Mutually Orthogonal Latin Squares: A Brief Survey of Constructions

Charles J. Colbourn Jeffrey H. Dinitz

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In Honour of S. S. Shrikhande, Combinatorial Pioneer

Abstract

In the two centuries since Euler first asked about mutually orthogonal latin squares, substantial progress has been made. The biggest breakthroughs came in 1960 with the celebrated theorems of Bose, Shrikhande, and Parker, and in 1974 in the research of Wilson. Current efforts have concentrated on refining these approaches, and finding new applications of the substantial theory opened. This paper provides a detailed list of constructions for MOLS, concentrating on the uses of pairwise balanced designs and transversal designs in recursive constructions as pioneered in the papers of Bose, Shrikhande, and Parker. In addition, several new lower bounds for MOLS are given and an up-to-date table of lower bounds for MOLS is provided.

1 An Historical Introduction

In 1779, Euler began a study of a simple mathematical puzzle, the 36 Officers Problem. Thirty-six officers drawn from six different ranks and six different regiments (one of each rank from each regiment) are to be arranged in a square so that in each horizontal and vertical line there are six officers from each rank and each regiment. Recording just the ranks of the officers, the square obtained is a latin square. Recording just the regiments, it is again a latin square. But the two latin squares have the property that, when superimposed, every ordered pair occurs exactly once. Thus the squares are *orthogonal*. A set of latin squares $L_1, \dots, L_m$ is *mutually orthogonal*, or a set of *MOLS*, if for every $1 \leq i < j \leq m$, L_i and L_j are orthogonal. $N(n)$ is the maximum number of latin squares in a set of MOLS of side n.

Euler [47] knew of course that $N(2) = 1$, and after much computation strongly suspected that $N(6) = 1$. He also established that $N(4n) \geq 2$ and that when n is odd, one also has $N(n) \geq 2$. On this basis, he made a conjecture that became the source of a huge literature:

Conjecture 1.1 *If* $n \equiv 2 \pmod 4$, *then* $N(n) = 1$.

Euler's conjecture rested comfortably for an extended period while the nascent field of combinatorics evolved tools to attack it. In 1850, Kirkman [63], for example, explored the existence of projective planes for prime order (in a different vernacular). At the turn of the century, Tarry [104] undertook a lengthy case analysis to prove that $N(6) = 1$. At the same time, Moore [73] established that $N(nm) \geq \max(N(n), N(m))$. This was later discovered independently by MacNeish [68]. MacNeish mistakenly believed that he had proved Euler's conjecture, and so advanced a stronger one:

Conjecture 1.2 *Let $n = q_1 q_2 \cdots q_s$ where $s \geq 1$ and each q_i is a power of a different prime. Then if $q_1 \leq q_i$ for all $2 \leq i \leq s$, $N(n) = q_1 - 1$.*

Levi [65] details the error in MacNeish's putative proof. MacNeish's conjecture strengthens Euler's because $q_1 = 2$ when $n \equiv 2 \pmod 4$. It is difficult to say who first noticed that $N(q) = q - 1$ when q is a prime power. Bose [19] derives this explicitly, but it appears already by Moore [73] in 1896.

Bose's work revealed dramatic and surprising connections with a wide variety of previous researches in algebra, geometry, number theory, and combinatorics, providing a fertile ground for combinatorial design theory to grow.

The challenge of the Euler and MacNeish conjectures was next taken up by Parker [82]. Parker devised a construction that uses balanced incomplete block designs to provide a framework (a "master design"); he established that $N(21) \geq 4$, and thanks Stein for pointing out that this disproves MacNeish's conjecture. (Indeed, if MacNeish's conjecture were true, one ought to have $N(21) = 2$.)

Bose and Shrikhande [22] saw Parker's work and made a remarkably astute generalization, replacing the block design by a pairwise balanced design. This yielded the first 'Euler spoiler', the proof that $N(22) \geq 2$ (indeed, they showed also that $N(66) \geq 5$, for example, demonstrating the power of their technique). Bose and Shrikhande introduced the notion of *Eulerian numbers*, those singly even numbers for which Euler's conjecture *does* hold. But, teaming up with Parker for the *coup de grace*, they [23] showed that $N(n) \geq 2$ for all $n \equiv 2 \pmod 4$, $n \geq 10$, and hence that there were *no* Eulerian numbers other than 2 and 6. The 10×10 mutually orthogonal latin squares are the real Euler spoiler. This got their picture in the *New York Times* on 26 April 1959.

The machinery developed by Bose, Shrikhande, and Parker provided an end to Euler's conjecture, but opened an even more challenging line of investigation: What then is $N(n)$? This problem is so well connected in mathematics, so rich in applications, and so easy to understand, that Mullen [74] has proposed it as the "next Fermat problem".

The ideas initiated by Parker, and extended in elegant ways with Bose and Shrikhande, led to a new class of constructions developed by Wilson [111]. Often we treat these as separate constructions entirely, but the remarkable developments in Wilson's methods are often heralded by special cases treated in the work of Bose, Shrikhande, and Parker.

Since Wilson's 1974 paper, much effort has gone into refining both the original Bose–Shrikhande–Parker technique, and the constructions of Wilson. Indeed so many generalizations of one kind or another have been devised that we cannot do much more than state them here, and refer the interested reader to the large and informative literature. We refer the reader to [35] for a more detailed presentation of constructions, and for a description of their use in determining lower bounds on $N(n)$ via a massive computer program.

In this paper we provide a detailed list of constructions for MOLS, concentrating on the uses of pairwise balanced designs and transversal designs in recursive constructions.

We begin, however, with an up-to-date table of values of lower bounds for $N(n)$, for $1 \leq n < 200$. This updates Table II.2.72 of *The CRC Handbook of Combinatorial Designs* [4] for values of $1 \leq n < 200$. The new values are indicated with a box around them and the constructions for these values can be found in §7.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	∞	∞	1	2	3	4	1	6	7	8	2	10	5	12	3	4	15	16	3	18
20	4	5	3	22	5	24	4	26	5	28	4	30	31	5	4	5	$\boxed{6}$	36	4	$\boxed{5}$
40	7	40	5	42	5	6	4	46	$\boxed{7}$	48	6	5	5	52	$\boxed{5}$	5	7	7	5	58
60	4	60	4	6	63	7	5	66	5	6	6	70	7	72	5	5	6	6	6	78
80	9	80	8	82	6	6	6	6	7	88	6	7	6	6	6	6	7	96	6	8
100	8	100	6	102	7	7	6	106	6	108	6	6	13	112	6	7	6	8	6	6
120	7	120	6	6	6	124	6	126	127	7	6	130	6	7	6	7	7	136	6	138
140	6	7	6	10	10	7	6	7	6	148	6	150	7	8	8	7	6	156	7	6
160	9	7	6	162	6	7	6	166	7	168	6	8	6	172	6	6	14	9	6	178
180	6	180	6	6	7	9	6	10	6	8	6	190	7	192	6	7	6	196	6	198

Table 1: Lower bounds on $N(n)$

2 Definitions

In order to describe the many constructions for sets of orthogonal latin squares, we must first establish the mathematical framework in which we work.

A *transversal design* of *order* or *groupsize* n, *blocksize* k and *index* λ, denoted $\mathrm{TD}_\lambda(k, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where

1. V is a set of kn elements;
2. $\mathcal{G}$ is a partition of V into k classes (called *groups*), each of size n;
3. $\mathcal{B}$ is a collection of k–subsets of V (called *blocks*);
4. every unordered pair of elements from V is either contained in exactly one group, or is contained in exactly λ blocks, but not both.

When $\lambda = 1$, one writes simply $\mathrm{TD}(k, n)$.

A $\mathrm{TD}(k,n)$ is equivalent to the existence of $k-2$ mutually orthogonal latin squares of order n, and the various generalizations of transversal designs all have reasonably natural interpretations in that formulation. An *orthogonal array* $\mathrm{OA}(k,s)$ is a $k \times s^2$ array with entries from an s-set S having the property that in any two rows, each (ordered) pair of symbols from S occurs exactly once. A $\mathrm{TD}(k,n)$ is also equivalent to an $\mathrm{OA}(k,n)$.

These equivalences are straightforward. That MOLS and OAs are equivalent can be seen as follows. Let $\{L_i : 1 \le i \le k\}$ be a set of k MOLS on symbols $\{1,\ldots,n\}$. Form a $(k+2) \times n^2$ array $A = (a_{ij})$ whose columns are $(i,j,L_1(i,j),L_2(i,j),\ldots,L_k(i,j))^T$ for $1 \le i,j \le k$. Then A is an orthogonal array, $\mathrm{OA}(k+2,n)$. This process can be reversed to recover k MOLS of side n from an $\mathrm{OA}(k+2,n)$, by choosing any two rows of the OA to index the rows and columns of the k squares. That OAs and TDs are equivalent can be seen as follows. Let A be an $\mathrm{OA}(k,n)$ on the n symbols in X. On $V = X \times \{1,\ldots,k\}$ (a set of size kn), form a set $\mathcal{B}$ of k-sets as follows. For $1 \le j \le n^2$, include $\{(a_{i,j},i) : 1 \le i \le k\}$ in $\mathcal{B}$. Then let $\mathcal{G}$ be the partition of V whose classes are $\{X \times \{i\} : 1 \le i \le k\}$. Then $(V,\mathcal{G},\mathcal{B})$ is a $\mathrm{TD}(k,n)$. This process can be reversed to recover an $\mathrm{OA}(k,n)$ from a $\mathrm{TD}(k,n)$. We choose to remain with one notation as much as possible, and use the language of transversal designs.

