Algorithmic Foundations 2

Exercises for drop-in tutorial in week 8

Mathematical Induction and Recursive Definitions

Note: questions marked with a * are additional questions and might not be covered in the tutorial

1. Use the Principle of Mathematical Induction to prove that

   $$1.1! + 2.2! + 3.3! + ... + n.n! = (n + 1)! – 1$$

   for all $n \geq 1$.

2. Use the Principle of Mathematical Induction to prove that $3^n < n!$ for all $n > 6$.

3. Use the Principle of Mathematical Induction to prove that $n^3 > n^2 + 3$ for all $n \geq 2$.

4. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Use the Second Principle of Mathematical Induction to prove that $a_n \leq 3^n$ for all positive integers n.

5. Use the Principle of Mathematical Induction to prove that a function f defined by specifying f(0) and a rule for obtaining f(n+1) from f(n) (for each $n \geq 0$) is well-defined.

6. Find f(1),f(2),f(3),f(4) if f(n) is defined recursively by f(0)=3 and for each $n \geq 0$,

   (a) $f(n+1) = -2f(n)$
   (b) $f(n+1) = 3f(n)+7$
   (c) $f(n+1) = \{f(n)\}^2-2f(n)-2$
   (d) $f(n+1) = 3^{f(n)/3}$

9. Give a recursive definition of the function f with initial condition, for each of the following non-recursive definitions:

   (a) $f(n) = 4.7^n$, for each $n \geq 0$
   (b) $f(n) = 3n + 5$, for each $n \geq 0$
   (c) $f(n) = n!$, for each $n \geq 1$
   (d) $f(n) = n^2$, for each $n \geq 0$

10. Give recursive definitions of the functions max and min, so that max($a_1,a_2,\ldots,a_n$) and min($a_1,a_2,\ldots,a_n$) are the maximum and minimum of the n real numbers $a_1,a_2,\ldots,a_n$ respectively.

11. Give a recursive definition of
(a) the set of odd positive integers
(b) the set of positive integer powers of 3
(c) the set of polynomials with integer coefficients.

12* Show that the set S defined by

$$5 \in S;$$
$$s \in S \text{ and } t \in S \rightarrow s+t \in S$$

is the set of positive integers divisible by 5.