If n is a positive integer then a G-space X is called (Σ, n)-*universal* if given any isovariant map f of a closed invariant subspace of a G-space Y into X there exists an extension of f to an isovariant map of Y into X, provided dim $(Y/G) \leq n$.

CLASSIFICATION THEOREM. *Let X be a $(\Sigma, n + 1)$-universal G-space and let Z be a Σ-space of dimension $\leq n$. Then the map which takes the strong (weak) Σ-homotopy class of $f^* \rightarrow$ strong (weak) equivalence class of $f^{*-1}(X)$ is a one-to-one correspondence between the strong (weak) Σ-homotopy classes of Σ-maps of Z into X/G and the strong (weak) equivalence classes of G-spaces over Z.*

To give content to the above theorem we must be able to construct (Σ, n)-universal G-spaces. This is done as follows. Given G-spaces $X_1, \ldots, X_n$ let $X_1 \circ \ldots \circ X_n$ denote their join (see J. Milnor, The construction of universal bundles II, *Annals of Math.*, **63**, no. 3, May, 1956) made into a G-space by $g(t_1x_1, \ldots, t_nx_n) = (t_1gx_1, \ldots, t_ngx_n)$. We define the *reduced join* $X_1^* \ldots * X_n$ of $X_1, \ldots X_n$ to be the invariant subset of their join consisting of those points $(t_1x_1, \ldots, t_nx_n)$ for which the set of isotropy groups G_{x_i} for which $t_i \neq 0$ has a smallest element under inclusion. We denote the k-fold join (reduced join) of a space X by $X^{(ok)}(X^{(*k)})$. Let $\Sigma = ((H_1), \ldots, (H_m))$ and let n be any positive integer. It follows from Lemma 2.3 (Milnor, *loc. cit.*) that we can find integers $k_1, \ldots, k_m$ such that $(N(H_i)/(H_i)^{(ok_i)}$ is n-connected, where $N(H_i)$ is the normalizer of H_i in G. Then

THEOREM. $(G/H_1)^{(*k_1)} * \ldots * (G/H_m)^{(*k_m)}$ *is (Σ, n)-universal.*

ORTHOGONAL LATIN SQUARES

BY E. T. PARKER

REMINGTON RAND UNIVAC, SPERRY RAND CORPORATION, ST. PAUL, MINNESOTA

Communicated by A. A. Albert, April 20, 1959

1. *Introduction.*—The principal purpose of this note is to display the first pair of orthogonal latin squares of order 10. A *latin square* of order n is an $n \times n$ matrix with exactly n distinct symbols, each symbol in each row (necessarily only once) and each symbol in each column (once). Two latin squares of order n are called *orthogonal* if each ordered pair of symbols occurs (once) in some cell, the first symbol in the first latin square and the second symbol in the second latin square. More generally, several latin squares of order n are a *mutually* (or *pairwise*) *orthogonal* set if each pair of latin squares is orthogonal. Unlike the somewhat similar magic squares, orthogonal latin squares are today no isolated combinatorial curiosity. For a set of $n - 1$ mutually orthogonal latin squares of order n is equivalent to an affine plane of order n. In turn, an affine plane determines a projective plane of the same order; and a projective plane is the completion of at least one affine plane of like order. The nonspecialist is referred to expository papers[1-3] and their bibliographies.

Eulre[4] introduced the concept of pair of orthogonal latin squares (also called Greco-Latin squares). He obtained several results, and conjectured that no pair exists of any order $n \equiv 2 \pmod 4$. Tarry[5] demonstrated by a lengthy case-by-case argument the truth of Euler's conjecture for order 6; his result made the conjecture seem more plausible. Bose and Shrikhande[6] have very recently obtained the first counterexamples to Euler's conjecture; they constructed pairs of orthogonal latin squares of infinitely many orders $n \equiv 2 \pmod 4$, their lowest order being 22.

Order 10 is of considerable interest in itself, being the lowest order for which the existence of a projective plane remains undecided, and having been also the lowest undecided order for existence of a pair of orthogonal latin squares. Considerable effort has been expended in searching by digital computer for a pair of orthogonal latin squares of order 10, but machines have proved too slow to cope with a search of such magnitude.

Section 2 contains a theorem and a corollary yielding construction of pairs of orthogonal latin squares of infinitely many orders $n \equiv 2 \pmod 4$, the first order being 10. In Section 3 a pair of orthogonal latin squares of order 10 is displayed. In Section 4 are some concluding remarks.

2. The Theorem.—First will be proved a lemma, which is familiar to some, but is apparently not in the literature. (This lemma in more general form is in reference 7).