An *incomplete transversal design* of *order* or *groupsize* n, *blocksize* k, *index* λ, and *holesizes* $b_1,\ldots,b_s$, denoted $\mathrm{ITD}_\lambda(k,n;b_1,\ldots,b_s)$ for short, is a quadruple $(V,\mathcal{G},\mathcal{H},\mathcal{B})$, where

1. V is a set of kn elements;
2. $\mathcal{G}$ is a partition of V into k classes (*groups*), each of size n;
3. $\mathcal{H}$ is a set of disjoint subsets $H_1,\ldots,H_s$ of V, with the property that, for each $1 \le i \le s$ and each $G \in \mathcal{G}$, $|G \cap H_i| = b_i$;
4. $\mathcal{B}$ is a collection of k–subsets of V (*blocks*);
5. every unordered pair of elements from V is
   - contained in a hole, and contained in no blocks; or
   - contained in a group, and contained in no blocks; or
   - contained in neither a hole nor a group, and contained in λ blocks.

When $\sum_{i=1}^{s} b_i = n$, an $\mathrm{ITD}(k,n;b_1,\ldots,b_n)$ is a *partitioned* ITD, or $\mathrm{PITD}(k,n;b_1,\ldots,b_s)$. We often write the list $b_1,\ldots,b_s$ in "exponential notation", so that $x_1^{y_1}\cdots x_s^{y_s}$ signifies that there are y_i holes of order x_i, for each $1 \le i \le s$.

Again, when $\lambda = 1$, it can be omitted from the notation. Another notation that we employ for an $\mathrm{ITD}_\lambda(k,n;b_1,\ldots,b_s)$ is $\mathrm{TD}_\lambda(k,n) - \sum_{i=1}^{s} \mathrm{TD}_\lambda(k,b_i)$. It is trivial that the hole sizes could in fact be written in any order, and so when one refers to a specific hole size such as b_1, one is really speaking of an arbitrary hole size.

3 Filling, Truncation and Inflation

First we introduce the easiest of the constructions, treating some basic equivalences.

Lemma 3.1 *A TD(k,n) is equivalent to an ITD$(k,n;b_1,\ldots,b_s)$ for any nonnegative integers $b_1,\ldots,b_s$ with $\sum_{i=1}^{s} b_i \leq 1$.*

Actually, holes of order 0 can be assumed present or absent to suit our purposes. Holes of size 1, on the other hand, can always be assumed absent if we choose, because the hole can always be replaced by a block; one cannot, however, assume them to be present unless there is a suitable block available for deletion to form the hole. It is convenient to make a simple convention which avoids treating holes of size 0 and 1 as special cases in each result, namely:

Convention 3.2 *For all nonnegative integers k, there exists a TD$(k,0)$ and a TD$(k,1)$.*

More than one hole of size one can occasionally be assumed:

Lemma 3.3 *[30] A ITD$(k,n;h)$ is equivalent to a ITD$(k,n;h,1,1)$ when $(k-1)h < n$. In particular, a TD(k,n) is equivalent to a TD$(k,n) - 3TD(k,1)$ when $k \leq n$.*

At this point, it is sensible to remark on the basic necessary condition:

Lemma 3.4 *An ITD$(k,n;h)$ exists only if $h = n$ or $(k-1)h \leq n$. When $(k-1)h = n$, all blocks have exactly one point in the hole.*

The case when $h = n$ corresponds to an incomplete transversal design which has no blocks at all, just one big hole. Nevertheless, to be explicit, we state the following:

Convention 3.5 *There exists a TD(k,n) $-$ TD(k,n) whenever n is a positive integer and k is a nonnegative integer.*

When $(k-1)h+1 = n$, simple counting shows that each element not in the hole lies on exactly one block that does not meet the hole. Deleting these blocks, we obtain:

Lemma 3.6 *An ITD$(k,(k-1)h+1;h)$ is equivalent to a PITD$(k,(k-1)h+1;h^1 1^{(k-2)h+1})$.*

Lemma 3.3 has a number of generalizations. One can remark, for example, that a simple greedy strategy always produces $1 + \lfloor\frac{n-1}{k-1}\rfloor$ disjoint blocks in a TD(k,n), which can improve upon Lemma 3.3 when k is "small" relative to n. In addition, we can examine what happens when there are two or more holes assumed:

Lemma 3.7 *An ITD$(k,n;b_1,\ldots,b_s)$ always satisfies $(k-1)b_1 + b_2 \leq n$ (in particular, this holds when b_1 and b_2 are orders of the largest and second largest hole, respectively). Moreover, the ITD always has a block missing the first two holes, unless $(k-1)b_1 + b_2 = n$, $b_2 = b_3 = \ldots = b_s$ and $n = \sum_{i=1}^{s} b_i$. Consequently, an ITD$(k,n;b_1,b_2)$ with $b_1 \geq b_2 > 0$ is equivalent to an ITD$(k,n;b_1,b_2,1)$.*

We collect some other easy constructions in three main categories.

3.1 Filling

The basic result for filling an incomplete transversal design is:

Lemma 3.8 *If an ITD*$(k,n;b_1,\ldots,b_s)$ *exists, and an ITD*$(k,b_1;a_1,\ldots,a_r)$ *exists, then an ITD*$(k,n;a_1,\ldots,a_r,b_2,\ldots,b_s)$ *exists.*

For partitioned ITDs, one can fill in a more general way:

Lemma 3.9 *Suppose there is a PITD*$(k,n;b_1,\ldots,b_s)$. *Let* ε *be a nonnegative integer, and suppose that, for each* $2 \leq i \leq s$, *there is a*

$$TD(k,b_i+\varepsilon) - TD(k,\varepsilon).$$

Then there exists a TD$(k,n+\varepsilon) - TD(k,b_1+\varepsilon)$.

3.2 Truncation

Truncation is the operation of removing some points from a group. Here we examine the simplest form of truncation, when all points in a single group are deleted.

Lemma 3.10 *If a TD*$(k+1,n)$ *exists, then a TD*$(k,n) - nTD(k,1)$ *exists.*

Lemma 3.11 *If a TD*$(k+1,n;h)$ *exists, then a TD*$(k,n) - TD(k,h) - (n-h)TD(k,1)$ *exists.*

Removing a level of an ITD also has quite a useful consequence, which has been little exploited previously:

Lemma 3.12 *Suppose that an ITD*$(k+1,n;b_1,\ldots,b_s)$ *exists and* $\sum_{i=1}^{s} b_i < n$. *Suppose further that, for* $1 \leq i \leq s$, *there exists TD*(k,b_i). *Then there exists a TD*$(k,n) - nTD(k,1)$.

If instead there exists TD(k,b_i) *for* $2 \leq i \leq s$, *then there exists a*

$$TD(k,n) - TD(k,b_1) - (n-b_1)TD(k,1).$$

A somewhat similar operation can be done with partitioned ITDs:

Lemma 3.13 *Suppose that a PITD*$(k+1,n;b_1,\ldots,b_s)$ *exists, with* $b_1 > 0$. *Suppose further that, for* $2 \leq i \leq s$, *there exists TD*(k,b_i). *Then there exists a TD*$(k,n) - TD(k,b_1) - (n-b_1)TD(k,1)$.

3.3 Inflation

The main form of inflation is a simple direct product:

Lemma 3.14 *Suppose that an ITD($k, n; b_1, \ldots, b_s$) and a TD(k, w) both exist. Then a ITD($k, wn; wb_1, \ldots, wb_s$) exists.*

It bears frequent repetition that filling followed by inflation is *weaker* than inflation followed by filling. To see that it is no stronger, it suffices that each ingredient can be inflated by the same factor and the filling operation remains possible. That it is on occasion weaker follows from the fact that the ITD on wn points may exist, even when the ITD on n points does not exist.

4 PBDs, GDDs and the Bose–Shrikhande–Parker Theorem

A *pairwise balanced design* of *order* v and *blocksizes* K, denoted (v, K)-PBD, is a pair $(X, \mathcal{D})$. X is a set of v elements, and $\mathcal{D}$ is a set of subsets (*blocks*) of X for which $|D| \in K$ for each $D \in \mathcal{D}$. For every 2-subset of elements $\{x, y\} \subset X$, there is exactly one block containing x and y.

A *clear set* in a PBD is a set of pairwise disjoint blocks (also called a *partial parallel class*). A *near clear set* in a PBD $(X, \mathcal{D})$ is a subset $\mathcal{D}^c \subseteq \mathcal{D}$ defined as follows. For every block $D \in \mathcal{D}^c$, there is a distinguished element $e_D \in D$, the *tip* of D. The set $\mathcal{D}^c$ is near clear if, for each $x \in X$ contained in ι_x blocks of $\mathcal{D}^c$, x is the tip of at least $\iota_x - 1$ blocks in $\mathcal{D}^c$.

This definition seems unnaturally complicated, so perhaps some examples are needed. A set of blocks that all intersect in a single element, and are otherwise pairwise disjoint, is near clear: Simply taking the common element to be the tip of each block. A different example arises from three blocks which pairwise intersect in one point, but the common intersection is empty. Choosing the three intersection points to be the tips of the three blocks shows that this structure is near clear.

Let us denote a (v, K)-PBD $(X, \mathcal{D})$ with a near clear set $\mathcal{D}^c$ as a (v, K_b, K_c)-PBD, where K_c is the sizes of the blocks that actually arise in the near clear set, and K_b is the sizes of the blocks that actually arise among the remaining blocks (note that $K_b \cup K_c \subseteq K$, but equality is *not* necessary, as a (v, K)-PBD need not in general realize every block size in K).

