Sequences and Summations

Sequences

Definition of a sequence

A sequence is a function from a subset of integers to a set S.

a_n the image of the value n

a_n the n^{th} term of the sequence

$\{a_n\}$ is the sequence

An example of a sequence

$$\{a_n\} \text{ where } a_n = \frac{1}{n}$$

Beginning with n=1 $a_1, a_2, a_3, \ldots$ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

A geometric progression

$$a, ar, ar^2, ar^3, \ldots$$

Beginning with n=0 $\{ar^n\}$

An arithmetic progression

$$a, a+d, a+2d, a+3d, \ldots$$

Beginning with n=0 $\{a+nd\}$

Example progression

$$\{s_n\} \; where \; s_n = 4n-1 \;\; and \;\; n \geq 0$$

$$-1, 3, 7, 11, \ldots$$

Summations

Summations

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

Sigma for "sum"

Product

$$\prod_{j=m}^{n} a_j = a_m \times a_{m+1} \times a_{m+2} \times \ldots \times a_n$$

Pi for "product"

Summations

$$\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j = \sum_{k=m}^{n} a_k$$

Upper limit

Lower limit

$$\sum_{i=m}^{n} a_i$$

Index of summation

Summations example

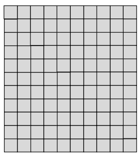
The sum of the first 100 integers

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \ldots + 100$$

What's the answer?

Summations example/picture

$$\sum_{i=1}^{10} i = 1+2+3+\ldots+10$$



$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{10} i = \frac{10\times 11}{2} = 55$$

$$\sum_{i=1}^{100} i = \frac{100\times 101}{2} = 5050$$

We now have a square 10 (n) by 11 (n+1) with area 110 units

We need half of that (10x11)/2

Summations challenge

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=50}^{100} i = 50+51+\ldots+100$$

Summations challenge

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=50}^{100} i = 50+51+\ldots+100$$

$$\sum_{i=50}^{100} i = \sum_{i=1}^{100} i - \sum_{i=1}^{49} i$$

$$\sum_{i=1}^{100} i - \sum_{i=1}^{49} i = \frac{100\times 101}{2} - \frac{49\times 50}{2}$$

Summations: shifting indices

$$\sum_{i=1}^{5} i^2 = \sum_{j=0}^{4} (j+1)^2$$

Summations: a theorem

$$\sum_{j=0}^{n} a.r^j = \begin{cases} \frac{a.r^{n+1}-a}{r-1} & if\ r \neq 1 \\ (n+1).a & if\ r = 1 \end{cases}$$

Proof: see the book

Summations: nesting

$$\sum_{i=1}^{4} \sum_{j=2}^{5} i.j$$

$$\begin{aligned} \sum_{i=1}^{25} \sum_{j=2}^{5} i.j &= \sum_{i=1}^{25} (i.2+i.3+i.4+i.5) \\ &= 14.\sum_{i=1}^{25} i \\ &= 14.\frac{25\times 26}{2} \\ &= 14.25.13 \\ &= 4550 \end{aligned}$$

Summations: implementing a nesting as a bit of program

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} i.j$$

```
sum =0
for i = 1 to n-1 do
   for j=i+1 to n do
      sum = sum + i*j
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Summations: just a wee tweak

$$\sum_{i\in\{2,4,6,8\}} i = 2+4+6+8$$

One More Example

$$\sum_{k=50}^{100} k^2$$

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$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

One More Example

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{i=1}^{49} k^2$$

Then use this →

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Challenge

$$\sum_{i=m}^{n}\sum_{j=p}^{q} i.j = ?$$