LEMMA. *A pair of orthogonal latin squares of order n is equivalent to a set of n^2 ordered quadruples $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$, $i = 1, \ldots, n^2$, with elements a_{ij} the numbers 1, $\ldots$, n, and such that for each pair u, v of integers, $1 \leq u < v \leq 4$, and each pair x, y of numbers from 1, $\ldots$, n, $a_{iu} = x$ and $a_{iv} = y$ both hold for some i ($i = 1, \ldots, n^2$).*

Proof: There being exactly n^2 ordered quadruples in the set, $a_{iu} = x$ and $a_{iv} = y$ are satisfied for a unique i. Associate the n^2 ordered quadruples with cells of two $n \times n$ matrices, a_{i1} and a_{i2} chosen as the row and column indices, respectively, a_{i3} and a_{i4} as the digits in the designated cell of the first and second matrix, respectively. The conditions imply that the matrices are orthogonal latin squares. For when u and v are 1 and 2, each cell is accounted for. When $u = 1$ and v is 3 or 4, each row of the appropriate matrix contains each digit—only once, of course. Similarly when $u = 2$ and v is 3 or 4, the same holds on columns. When u and v are 3 and 4, each ordered pair of digits occurs (exactly once) in some cell of the matrices. The converse construction of the set of ordered quadruples from the pair of orthogonal latin squares is carried out similarly.

THEOREM. *There exists a pair of orthogonal latin squares of order $(3q - 1)/2$, where q is a prime-power, and $q \equiv 3 \pmod 4$. (While there exists a pair of orthogonal latin squares of order 4, the construction below fails for $q = 3$.)*

Proof: A set of ordered quadruples satisfying the conditions of the Lemma will be constructed. In all four positions, the $(3q - 1)/2$ symbols are chosen as the q elements of $GF[q]$, and X_i, $i = 1, \ldots, (q - 1)/2$. Form the ordered quadruples,

$$\begin{aligned}
&(X_i,\ a,\ r^{2i} + a,\ r^{2i}(r + 1) + a)\\
&(r^{2i}(r + 1) + a,\ X_i,\ a,\ r^{2i} + a)\\
&(r^{2i} + a,\ r^{2i}(r + 1) + a,\ X_i,\ a)\\
&(a,\ r^{2i} + a,\ r^{2i}(r + 1) + a,\ X_i),
\end{aligned}$$

where $i = 1, \ldots, (q - 1)/2$, a ranges over the elements of $GF[q]$, and r is a fixed primitive element (i.e., a generator of the cyclic multiplicative group) of $GF[q]$. The above ordered quadruples are related by cyclic permutation of the four positions. Selecting all pairs i, a generates a list of $4 \cdot q \cdot (q - 1)/2 = 2q(q - 1)$ ordered quadruples. It is easily checked that each X_i occurs once in the list with each element of $GF[q]$ in each ordered pair of positions. No two like or distinct X_i occur in any of these quadruples; no element of $GF[q]$ is repeated in a quadruple. Since $(r^{2i} + a) - a = r^{2i}$ ranges over (nonzero) square elements of $GF[q]$, and $\{r^{2i}(r + 1) + a\} - (r^{2i} + a) = r^{2i+1}$ ranges over nonsquare elements of $GF[q]$, each pair of distinct elements of $GF[q]$ occurs once in each adjacent pair of positions once in either order—here the first and fourth positions are considered adjacent. Since $q \equiv 3 \pmod 4$, -1 is a nonsquare in $GF[q]$. It follows that $\{r^{2i}(r + 1) + a\} - a$ and $a - \{r^{2i}(r + 1) + a\}$ are one square and one nonsquare in $GF[q]$. Hence each ordered pair of distinct elements of $GF[q]$ occurs once in the first and third positions; once in the second and fourth positions. (Since -1 is the only primitive element of $GF[3]$, and $-1 + 1 = 0$, the construction fails for $q = 3$.)

Form the ordered quadruples, (a, a, a, a), where again a ranges over the elements of $GF[q]$. These q quadruples account for one occurrence in each pair of positions of each a with itself. Finally, generate a set of $(q - 1)^2/4$ ordered quadruples equivalent to any pair of orthogonal latin squares of order $(q - 1)/2$, with the X_i in all four positions. (A pair of orthogonal latin squares of any odd order exists.) In this latter set of ordered quadruples, each pair of like or distinct X_i occurs in each pair of positions once. Collectively, the three classes of ordered quadruples form a set of $\{(3q - 1)/2\}^2$ satisfying the conditions of the Lemma.

By the celebrated theorem of Dirichlet, there exist infinitely many prime-powers $q \equiv 7 \pmod 8$. Each such q in the above construction produces a pair of orthogonal latin squares of order $(3q - 1)/2 \equiv 10 \pmod{12}$.

COROLLARY. *There exists a pair of orthogonal latin squares of order any odd multiple of any order constructed in the Theorem.*

Proof: MacNeish[8] has shown the following by a straightforward constructive method: If there exist sets of t mutually orthogonal latin squares of orders m and n, respectively, then there exists a set of t mutually orthogonal latin squares of order mn. Specializing to $t = 2$, and noting a pair of latin squares exists of any odd order, the Corollary is immediate.

3.—A pair of orthogonal latin squares of order 10, constructed from the Theorem is displayed. Here $q = 7$, $r = 3$, and digits 7, 8, 9 correspond to the X_i. The horizontal and vertical divisions are intended to bring out relations.