4.1 First Constructions Using PBDs and GDDs

Now to the basic (Bose–Shrikhande–Parker) construction:

Theorem 4.1 *Suppose that a (v, K_b, K_c)-PBD exists. Suppose that, for every $m \in K_b$, there exists a PITD($k, m; 1^m$). Further suppose that, for every $m \in K_c$, there exists a TD(k, m). Then there exists a TD(k, v).*

Theorem 4.1 is a fairly standard Wilson-type construction using weight k for pairwise balanced designs [112]. The unusual feature is the use of near clear sets rather than clear sets. We content ourselves with remarking that for blocks in the near clear set, the ingredient used is actually a $\mathrm{TD}(k,m) - \mathrm{TD}(k,1)$, and the $\mathrm{TD}(k,1)$ is chosen to coincide with the k copies of the tip element. In this vein, when ingredients exist with more than one hole of size 1, one could permit the blocks of the near clear set to have more than one tip; this would extend the definition of near clear set. However, we know of no applications of this generalization, so we omit it.

When we have a certain types of near clear sets (clear sets being one example), we can say something about incomplete TDs as well:

Theorem 4.2 *Let $(X,\mathcal{D})$ be a PBD of order v. Suppose that for some subset $\mathcal{D}^c \subseteq \mathcal{D}$ of blocks, we have that there is one element $x \in X$, so that for $D, D' \in \mathcal{D}^c$, $D \cap D' \subseteq \{x\}$. Suppose that for every $D \in \mathcal{D} \setminus \mathcal{D}^c$, there exists a PITD$(k,|D|;1^{|D|})$. Fix a block $F \in \mathcal{D}^c$, and suppose that for every block $D \in \mathcal{D}^c \setminus \{F\}$, there exists a TD$(k,|D|)$.*

Then there exists a TD$(k,v) - TD(k,|F|)$.

Next we constrain the near clear set to be a clear set to obtain:

Theorem 4.3 *Let $(X,\mathcal{D})$ be a PBD of order v. Suppose that some subset $\mathcal{D}^c \subseteq \mathcal{D}$ of blocks is a clear set. Suppose that for every $D \in \mathcal{D} \setminus \mathcal{D}^c$, there exists a PITD$(k,|D|;1^{|D|})$. Then there exists a TD$(k,v) - \sum_{F\in\mathcal{D}^c} TD(k,|F|)$.*

In fact, letting $f = \sum_{F\in\mathcal{D}^c} |F|$, we obtain the stronger conclusion that there exists the partitioned ITD

$$TD(k,v) - (v-f)TD(k,1) - \sum_{F\in\mathcal{D}^c} TD(k,|F|).$$

Theorem 4.4 *Let $(X,\mathcal{D})$ be a PBD of order v. Suppose that some subset $\mathcal{D}^c \subseteq \mathcal{D}$ of blocks is a clear set. Let $F \in \mathcal{D} \setminus \mathcal{D}^c$. Suppose that for every $D \in \mathcal{D} \setminus \mathcal{D}^c$, there exists a PITD$(k,|D|;1^{|D|})$. Further suppose that for every $D \in \mathcal{D} \setminus (\mathcal{D}^c \cup \{F\})$, there exists a TD$(k,|D|)$. Then there exists a TD$(k,v) - TD(k,|F|)$.*

When the clear set is spanning (i.e., the union of the blocks is the set X of all elements in the PBD, or it is a *parallel class*), more flexibility exists. We introduce the appropriate language. A *group-divisible design* of *order* v, *blocksizes* K, and *type* $T = (t_1, \ldots, t_g)$, denoted (v,K)-GDD of type T, is a triple $(X,\mathcal{C},\mathcal{D})$, where X is a set of v elements, $\mathcal{C}$ is a partition of X into g classes (*groups*) $G_1, \ldots, G_g$, where $|G_i| = t_i$; and $\mathcal{D}$ is a set of subsets (*blocks*) of X, with the property that when $D \in \mathcal{D}$, we find $|D| \in K$. Moreover, every pair of elements appears together exactly once, either in a group or in a block. Often the type is written in exponential notation.

Now a (v,K)-PBD is equivalent to a (v,K)-GDD of type 1^v. A $\mathrm{TD}(k,n)$ is equivalent to a $(kn,\{k\})$-GDD of type n^k. Also, converting the groups of a (v,K)-GDD of type T into blocks, we obtain a $(v, K \cup T)$-PBD in which the images of the groups form a parallel class of blocks. Then restating Theorem 4.3 when the clear set is a parallel class is equivalent to:

Theorem 4.5 *Suppose that there is a (v, K)-GDD of type $T = (t_1, \ldots, t_s)$. Suppose that, for each $m \in K$, there is a PITD$(k, m; 1^m)$. Then there exists a PITD$(k, v; t_1, \ldots, t_s)$.*

Theorem 4.5 shows that one can employ the presence of a single parallel class. How can we use the presence of further parallel classes?

A PBD or GDD with element set X and block set $\mathcal{B}$ is *resolvable* if $\mathcal{B}$ can be partitioned into parallel classes. The partitioning into parallel classes is a *resolution.* Intermediate between GDDs and resolvable PBDs, we may have a PBD in which some, but not all, of the blocks are partitioned into parallel classes. Since resolvable PBDs form a special case, we treat this more general situation.

Theorem 4.6 *Let $(X, \mathcal{D})$ be a PBD of order v. Suppose that $\mathcal{D}$ is partitioned into $r+1$ classes $\mathcal{D}_1, \ldots, \mathcal{D}_{r+1}$, where $\mathcal{D}_i$ is a parallel class for $1 \leq i \leq r$, and $\mathcal{D}_{r+1}$ is arbitrary (possibly even empty). Suppose that, for each $D \in \mathcal{D}_{r+1}$, there is a PITD$(k, |D|; 1^{|D|})$.*

Now for $2 \leq i \leq r$, let ε_i be a nonnegative integer. Suppose that, for each $2 \leq i \leq r$, and each $D \in \mathcal{D}_i$, there exists a

$$TD(k, |D| + \varepsilon_i) - TD(k, \varepsilon_i) - |D|\ TD(k, 1).$$

Let $\sigma = \sum_{i=2}^{r} \varepsilon_i$. Then there exists a

$$TD(k, v + \sigma) - TD(k, \sigma) - \sum_{D \in \mathcal{D}_1} TD(k, |D|),$$

a partitioned ITD.

Theorem 4.6 applies equally well to resolvable GDDs, or GDDs with parallel classes; simply treat the groups as blocks forming a parallel class of an equivalent PBD.

4.2 Incomplete PBDs

Group-divisible designs are pairwise balanced designs with a spanning set of holes (the groups). Here we treat pairwise balanced designs with one hole. An *incomplete PBD* of *order n*, *blocksizes K*, and a *hole* of order h ((v, h, K)-IPBD) is a triple $(V, H, \mathcal{B})$. $|V| = v$, $|H| = h$, and $H \subset V$. $\mathcal{B}$ is a set of subsets of V, for which $B \in \mathcal{B}$ implies $|B| \in K$. Moreover, $(V, \mathcal{B} \cup \{H\})$ is a $(v, K \cup \{h\})$-PBD. Since any single block can be taken to form a clear set, we obtain from Theorem 4.3:

Corollary 4.7 *Suppose there exists a (v, h, K)-IPBD. Suppose that for each $m \in K$, there exists a PITD$(k, m; 1^m)$. Then there exists an ITD$(k, v; h)$, and in fact there exists a PITD$(k, v; h^1 1^{v-h})$.*

In a (v, h, K)-IPBD $(V, H, \mathcal{B})$, a *holey parallel class* is a set $\mathcal{P}$ of disjoint blocks, none of which meet the hole, and for which $V = H \cup \bigcup_{P \in \mathcal{P}} P$. One simple way to produce IPBDs with a holey parallel class is the following:

Lemma 4.8 *If there exists a* (v, K)*-GDD of type* $T = (t_1, t_2, \ldots, t_s)$*, then there exists a* $(v, t_1, K \cup \{t_2, \ldots, t_s\})$*-IPBD with a holey parallel class with block sizes in* $\{t_2, \ldots, t_s\}$*.*

Later we see other ways to produce IPBDs that have many holey parallel classes, so here we examine a method to use their presence:

Theorem 4.9 *Let* $(V, H, \mathcal{B})$ *be an* (v, h, K)*-IPBD, with blocks partitioned into classes* $\mathcal{P}_1, \ldots, \mathcal{P}_r, \mathcal{Q}$ *so that, for* $1 \leq i \leq r$*,* $\mathcal{P}_i$ *is a holey parallel class. For* $2 \leq i \leq r$*, let* ε_i *be a nonnegative integer. Now suppose that, for each* $B \in \mathcal{Q}$*, there exists a PITD*$(k, |B|; 1^{|B|})$*. Further suppose that, for each* $2 \leq i \leq r$*, and each* $B \in \mathcal{P}_i$*, there exists a PITD*$(k, |B| + \varepsilon_i; \varepsilon_i^1 1^{|B|})$*. Let* $\sigma = \sum_{i=2}^{r} \varepsilon_i$*. Then there exists a partitioned ITD*

$$TD(k, v + \sigma) - TD(k, \sigma + h) - \sum_{B \in \mathcal{P}_1} TD(k, |B|).$$

Actually, we could take $h = 0$; then the IPBD would be a PBD and the holey parallel classes would be parallel classes. Theorem 4.9 would then reduce to Theorem 4.6.

