion must follow from the truth of the preceding statements or premises

We use *rules of inference* to construct valid arguments

Valid Arguments in Propositional Logic

Is this a valid argument?

If you listen you will hear what I'm saying
You are listening
Therefore, you hear what I am saying

Let p represent the statement "you listen"
Let q represent the statement "you hear what I am saying"

The argument has the form:

$$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

Valid Arguments in Propositional Logic

$((p \to q) \wedge p) \to q$ is a tautology (always true)

$$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

This is another way of saying that

$\therefore$ *therefore*

Valid Arguments in Propositional Logic

When we replace statements/propositions with propositional variables we have an ***argument form***.

Defn:
An argument (in propositional logic) is a sequence of propositions.
All but the final proposition are called ***premises***.
The last proposition is the ***conclusion***
The argument is valid iff the truth of all premises implies the conclusion is true
An argument form is a sequence of compound propositions

Valid Arguments in Propositional Logic

The argument form with premises $p_1, p_2, \cdots, p_n$
and conclusion q
is valid when $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is a tautology

We prove that an argument form is valid by using the laws of inference

But we could use a truth table. Why not?

Rules of Inference for Propositional Logic — The 1[st] law

$$p \to q$$
$$\underline{p \quad\quad}$$
$$\therefore q$$

modus ponens
aka
law of detachment

modus ponens (Latin) translates to "***mode that affirms***"

Rules of Inference for Propositional Logic — modus ponens

$$p \to q$$
$$\underline{p \quad\quad}$$
$$\therefore q$$

If it's a nice day we'll go to the beach. Assume the hypothesis "it's a nice day" is true. Then by modus ponens it follows that "we'll go to the beach".

Rules of Inference for Propositional Logic — modus ponens

$$p \to q$$
$$\underline{p \quad\quad}$$
$$\therefore q$$

A valid argument can lead to an incorrect conclusion if one of its premises is wrong/false!

$$\sqrt{2} > \frac{3}{2} \to (\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$$
$$\underline{\sqrt{2} > \frac{3}{2} \quad\quad\quad\quad\quad\quad}$$
$$\therefore 2 > \left(\frac{3}{2}\right)^2$$

Rules of Inference for Propositional Logic — modus ponens

$$\sqrt{2} > \frac{3}{2} \to (\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$$
$$\underline{\sqrt{2} > \frac{3}{2} \quad\quad\quad\quad}$$
$$\therefore 2 > \frac{9}{4}$$

A valid argument can lead to an incorrect conclusion if one of its premises is wrong/false!

$$p \to q$$
$$\underline{p \quad\quad}$$
$$\therefore q$$

$$p:\ \sqrt{2} > \frac{3}{2}$$
$$q:\ 2 > \left(\frac{3}{2}\right)^2$$
$$p \to q$$

The argument is valid as it is constructed using modus ponens
But one of the premises is false (p is false)
So, we cannot derive the conclusion

The rules of inference — Page 66

Rule of inference	Tautology	Name
$p \to q$ $\underline{p}$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $\underline{p \to q}$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollen
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
$\underline{p}$ $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p $\underline{q}$ $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (p \vee r)$	Resolution

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Another view on what we are doing

You might think of this as some sort of game.

You are given some statement, and you want to see if it is a valid argument and true

You translate the statement into argument form using propositional variables, and make sure you have the premises right, and clear what is the conclusion

You then want to get from premises/hypotheses (A) to the conclusion (B) using the rules of inference.

So, get from A to B using as "moves" the rules of inference

Using the rules of inference to build arguments | An example

It is not sunny this afternoon and it is colder than yesterday.
If we go swimming it is sunny.
If we do not go swimming then we will take a canoe trip.
If we take a canoe trip then we will be home by sunset.
We will be home by sunset

Using the rules of inference to build arguments | An example

1. It is not sunny this afternoon and it is colder than yesterday.
2. If we go swimming it is sunny.
3. If we do not go swimming then we will take a canoe trip.
4. If we take a canoe trip then we will be home by sunset.
5. We will be home by sunset

p	It is sunny this afternoon
q	It is colder than yesterday
r	We go swimming
s	We will take a canoe trip
t	We will be home by sunset (the conclusion)

propositions

1. $\neg p \wedge q$
2. $r \rightarrow p$
3. $\neg r \rightarrow s$
4. $s \rightarrow t$
5. t

hypotheses

Using the rules of inference to build arguments | An example

p	It is sunny this afternoon
q	It is colder than yesterday
r	We go swimming
s	We will take a canoe trip
t	We will be home by sunset (the conclusion)

1. $\neg p \wedge q$
2. $r \rightarrow p$
3. $\neg r \rightarrow s$
4. $s \rightarrow t$
5. t

	Step	Reason
1.	$\neg p \wedge q$	Hypothesis
2.	$\neg p$	Simplification using (1)
3.	$r \rightarrow p$	Hypothesis
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \rightarrow s$	Hypothesis
6.	s	Modus ponens using (4) and (5)
7.	$s \rightarrow t$	Hypothesis
8.	t	Modus ponens using (6) and (7)

Rule of inference	Tautology	Name
$p \rightarrow q$, p, $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\neg q$, $p \rightarrow q$, $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollen
$p \rightarrow q$, $q \rightarrow r$, $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$, $\neg p$, $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
p, $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$, $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p, q, $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$, $\neg p \vee r$, $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (p \vee r)$	Resolution

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Using the resolution rule (an example)

1. Anna is skiing or it is not snowing.
2. It is snowing or Bart is playing hockey.
3. Consequently Anna is skiing and Bart is playing hockey.

We want to show that (3) follows from (1) and (2)

Using the resolution rule (an example)

1. Anna is skiing or it is not snowing.
2. It is snowing or Bart is playing hockey.
3. Consequently Anna is skiing or Bart is playing hockey.

hypotheses

1. $p \vee \neg r$
2. $r \vee q$

propositions

p	Anna is skiing
q	Bart is playing hockey
r	it is snowing

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore q \vee r \end{array}$$

Resolution rule

Consequently Anna is skiing or Bart is playing hockey

Rules of Inference & Quantified Statements

All men are £$%^$*(%, said Jane
John is a man
Therefore John is a £$%^$*(

Above is an example of a rule called "Universal Instantiation".
We conclude P(c) is true, where c is a particular/named element in the domain of discourse, given the premise $\forall x\, P(x)$

Rules of Inference & Quantified Statements

Rule of Inference	Name
$\forall x\ P(x)$ / $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c / $\therefore \forall x\ P(x)$	Universal generalisation
$\exists x\ P(x)$ / $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c / $\therefore \exists x\ P(x)$	Existential generalisation

Rules of Inference & Quantified Statements

All men are £$%^$*(%, said Jane
John is a man
Therefore John is a £$%^$*(

premises

M(x) x is a man
B(x) x is a £$%^$*(

premises

$\forall x\ (M(x) \rightarrow B(x))$

	Step	Reason
1.	$\forall x\ (M(x) \rightarrow B(x))$	Premise
2.	$M(John) \rightarrow B(John)$	Universal instantiation from (1.)
3.	M(John)	Premise
4.	B(John)	Modus ponens from (2.) and (3.)

Rule of Inference	Name
$\forall x\ P(x)$ / $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c / $\therefore \forall x\ P(x)$	Universal generalisation
$\exists x\ P(x)$ / $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c / $\therefore \exists x\ P(x)$	Existential generalisation

Rules of Inference & Quantified Statements

Maybe another example?