0 4 1 7 2 9 8	3 6 5	0 7 8 6 9 3 5	4 1 2
8 1 5 2 7 3 9	4 0 6	6 1 7 8 0 9 4	5 2 3
9 8 2 6 3 7 4	5 1 0	5 0 2 7 8 1 9	6 3 4
5 9 8 3 0 4 7	6 2 1	9 6 1 3 7 8 2	0 4 5
7 6 9 8 4 1 5	0 3 2	3 9 0 2 4 7 8	1 5 6
6 7 0 9 8 5 2	1 4 3	8 4 9 1 3 5 7	2 6 0
3 0 7 1 9 8 6	2 5 4	7 8 5 9 2 4 6	3 0 1
1 2 3 4 5 6 0	7 8 9	4 5 6 0 1 2 3	7 8 9
2 3 4 5 6 0 1	8 9 7	1 2 3 4 5 6 0	9 7 8
4 5 6 0 1 2 3	9 7 8	2 3 4 5 6 0 1	8 9 7

4. Remarks.—The author is indebted to Professors Bose and Shrikhande for the crucial idea[6] of applying a group of automorphisms to the set of ordered quadruples *except* for a subset equivalent to a pair of orthogonal latin subsquares. The author failed to handle Euler's conjecture with a group of automorphisms over all ordered quadruples; he did, however, disprove[7] MacNeish's[8] generalization of the Euler conjecture.

The above pair of orthogonal latin squares of order 10 cannot be extended to a set of 9 mutually orthogonal latin squares of order 10, generating a plane of order 10. For the plane would have a subplane of order 3 formed by the set of $3 - 1 = 2$ orthogonal latin squares of order 3; this is impossible by a theorem of Bruck.[9] Whether the techniques referred to[7,6] and this note will lead to more significant results can hardly be predicted so soon. There is less reason than before to believe that all finite projective planes are of prime-power orders. The author hopes that more effort will now be put into attempts to construct projective planes of nonprime-power orders, and that someone will succeed.

[1] Bruck, R. H., "Recent Advances in the Foundations of Euclidean Geometry," Slaught Mem. Papers, No. 4, Math. Assn. of Amer., 1955, 2–17.

[2] Hall, Jr., Marshall, "Finite Projective Planes," Slaught Mem. Papers, No. 4, Math. Assn. of Amer., 1955, 18–24.

[3] Ryser, Herbert J., "Geometries and Incidence Matrices," Slaught Mem. Papers, No. 4, Math. Assn. of Amer., 1955, 25–31.

[4] Euler, Leonhard, "Recherches sur une Nouvelle Espece de Quarres Magiques," *Verh. Genootsch. der Wet. Vlissingen*, **9**, 85–232 (1782).

[5] Tarry, G., "Le Problème de 36 Officiers," *Comptes Rendu de l'Association Française pour l'Avancement de Science Naturel* **1**, 122–123 (1900); **2**,170–203 (1901).

[6] Bose, R. C., and S. S. Shrikhande, "On the Falsity of Euler's Conjecture About the Non-existence of Two Orthogonal Latin Squares of Order 4t + 2," these PROCEEDINGS, **45**, 734–737 (1959). A longer paper containing these and further results will appear in *Trans. Am. Math. Soc.*

[7] Parker, E. T., "Construction of Some Sets of Mutually Orthogonal Latin Squares," *Proc. Am. Math. Soc.* pending. (Am. Math. Soc. abstract title: "Construction of Some Sets of Pairwise Orthogonal Latin Squares.")

[8] MacNeish, Harris F., "Euler Squares," *Ann. Math.* (ser. 2), **23**, 221–227 (1921–22).

[9] Hall, Jr., Marshall, "Correction to 'Uniqueness of the Projective Plane with 57 Points,'" *Proc. Am. Math. Soc.*, **5**, 994–997 (1954).

---

*A DEMONSTRATION OF ANTIBODY ACTIVITY ON MICROSOMES**

BY MILTON KERN, ERNST HELMREICH, AND HERMAN N. EISEN

DIVISION OF DERMATOLOGY, DEPARTMENT OF INTERNAL MEDICINE, WASHINGTON UNIVERSITY SCHOOL OF MEDICINE, AND THE BARNARD FREE SKIN AND CANCER HOSPITAL, ST. LOUIS

Communicated by Colin M. MacLeod, April 13, 1959

Lymph nodes are well known to be the principal site for the synthesis of antibodies.[1] Since there is a growing body of evidence that an important stage in protein synthesis involves the condensation of amino acids on microsomes,[2,3] it was of interest to determine whether antibody activity could be demonstrated on microsomes isolated from the lymph nodes of antigenically stimulated animals. The assay devised for this purpose was based on the reversible binding of a soluble antigen by antibodies which were insoluble due to their association with microsomal particles. I^{131} labeled antigen was used to provide an indicator of high sensitivity. In order to increase the opportunities for evaluating the specificity of the assay procedure, the immune system used was specific for a simple chemical group (2,4-