A (v, h, K)-IPBD can have both parallel classes *and* holey parallel classes. If such an event occurs, we can proceed as follows:

Theorem 4.10 *Let* $(V, H, \mathcal{B})$ *be an IPBD with* $|V| = v$ *and* $|H| = h$*. Suppose that* $\mathcal{B}$ *has a partition into classes* $\{\mathcal{P}_1, \ldots, \mathcal{P}_r, \mathcal{Q}_1, \ldots, \mathcal{Q}_s, \mathcal{R}\}$*, where the* $\{\mathcal{P}_i\}$ *are parallel classes, the* $\{\mathcal{Q}_i\}$ *are holey parallel classes, and* $\mathcal{R}$ *is the remaining set of blocks (possibly empty). Suppose that* $s \geq 1$*. Suppose that, for every* $B \in \mathcal{R}$*, there exists a PITD*$(k, |B|; 1^{|B|})$*.*

Choose nonnegative integers ε_i *for* $1 \leq i \leq r$*, and suppose that, for every* $B \in \mathcal{P}_i$*, there exists a PITD*$(k, |B| + \varepsilon_i; \varepsilon_i^1 1^{|B|})$*. Let* $\sigma = \sum_{i=1}^{r} \varepsilon_i$*.*

Choose nonnegative integers γ_i *for* $2 \leq i \leq s$*, and suppose that, for every* $B \in \mathcal{Q}_i$*, there exists a PITD*$(k, |B| + \gamma_i; \gamma_i^1 1^{|B|})$*. Let* $\sigma' = \sum_{i=2}^{s} \gamma_i$*.*

Then two outcomes are possible:

1. *If there exists a TD*$(k, \sigma + \sigma') - TD(k, \sigma')$*, then there exists a*

$$TD(k, v + \sigma + \sigma') - TD(k, h + \sigma') - \sum_{B \in \mathcal{Q}_1} TD(k, |B|).$$

2. *If there exists a TD*$(k, h + \sigma') - TD(k, \sigma')$*, then there exists a*

$$TD(k, v + \sigma + \sigma') - TD(k, \sigma + \sigma') - \sum_{B \in \mathcal{Q}_1} TD(k, |B|).$$

Some variants are possible. Prior to choosing the two outcomes, we find that two holes, one of size $h + \sigma'$ and the other of size $\sigma + \sigma'$, intersect in σ' elements. The last ingredients used to "break the tie" could themselves have holes, which would lead to even more holes in the final result. We do not pursue this.

However, it is necessary to explore what happens when we save back a parallel class instead of a holey parallel class. That leads to the next result:

Theorem 4.11 *Let $(V, H, \mathcal{B})$ be an IPBD with $|V| = v$ and $|H| = h$. Suppose that $\mathcal{B}$ has a partition into classes $\{\mathcal{P}_1, \dots, \mathcal{P}_r, \mathcal{Q}_1, \dots, \mathcal{Q}_s, \mathcal{R}\}$, where the $\{\mathcal{P}_i\}$ are parallel classes, the $\{\mathcal{Q}_i\}$ are holey parallel classes, and $\mathcal{R}$ is the remaining set of blocks (possibly empty). Suppose that $r \geq 1$. Suppose that, for every $B \in \mathcal{R}$, there exists a PITD$(k, |B|; 1^{|B|})$.*

Choose nonnegative integers ε_i for $2 \leq i \leq r$, and suppose that, for every $B \in \mathcal{P}_i$, there exists a PITD$(k, |B| + \varepsilon_i; \varepsilon_i^1 1^{|B|})$. Let $\sigma = \sum_{i=2}^{r} \varepsilon_i$.

Choose nonnegative integers γ_i for $1 \leq i \leq s$, and suppose that, for every $B \in \mathcal{Q}_i$, there exists a PITD$(k, |B| + \gamma_i; \gamma_i^1 1^{|B|})$. Let $\sigma' = \sum_{i=1}^{s} \gamma_i$.

Suppose that a PITD$(k, h + \sigma'; (\sigma')^1 1^h)$ exists.

Then there exists a partitioned ITD,

$$TD(k, v + \sigma + \sigma') - TD(k, \sigma + \sigma') - \sum_{B \in \mathcal{P}_1} TD(k, |B|).$$

One way to construct suitable IPBDs for Theorems 4.10 and 4.11 is to use the following result:

Lemma 4.12 *Let $(V, \mathcal{B})$ be a resolvable PBD with resolution $\{\mathcal{P}_1, \dots, \mathcal{P}_r\}$. Choose $B \in \mathcal{P}_1$. Then $(V, B, \mathcal{B} \setminus \{B\})$ is an IPBD whose blocks are partitioned into one holey parallel class $\mathcal{P}_1 \setminus \{B\}$, and $r - 1$ parallel classes $\mathcal{P}_2, \dots, \mathcal{P}_s$.*

One can go further, and consider structures in which there are many holes, and holey parallel classes associated with each. In this direction, one might consider "frames", for example. However, we do not explore this extension.

4.3 Making PBDs and GDDs

Making pairwise balanced designs and group-divisible designs is an industry in itself. Since Wilson's pioneering work on the asymptotic existence of designs (see [112]), constructions of PBDs and GDDs have flourished. Indeed one of the main reasons to construct incomplete transversal designs is to use them in constructing various other classes of designs.

We make no effort in this paper to describe all of the available constructions for PBDs and GDDs. Instead, we describe here some constructions from ITDs; in §5 we see a number of other constructions from other classes of designs.

Let us start with easy things. A TD(k, n) is itself a $(nk, \{k, n\})$-PBD; in fact, it is a $(nk, \{k\})$-GDD of type n^k.

Deleting any set of elements from a PBD produces another PBD, in which each deleted element is simply omitted from each block in which it occurred (blocks of size 0, 1, or 2 may result; blocks of size 0 or 1 can be omitted if we choose). Thus every PBD gives an enormous variety of smaller PBDs by this *puncturing* process. However, it should be clear that puncturing a TD randomly typically leads to a PBD with many block sizes. Since we are interested in being able to apply the theorems given earlier, we are concerned primarily with the cases when puncturing leads to relatively few block sizes.

We describe some concrete instances here. The first is obtained by puncturing points from ℓ of the groups.

Lemma 4.13 *Suppose that a TD($k+\ell, n$) exists. Choose integers $b_1, \ldots, b_\ell$ so that $0 \leq b_i \leq n$ for $1 \leq i \leq \ell$. Then there exists a $(kn + \sum_{i=1}^{\ell} b_i, \{k, k+1, \ldots, k+\ell\})$-GDD of type $n^k b_1^1 b_2^1 \cdots b_\ell^1$.*

Of course, in Lemma 4.13, blocks of sizes $\{k, k+1, \ldots, k+\ell\}$ are all *possible*. But whether a block of a particular size arises depends on the structure of the TD and the actual points deleted. Nevertheless, we can apply Theorem 4.5 to this GDD. Knowing the actual block sizes could result in a stronger application of that theorem. We return to this point in §6.7.

Truncating a single group can yield useful IPBDs:

Lemma 4.14 *Suppose that a TD($k+1, n$) exists. Let $0 \leq \rho \leq n-1$. Then a $(kn+\rho, \rho, \{k, k+1, n\})$-IPBD exists having one holey parallel class of type n^k, $n-\rho$ parallel classes of type k^n, and the remaining ρn blocks of size k.*

An extreme case of Lemma 4.14 is when the whole group is deleted. This is equivalent to the following well–known result:

Lemma 4.15 *A resolvable TD(k, n) is equivalent to a TD($k+1, n$).*

Puncturing partitioned ITDs leads to GDDs with groups arising from the holes:

Lemma 4.16 *Suppose that there exists a PITD($k+1, n; 1^n$). Let $0 \leq \alpha \leq n$. Then there exists a $(nk+\alpha, \alpha, \{n, k+1, k\})$-IPBD with a holey parallel class of type n^k, a parallel class of type $(k+1)^\alpha k^{n-\alpha}$, and all other blocks of sizes k (whenever $\alpha < n$) and $k+1$ (whenever $\alpha > 0$).*

Puncturing one group of an incomplete TD with one hole leads to:

Lemma 4.17 *Suppose that there exists an ITD($k+1, n; h$). Let $0 \leq \alpha < h$. Suppose that there exists a PITD($k, k+1; 1^{k+1}$) and a PITD($k, k; 1^k$). Then there exists a PITD($k, nk+\alpha, (kh+\alpha)^1 k^{n-h}$).*

Actually, more can be said since $h-\alpha$ holey parallel classes of blocks of size k missing the hole of size $kh+\alpha$ are present, and we have used only a single one here.

We see more sophisticated ways to puncture a TD in §6.7; we give one of the simpler cases here:

Lemma 4.18 *Suppose that a TD($k+\ell, n$) exists with $\ell \geq 2$. Let $1 \leq \alpha \leq n$. Then a $(nk+\alpha+\ell-1, \{k, k+1, k+2, k+\ell\})$-GDD of type $n^k \alpha^1 1^{\ell-1}$ exists.*

Using resolvable TDs, we also obtain:

Lemma 4.19 *Suppose that a TD$(k+\ell+1,n)$. Then there is a $(nk+\ell,k+\ell,\{k,k+1,n\})$-IPBD having a holey parallel class of type k^{n-1}, a parallel class of type n^k1^ℓ, and $n-1$ parallel classes of type $(k+1)^\ell k^{n-\ell}$.*

Another useful puncture is to delete points from a block, rather than from a group:

Lemma 4.20 *Suppose that a TD(k,n) exists. Let ρ be an integer satisfying $0 \le \rho \le k$. Then there exists a $(k(n-1)+\rho,\rho,\{k,k-1,n,n-1\})$-IPBD and a $(k(n-1)+\rho,\{k,k-1,\rho\})$-GDD of type $n^\rho(n-1)^{k-\rho}$.*

In the IPBD, blocks of size n appear only if $\rho > 0$ and blocks of size $n-1$ appear only if $\rho < k$. In both the IPBD and the GDD, blocks of size $k-1$ appear if and only if $\rho < k$; blocks of size k appear if and only if $\rho > 0$ or *$k \le n$.*

Deleting a whole block gives, on two occasions, PBDs that ought to be noted.

Lemma 4.21 *If a TD$(n+1,n)$ exists, then there is a $(n^2-1,\{n\})$-GDD of type $(n-1)^{n+1}$. (In fact, they are equivalent.) Further deleting all elements in one group, we obtain a resolvable $(n(n-1),\{n-1\})$-GDD of type n^{n-1}.*

When a block is deleted from a resolvable TD, information about parallel classes can be retained:

Lemma 4.22 *Suppose that a TD$(k+1,n)$ exists. Let ρ be an integer satisfying $0 \le \rho \le k$. Then there exists a $(k(n-1)+\rho,\rho,\{k,k-1,n,n-1\})$-IPBD having one holey parallel class of type k^{n-1}, one parallel class of type $(n-1)^\rho n^{k-\rho}$, and $n-1$ parallel classes of type $k^{n-k+\rho}(k-1)^{k-\rho}$.*

4.4 The Bose–Shrikhande–Parker Theorem

The (general form of the) Bose-Shrikhande-Parker theorem [23, 30] exploits additional structure occurring in some PBD. We generalize the notion of parallel class. An *α-parallel class* in a PBD $(V,\mathcal{B})$ is a set $\mathcal{C} \subseteq \mathcal{B}$ of blocks, with the property that every $x \in V$ appears in *exactly* α blocks of $\mathcal{C}$. Evidently, a 1–parallel class is just a parallel class.

An α-parallel class $\mathcal{C}$ is *symmetric* if every block in $\mathcal{C}$ has size α. It is easy to verify in this case that the number of blocks in $\mathcal{C}$ coincides with the number of elements in V — hence the term symmetric.

A *separable PBD* is one whose blocks can be partitioned into 1–parallel classes and symmetric parallel classes; within each class, all blocks have the same size.

Now we can state the Bose–Shrikhande–Parker theorem:

Theorem 4.23 *Let $(V,\mathcal{B})$ be a (v,K)-PBD, and suppose that $\mathcal{B}$ can be partitioned into classes $\mathcal{P}_1,\ldots,\mathcal{P}_r,\mathcal{S}_1,\ldots,\mathcal{S}_s$. For $1 \le i \le r$, $\mathcal{P}_i$ is a parallel class. For $1 \le i \le s$, $\mathcal{S}_i$ is a symmetric α_i-parallel class. Now let $\varepsilon_i \in \{0,1\}$ for $1 \le i \le s$, and suppose that a*

PITD$(k,\alpha_i;1^{\alpha_i})$ *exists if* $\varepsilon_i = 0$*, and that a PITD*$(k,\alpha_i+1;1^{\alpha_i+1})$ *exists if* $\varepsilon_i = 1$*. Let* $\sigma = \sum_{i=1}^{s}\varepsilon_i\alpha_i$*.*

Let γ_i *be a positive integer, for* $2 \le i \le r$*. Suppose that, for* $2 \le i \le r$ *and for each* $B \in \mathcal{P}_i$*, there exists a PITD*$(k,|B|+\gamma_i;\gamma_i^1 1^{|B|})$ *exists. Let* $\sigma' = \sum_{i=2}^{r}\gamma_i$*.*

Then, if $r \ge 1$*, there exists the partitioned ITD*

$$TD(k, v+\sigma+\sigma') - TD(k,\sigma+\sigma') - \sum_{B\in\mathcal{P}_1} TD(k,|B|).$$

If $r = 0$*, we instead obtain the partitioned ITD*

$$TD(k, v+\sigma) - TD(k,\sigma) - vTD(k,1).$$

4.5 Making Separable PBDs

The Bose–Shrikhande–Parker theorem is a generalization of Theorem 4.6 to separable PBDs. However, it is difficult to find examples of separable PBDs that are not resolvable. We see in §5 that examples arise from cyclic block designs and symmetric block designs. The only other general construction for separable PBDs is due to Brouwer [27]:

Theorem 4.24 *Let* q *be a prime power, and let* $0 < t \le q^2-q+1$*. Then there exists a separable* $(t(q^2+q+1),\{t,q+t\})$*-PBD which has a partitioning with* $q^2-q+1-t$ *parallel classes of blocks of size* t*, and one symmetric* $(q+t)$*-parallel class.*

5 Steiner Systems, Symmetric Designs and Difference Sets

A *Steiner system* of *order* v and *blocksize* κ, denoted $S(2,\kappa,v)$, is a $(v,\{\kappa\})$-PBD. (This is actually a *Steiner 2-design*, but we only have occasion to use the case of $t = 2$ here; it is also a (balanced incomplete) block design, but we only treat the case when $\lambda = 1$. For these reasons, we have adopted the Steiner system notation here.)

An $S(2,\kappa,v)$ is *symmetric* when the number of blocks in the design, namely $\frac{v(v-1)}{\kappa(\kappa-1)}$ is equal to v (i.e., $v = \kappa(\kappa-1)+1$). A symmetric $S(2,\kappa,v)$ is equivalent to a *projective plane* of *order* $n = \kappa - 1$, with $v = n^2+n+1$ elements (and n^2+n+1 blocks or lines). In a projective plane, every two distinct blocks intersect in one element.

An $S(2,\kappa,v)$ is *cyclic* when there is an automorphism of the design that is a v–cycle.

First, the basics:

Lemma 5.1 *Removing one element from a projective plane of order* n*, and treating the resulting blocks of size* n *as groups, a TD*$(n+1,n)$ *is produced.*

We can also remove a whole block:

Lemma 5.2 *Removing one block from a projective plane of order n (or one group from a TD$(n+1,n)$, a resolvable TD(n,n) (*affine plane of order *n, or $S(2,n,n^2)$) results.*

Moreover, every TD(n,n) is resolvable and can be extended to a projective plane.

5.1 Arbitrary $S(2,\kappa,v)$

An $S(2,\kappa,v)$ is, of course, itself a PBD. However, sometimes truncating this special type of PBD can lead to extra information. We can truncate points from a single block:

Lemma 5.3 *If an $S(2,\kappa,v)$ exists, then for $0 \le x \le \kappa$, a $(v-x,\kappa-x,\{\kappa,\kappa-1\})$-IPBD exists having x holey parallel classes of type $(\kappa-1)^{(v-\kappa)/(\kappa-1)}$. It has blocks of size κ unless $x=\kappa$ and the $S(2,\kappa,v)$ is symmetric.*

Next we delete a small number of points, not all from the same block.

Lemma 5.4 *If an $S(2,\kappa,v)$ exists, then there exists a $(v-3,\{\kappa-2,\kappa-1,\kappa\})$-PBD in which there are exactly three blocks of size $\kappa-2$, and they form a near clear set.*

If an $S(2,\kappa,v)$ exists, then there exists a $(v-4,\{\kappa-2,\kappa-1,\kappa\})$-PBD in which there are exactly four blocks of size $\kappa-2$, and they form a near clear set.

We can naturally delete points all over if we so desire, but to obtain useful results we want to minimize the number of different block sizes. With this in mind, we give another definition. If A is a set of s points, no three of which lie on a block, then A is an *s–arc*.

Lemma 5.5 *If an $S(2,\kappa,v)$ exists having an s-arc, then for all $1 \le x \le s$, there exists a $(v-x,\{\kappa-2,\kappa-1,\kappa\})$-PBD. Blocks of size $\kappa-2$ always occur when $x>1$. Blocks of size $\kappa-1$ appear unless $v=1+(x-1)(\kappa-1)$. Blocks of size κ always appear.*

Examples of designs with useful arcs appear in §5.3.

Existence of block designs is a central problem in combinatorial design theory, and there is a huge literature. For existence results, see [71]. Much is known about resolvability of block designs, furnishing many examples of resolvable PBDs.

5.2 Cyclic $S(2,\kappa,v)$

Cyclic $S(2,\kappa,v)$s have been studied extensively; see [3]. A cyclic $S(2,\kappa,v)$ can exist only if $v \equiv 1,\kappa \pmod{\kappa(\kappa-1)}$. When $v \equiv 1 \pmod{\kappa(\kappa-1)}$, all block orbits under the cyclic automorphism have length v (they are *full*). When $v \equiv \kappa \pmod{\kappa(\kappa-1)}$, one block orbit has length $\frac{v}{\kappa}$ (it is *short*), and the rest are full.

A full orbit of blocks can be easily checked to be a κ-parallel class. Hence every cyclic $S(2,\kappa,v)$ is separable, with $\lfloor\frac{v}{\kappa(\kappa-1)}\rfloor$ κ-parallel classes, and one parallel class if the short orbit is present, none otherwise. We can therefore apply the Bose–Shrikhande–Parker theorem to cyclic $S(2,\kappa,v)$s.

5.3 Symmetric Designs

First we remark on a basic filling result that does not follow from filling the corresponding PBD [67].

Lemma 5.6 *If a symmetric $S(2,\kappa,v)$ exists and a TD(k,κ) exists, then a TD(k,v) exists.*

Next observing that a symmetric design $S(2,\kappa,v)$ is itself a single $((v-1)/(\kappa-1))$-parallel class, we can apply the Bose–Shrikhande–Parker theorem to obtain:

Lemma 5.7 *If a symmetric $S(2,\kappa,v)$ exists and a PITD$(k,\kappa+1;1^{\kappa+1})$ exists, then a TD$(k,v+\frac{v-1}{\kappa-1})-TD(k,\frac{v-1}{\kappa-1})$ exists.*

See also [67].

Certain projective planes have large arcs:

Theorem 5.8 *The desarguesian projective plane of order q (a prime power) contains a $(q+1)$–arc (an* oval*) if q is odd, and contains a $(q+2)$–arc (a* hyperoval*) if q is even.*

5.4 Line–flips in Affine Planes

Suppose that a TD(n,n) exists; this is an affine plane of order n. Now choose an integer x with $1 \leq x < n$, and choose one block B. Delete all points from x of the groups *except* those on block B. Next delete all points on block B in the remaining $n-x$ groups. The resulting PBD has blocks of five types:

1. a single block on x points, which is the truncation of B;
2. $n-1$ disjoint blocks each of size $n-x$, which are the truncations of the blocks disjoint from B in the affine plane;
3. $n-x$ disjoint blocks each of size $n-1$, which are the truncations of the remaining groups;
4. blocks of size $n-x-1$ that do not intersect the truncation of B (in the affine plane, they did intersect B);
5. blocks of size $n-x+1$ that do intersect the truncation of B.

Types (1) and (2) form a parallel class; so also do types (1) and (3). Thus adding a point at infinity to the blocks of type (1) and (2), called a *type A extension*, gives a $\{n-x-1,n-x+1\}$-GDD of type $(n-1)^{n-x}(x+1)^1$. On the other hand, adding a point at infinity to blocks of types (1) and (3) gives a $\{n-x-1,n,n-x+1\}$-GDD of type $(n-x)^{n-1}(x+1)^1$ (a *type B extension*). Greig [49] observes that either GDD can be extended with a further point at infinity to form a PBD on $(n-x)(n-1)+x+2$ with block sizes $\{n,n-x-1,n-x+1,(x+2)^\star\}$. The superscript $\star$ indicates that a block of size $x+2$ is present, and that all other blocks have sizes from $\{n,n-x-1,n-x+1\}$.

5.5 Difference Sets

Singer [94] showed that the desarguesian projective plane of order q (a prime power) has a representation as a cyclic difference set. This provides a mechanism for finding other configurations in desarguesian planes.

Let ? be an additively written group of order v. A κ-subset D of ? is a (v,κ,λ)-*difference set* of *order* $n = \kappa - \lambda$ if every nonzero element of ? has exactly λ representations as a difference $d - d'$ of distinct elements from D. The difference set is *abelian* or *cyclic* if the group ? has the corresponding property.

The development of a difference set D under the action of the group ? is a symmetric design; when $\lambda = 1$, it is a projective plane of order n. Thus our earlier remarks apply to the symmetric design. But here we may obtain more information.

We consider a $(q^2+q+1, q+1, 1)$–difference set D over the cyclic group Z_{q^2+q+1}, using the usual representation over the integers modulo q^2+q+1. For any divisor d of q^2+q+1, denote by $D_{i,d}$ the elements of D that are congruent to i modulo d. For an arbitrary subset $\mathcal{I} \subset \{0,1,\ldots,d-1\}$, let $D_{\mathcal{I},d} = \bigcup_{i\in\mathcal{I}} D_{i,d}$. Then we have the following result, first studied by Brouwer [27] and later extended by Greig [49]:

Theorem 5.9 *Let D be a $(q^2+q+1, q+1, 1)$–difference set over the integers modulo q^2+q+1. Let d be a divisor of q^2+q+1, and $\mathcal{I} \subseteq \{0,1,\ldots,d-1\}$. Then the collection of blocks*

$$\{\{x+i\} : 0 \le i \le q^2+q,\ x+i \bmod d \in \mathcal{I}\}$$

is a pairwise balanced design on $|\mathcal{I}|\frac{q^2+q+1}{d}$ elements.

The relevance of Theorem 5.9 is that it produces a PBD having at most d different block sizes.

5.6 Configurations in Projective Planes

In §5.5, we saw that projective planes arising from difference sets can embed a pairwise balanced design that often has "few" block sizes. We are interested in this phenomenon for a number of reasons. It provides a way to construct pairwise balanced designs, of course. But what is more critical for us is that it tells us something about the structure of the TD$(n+1,n)$ that arises from the plane — and this information can be helpful in predicting the block sizes that result when we puncture the TD. There is a third reason as well, namely that when PBDs live in a projective plane, we can use this to produce more PBDs. We pursue this in §5.7, but for now we explore results on when PBDs live in projective planes.

Ovals, Hyperovals and Denniston Arcs

Arcs (ovals and hyperovals) form one important class of pairwise balanced designs inhabiting projective planes (Theorem 5.8), although the PBDs themselves are quite trivial. However, associated with the exterior lines of a hyperoval are a number of important PBDs contained in the plane [92]:

Theorem 5.10 *The desarguesian projective plane of order $q = 2^\alpha$ contains*

1. *a resolvable $(\binom{q}{2}, \{\frac{q}{2}\})$-PBD;*
2. *a $(\binom{q+2}{2}, \{\frac{q}{2}+1, q+1\})$-PBD; and*
3. *a resolvable $(\binom{q+1}{2}, \{\frac{q}{2}, q\})$-PBD.*

Denniston arcs [41] provide a generalization of these:

Theorem 5.11 *The desarguesian projective plane of order $q = 2^\alpha$ contains, for every $1 \leq \beta < \alpha$,*

1. *a resolvable $(2^{\alpha+\beta} - 2^\alpha + 2^\beta, \{2^\beta\})$-PBD;*
2. *a $(2^{2\alpha} + 2^{\alpha+1} - 2^{\alpha+\beta} - 2^\beta + 1, \{2^\alpha - 2^\beta + 1, 2^\alpha + 1\})$-PBD; and*
3. *a resolvable $((2^\alpha - 2^\beta)(2^\alpha + 1), \{2^\alpha - 2^\beta, 2^\alpha\})$-PBD.*

Greig [49] employs ovals in planes of odd order to prove:

Theorem 5.12 *If q is an odd prime power, the desarguesian plane of order q contains*

1. *a GDD on $\binom{q}{2}$ points with uniform group size $\frac{q-1}{2}$, and block sizes in $\{\frac{q-1}{2}, \frac{q+1}{2}\}$; and*
2. *a GDD on $\binom{q+1}{2}$ points with uniform group size $\frac{q+1}{2}$, and block sizes in $\{\frac{q+1}{2}, \frac{q+3}{2}\}$.*

Subplanes and Baer Subplanes

More complex examples are given by subplanes of a plane. Simple numerical arguments show that a projective plane of order q can have a projective subplane of order p only if $q \geq p^2$. In the positive direction, we have [18]:

Lemma 5.13 *The desarguesian projective plane of order p^α has a subplane of order p^β whenever $\beta \mid \alpha$.*

The extreme case when $\alpha = 2\beta$ is especially important. In this case, the subplane is a *Baer subplane*, and some elementary counting arguments provide us with useful information. Let $q = p^\beta$ and $q^2 = p^{2\beta}$. Let $(V, \mathcal{B})$ be the plane of order q^2, and $(X, \mathcal{D})$ be its Baer subplane of order q.

Lemma 5.14 1. *Every point $x \in X$ lies on $q+1$ lines of $\mathcal{B}$ that intersect X in $q+1$ points, and on $q^2 - q$ lines of $\mathcal{B}$ that contain only x from X.*

2. *Every point of $V \setminus X$ lies on one line of $\mathcal{B}$ that intersects X in $q+1$ points, and lies on q^2 lines of $\mathcal{B}$ that intersect X in one point.*

3. Hence, all lines of $\mathcal{B}$ intersect X in either 1 or $q+1$ points.

Removing the points in X from the plane yields a $(q^4-q,\{q^2-q,q^2\})$-PBD. Considering any point $x \in X$, we find that the blocks containing X form a parallel class of this PBD, and hence we in fact obtain a $(q^4-q,\{q^2-q,q^2\})$-GDD of type $(q^2-q)^{q+1}(q^2)^{q^2-q}$, $(Y,\mathcal{C})$.

Now consider a block of size q^2-q in $\mathcal{C}$. It cannot intersect any group of size q^2-q (lines meet at a single point in the projective plane of order q^2), so it must intersect all groups of size q^2. In fact, all blocks and groups of size q^2-q are disjoint, so we have [95]:

Lemma 5.15 *If a projective plane of order q^2 has a Baer subplane of order q, there exists a $(q^4-q,\{q^2\})$-GDD of type $(q^2-q)^{q^2+q+1}$.*

Now a block of size q^2 from $\mathcal{C}$ must intersect all groups of size q^2, and precisely q of the groups of size q^2-q. Thus we can delete all but x of the groups of size q^2 to obtain:

Lemma 5.16 *If a projective plane of order q^2 has a Baer subplane of order q, then for all $0 \leq x \leq q^2-q$, there exist*

1. *a $((q^2-q)(q+1)+xq^2,\{q+x,x\})$-GDD of type $(q^2-q)^{q+1}(q^2)^x$; and*
2. *a $((q^2-q)(q+1)+xq^2,\{q+x,q^2\})$-GDD of type $(q^2-q)^{q+1}x^{(q^2)}$.*

Baer subplanes can be exploited further yet; see, for example, [59] for the following:

Lemma 5.17 *The desarguesian projective plane of order q^2 can be partitioned into q^2-q+1 element–disjoint Baer subplanes (each on q^2+q+1 points).*

Considering any line of the plane, simple counting shows that it intersects one Baer subplane of this partition in $q+1$ points, and the remaining q^2-q subplanes in one point each. So retaining points of t of the subplanes in the partition, we obtain:

Lemma 5.18 *Using the desarguesian projective plane of order q^2 (q a prime power), for each $1 \leq t \leq q^2-q$, we obtain a $(t(q^q+q+1),\{q+t,t\})$-PBD in which the blocks of size t are partitioned into $q^2-q+1-t$ parallel classes.*

Other specific planes have subplanes of interest: the Hughes plane of order 9 has a subplane of order 2 [40]; indeed it has a partition into subplanes of order two [62]. The Hughes plane of order 25 contains subplanes of orders 2 and 3 [85]. For every odd prime power q, there is a non-desarguesian plane of order q^2 (the Hall plane) that contains a subplane of order 2 [80]. A complete survey of subplanes is *not* attempted here.

Affine Subplanes

Now we examine other structures in planes. In the direction of affine planes residing in projective planes, Ostrom and Sherk [81] and Rigby [86] proved:

Theorem 5.19 *The desarguesian projective plane of order q (a prime power) contains an affine plane of order 3 (an $S(2,3,9)$) if and only if $q \equiv 0, 1 \pmod 3$.*

The notion of "containment" in Theorem 5.19 is that a subset of the points is selected, and the intersections of all lines with these points induce shorter lines; then keeping all such truncated lines on two or more points gives the affine plane.

Subsquares

Often a subplane (projective or affine) is not present, but useful portions are. For example, considering the standard construction of the desarguesian plane, we find [37]:

Lemma 5.20 *In the desarguesian plane of order p^α, for each $0 \leq \beta \leq \alpha$, there is embedded a $(3p^\beta + 1, \{3, p^\beta + 1\})$-PBD having three blocks of size $p^\beta + 1$ meeting in a single point, and all other blocks of size three.*

Actually, a more convenient way to express this is to observe that, when we remove the point common to the three "long" blocks, we form a $\mathrm{TD}(p^\alpha + 1, p^\alpha)$. Truncating to the $\mathrm{TD}(3, p^\alpha)$ on the three special groups, and interpreting this TD as a latin square, we are essentially noting in Lemma 5.20 that this latin square of size p^α has a subsquare of size p^β.

Using the structure of the finite field, one can extend this to obtain:

Lemma 5.21 *In the desarguesian plane of order p^α, for each $0 \leq \beta \leq \alpha$, there is embedded a $((p+1)p^\beta + 1, \{p + 1, p^\beta + 1\})$-PBD having $p + 1$ blocks of size $p^\beta + 1$ meeting in a single point, and all other blocks of size $p + 1$.*

When $\beta = \alpha - 1$ in Lemma 5.21, all lines of the plane meet the sub-TD in $p^\beta + 1$, $p + 1$, or 1 points. The latter tangent lines induce a design in the dual plane; this tangent design is a $\{p^\alpha, p^\alpha - p^{2\beta-\alpha} + 1\}$-GDD of type $(p^\alpha - p^\beta)^{p^\alpha + p^\beta}(p^\alpha - p^{\alpha-\beta})^1$.

Blocking Sets and Generalizations

We can represent the desarguesian plane of order q using elements $((\mathsf{GF}(q) \cup \{\infty\}) \times \mathsf{GF}(q)) \cup \{\infty\}$. Taking ω as a primitive element, the lines are:

$$
\begin{array}{ll}
\{\infty, (\xi, 0), (\xi, \omega^0), \ldots, (\xi, \omega^{q-2}\} & \text{for} \quad \xi \in \mathsf{GF}(q) \\
\{(\infty, 0), (0, \xi), (\omega^0, \xi), \ldots, (\omega^{q-2}, \xi)\} & \text{for} \quad \xi \in \mathsf{GF}(q) \\
\{(\infty, \omega^j), (0, \xi), (\omega^0, \xi + \omega^{0+j}), \ldots, (\omega^{q-2}, \xi + \omega^{q-2+j})\} & \text{for} \quad \xi \in \mathsf{GF}(q),\ 1 \leq j < q - 1 \\
\{\infty, (\infty, 0), (\infty, \omega^0), \ldots, (\infty, \omega^{q-2}\} &
\end{array}
$$

Write $q = ef + 1$, and consider the elements

$$\{(0, \omega^i), (\omega^i, 0), (\infty, -\omega^i) : i \equiv 0 \pmod{f}\}.$$

When the lines of the plane are restricted to this set of $3s$ points, every line is truncated to 0, 1, 3, or e points. Indeed the structure is a TD$(3, e)$. It follows that a $(q^2 + q + 1 - 3e, \{q + 1, q, q - 2, q + 1 - e\})$-PBD exists in which the three blocks of size $q + 1 - e$ form a near clear set. Applications of this are described in [36] and [50]. When $f = 2$, adjoining $\{\infty, (\infty, 0), (0, 0)\}$ to the $3e$ points yields a minimal blocking set [31].

5.7 Line-flips in Projective Planes

When a PBD is embedded in a projective plane, we can exploit the structure of the enclosing plane to form other PBDs. Simply taking all points of the plane *not* in the PBD, for example, gives:

Lemma 5.22 *If a (v, K)-PBD is embedded in a projective plane of order n, and there is a one–to–one correspondence between blocks of the PBD and lines of the plane so that each block is extended to the corresponding line (which may require adding blocks of sizes 0 and 1 to the PBD), then there exists a $(n^2 + n + 1 - v, \overline{K})$-PBD, where $\overline{K} = \{n + 1 - s : s \in K\}$.*

In fact, the number of blocks of size $n + 1 - s$ in the resulting PBD is the same as the number of blocks of size s in the original PBD.

One can also do a "line–flip", by choosing some block of the PBD, deleting the points on this block and instead adding the points on the line of the plane which extends this block, but not on the block itself [49]. One obtains the following:

Lemma 5.23 *Suppose that a (v, K)-PBD is embedded in a projective plane of order n, and there is a one–to–one correspondence between blocks of the PBD and lines of the plane so that each block is extended to the corresponding line (which may require adding blocks of sizes 0 and 1 to the PBD). Suppose further that the embedded PBD has a block of size s. Let $\widehat{K} = \{s - 1, s + 1 : s \in K\}$. Then there exists a $(v + n + 1 - 2s, n + 1 - s, \widehat{K})$-IPBD.*

Examples of PBDs that inhabit projective planes are given in §5.6, and also arise from Theorem 5.9.

6 Wilson's Theorem

Wilson's theorem, and all of its variants discussed here, start with a transversal design (or incomplete transversal design) of order t of blocksize $k + \ell$. We refer to this design as the *master design.*

The master design is always taken to be $(V, \mathcal{G}, \mathcal{B})$, although it may have additional structure, or holes. Let $\mathcal{G} = \{G_1, \ldots, G_k, E_1, \ldots, E_\ell\}$. Let $E_i = \{x_{i1}, \ldots, x_{it}\}$. For each $x_{ij} \in \bigcup_{i=1}^{\ell} E_i$, let w_{ij} be a nonnegative integer, the *weight* of x_{ij}. For each block $B \in \mathcal{B}$, let $w_i^B = w_{ij}$ when $B \cap E_i = \{x_{ij}\}$.

6.1 Transversal Designs as Master Designs

First we give what has come to be accepted as the basic form of Wilson's theorem, although Wilson [111] gave it in the case that $w_{ij} \in \{0,1\}$ for all $1 \le i \le \ell$, $1 \le j \le t$, and $\lambda = \mu = 1$.

Theorem 6.1 *Suppose that a $TD_\mu(k+\ell,t)$ exists. Suppose that for each $B \in \mathcal{B}$, there exists*

$$TD_\lambda(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{\ell} TD_\lambda(k, w_i^B).$$

Then there exists a

$$TD_{\lambda\mu}(k, mt + \sum_{i=1}^{\ell}\sum_{j=1}^{t} w_{ij}) - \sum_{i=1}^{\ell} TD_{\lambda\mu}(k, \sum_{j=1}^{t} w_{ij})$$

Of course, if we can fill some or all of the holes, further incomplete transversal designs result. When $\mu = 1$, we can obtain different holes as well:

Theorem 6.2 *Suppose that a $TD_1(k+\ell,t)$ exists. Let $F \in \mathcal{B}$. Suppose that for each $B \in \mathcal{B} \setminus \{F\}$, there exists*

$$TD_\lambda(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{\ell} TD_\lambda(k, w_i^B).$$

Suppose that, for $1 \le i \le \ell$, there exists

$$TD_\lambda(k, \sum_{j=1}^{t} w_{ij}) - TD_\lambda(k, w_i^F).$$

Then there exists

$$TD_\lambda(k, mt + \sum_{i=1}^{\ell}\sum_{j=1}^{t} w_{ij}) - TD_\lambda(k, m + \sum_{i=1}^{\ell} w_i^F).$$

We now change the structure of some of the ingredients. Let $\mathcal{B}_1$ be the blocks B for which $x_{11} \in B \in \mathcal{B}$, and let $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$.

Theorem 6.3 *Suppose that a $TD_1(k+\ell,t)$ exists. Suppose that for each $B \in \mathcal{B}_2$, there exists*

$$TD_\lambda(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{\ell} TD_\lambda(k, w_i^B).$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive integers with $\sum_{i=1}^r \alpha_i \leq m$. Suppose that for each $B \in \mathcal{B}_1$, there exists

$$TD_\lambda(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{r} TD_\lambda(k, \alpha_i) - \sum_{i=1}^{\ell} TD_\lambda(k, w_i^B).$$

Then there exists

$$TD_\lambda(k, mt + \sum_{i=1}^{\ell}\sum_{j=1}^{t} w_{ij}) - \sum_{i=1}^{r} tTD_\lambda(m, \alpha_i) - \sum_{i=1}^{\ell} TD_\lambda(k, \sum_{j=1}^{t} w_{ij}).$$

Finally we change the structure of all of the ingredients:

Theorem 6.4 *Suppose that a $TD_\mu(k+\ell, t)$ exists. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive integers with $\sum_{i=1}^r \alpha_i \leq m$. Suppose that for each $B \in \mathcal{B}$, there exists*

$$TD_\lambda(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{r} TD_\lambda(k, \alpha_i) - \sum_{i=1}^{\ell} TD_\lambda(k, w_i^B).$$

Then there exists

$$TD_{\lambda\mu}(k, mt + \sum_{i=1}^{\ell}\sum_{j=1}^{t} w_{ij}) - \sum_{i=1}^{r} TD_{\lambda\mu}(m, \alpha_i t) - \sum_{i=1}^{\ell} TD_{\lambda\mu}(k, \sum_{j=1}^{t} w_{ij}).$$

6.2 Incomplete Transversal Designs as Master Designs

In the preceding constructions, we saw how incomplete transversal designs can be used in conjunction with a master design that is a transversal design. Here we examine variants where the master design itself is incomplete.

Theorem 6.5 *Let $\beta_1, \ldots, \beta_u$ be positive integers with $\sum_{a=1}^u \beta_a \leq t$. Suppose that there exists a master design, a*

$$TD_\mu(k+\ell, t) - \sum_{a=1}^{u} TD_\mu(k+\ell, \beta_a).$$

For $1 \leq a \leq u$, let O_a be the points in the ITD that lie in the a^{th} hole of size β_a (so that O_a contains $\beta_a(k+\ell)$ elements in total). Let $z_{ia} = \sum_{x_{ij} \in (E_i \cap O_a)} w_{ij}$. Suppose that for each $B \in \mathcal{B}$, there exists

$$TD_\lambda(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{\ell} TD_\lambda(k, w_i^B).$$

Suppose further that, for $1 \leq a \leq u$, there exists

$$TD_{\lambda\mu}(k, m\beta_a + \sum_{i=1}^{\ell} z_{ia}) - \sum_{i=1}^{\ell} TD_{\lambda\mu}(k, z_{ia}).$$

Then there exists a

$$TD_{\lambda\mu}(k, mt + \sum_{i=1}^{\ell}\sum_{j=1}^{t} w_{ij}) - \sum_{i=1}^{\ell} TD_{\lambda\mu}(k, \sum_{j=1}^{t} w_{ij})$$

Actually, the holes that arise on the extra ℓ levels, and the holes that arise from the holes in the master design in the construction are not disjoint. In Theorem 6.5, we have elected to fill the latter and leave the former. In Theorem 6.6, we do the opposite. Since the theorems differ only in the last set of ingredients, they look cosmetically similar. Nevertheless, we state the conditions of the theorem in their entirety.

Theorem 6.6 *Let $\beta_1, \ldots, \beta_u$ be positive integers with $\sum_{a=1}^{u} \beta_a \leq t$. Suppose that there exists a master design, a*

$$TD_{\mu}(k+\ell, t) - \sum_{a=1}^{u} TD_{\mu}(k+\ell, \beta_a).$$

For $1 \leq a \leq u$, let O_a be the points in the ITD that lie in the a^{th} hole of size β_a (so that O_a contains $\beta_a(k+\ell)$ elements in total). Let $z_{ia} = \sum_{x_{ij} \in (E_i \cap O_a)} w_{ij}$. Suppose that for each $B \in \mathcal{B}$, there exists

$$TD_{\lambda}(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{\ell} TD_{\lambda}(k, w_i^B).$$

Suppose further that, for $1 \leq i \leq \ell$, there exists

$$TD_{\lambda\mu}(k, \sum_{j=1}^{t} w_{ij}) - \sum_{a=1}^{u} TD_{\lambda\mu}(k, z_{ia}).$$

Then there exists a

$$TD_{\lambda\mu}(k, mt + \sum_{i=1}^{\ell}\sum_{j=1}^{t} w_{ij}) - \sum_{a=1}^{u} TD_{\lambda\mu}(k, m\beta_a + \sum_{i=1}^{\ell} z_{ia}).$$

6.3 Du Variations

Du [45] considers another use of incomplete transversal designs in Wilson's theorem. Denote by ITD$^s(k,t;h)$ an ITD$(k,t;h)$ that has s disjoint holey parallel classes of blocks. An ITD$^1(k,t;h)$ is equivalent to a PITD$(k,t;h^1 1^{t-h})$. Then we have:

Theorem 6.7 *Suppose that an ITD$^s(k,t;h)$ exists. Suppose that a TD(k,m) exists. Let $w_1, \ldots, w_s$ be nonnegative integers, and suppose that an ITD$(k, m+w_i; w_i)$ exists for each $i = 1, \ldots, s$. Then a TD$(k, mt + \sum_{i=1}^{s} w_i) - TD(k, mh + \sum_{i=1}^{s} w_i)$ exists.*

If $w_i = 0$ for some i, $1 \leq i \leq s$, then the stronger result is obtained that a

$$TD(k, mt + \sum_{i=1}^{s} w_i) - TD(k, mh + \sum_{i=1}^{s} w_i) - (t-h)TD(k,m)$$

exists.

6.4 Another Variant

Colbourn [34] establishes the following:

Theorem 6.8 *If there exists a ITD($k, n+h; h$) for which $(k-2)h = n$, and there exists a TD(k, m), then there exists an ITD($k, mn + (m-1)h; n^m(h(m-1))^1$).*

6.5 Wojtas Structures

For ease of exposition, we assume henceforth that $\lambda = \mu = 1$; the extensions to higher index are, for the most part, routine.

An examination of the propositions used in Brouwer [25] reveals that many, due to Wojtas, arise by inflating objects before filling them. We develop a framework here for presenting such constructions generally.

A *partial transversal design* of *order* or *groupsize* n, *blocksize* k, denoted here by $PTD(k, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where

1. V is a set of kn elements;
2. $\mathcal{G}$ is a partition of V into k classes (called *groups*), each of size n;
3. $\mathcal{B}$ is a collection of k–subsets of V (called *blocks*);
4. every unordered pair of elements from V is either contained in exactly one group, or is contained in *at most one* block. Elements appearing together in a group do not appear together in a block.

A *hole* H of order h in a $PTD(k, n)$ $(V, \mathcal{G}, \mathcal{B})$ is a set $H \subseteq V$ with $|H \cap G| = h$ for each $G \in \mathcal{G}$, and $H \cap B = \emptyset$ for each $B \in \mathcal{B}$.

Two holes H_1 and H_2 are *compatible* if $H_1 \cap H_2 \subset G$ for some $G \in \mathcal{G}$. Compatibility is a *weaker* condition than disjointness. A *Wojtas structure* of *order* n, *blocksize* k, and *holesizes* $\mathcal{M}$, denoted WS($k, n, \mathcal{M}$), is a $PTD(k, n)$ $(V, \mathcal{G}, \mathcal{B})$ together with a set $\mathcal{H}$ of holes, so that

$$\mathcal{M} = \{\frac{|H|}{k} : H \in \mathcal{H}\}.$$

In addition, every pair of distinct holes from $\mathcal{H}$ are compatible. Moreover, every pair $\{x, y\} \subset V$ that does not appear in a group, *either* appears in exactly one block of $\mathcal{B}$, *or* appears in exactly one hole of $\mathcal{H}$, *but not both.*

If, in the definition, we replaced the single word "compatible" by the stronger word "disjoint", we would repeat the definition of incomplete transversal design. We make some simple (but important) observations.

Lemma 6.9 *A WS($k, n, \{1\}$) is equivalent to a TD(k, n).*

Lemma 6.10 *If a WS*$(k,n,\mathcal{M})$ *exists, and a TD*(k,m) *exists for each* $m \in \mathcal{M}$*, then a TD*(k,n) *exists. Indeed, a TD*$(k,n) - TD(k,m)$ *exists for every* $m \in \mathcal{M}$*.*

Lemma 6.11 *If WS*$(k,n,\mathcal{M})$ *and TD*(k,w) *both exist, a WS*$(k,wn,\{wm : m \in \mathcal{M}\})$ *exists.*

Of course, when the holes are all disjoint, we have incomplete transversal designs. Sometimes we can employ other useful patterns of holes as well. Many variations are possible, but we just develop one generalization here.

A Wojtas structure WS$(n,k,\mathcal{M})$ $(V,\mathcal{G},\mathcal{H},\mathcal{B})$ is *partitionable of type*

$$(e_1,\ldots,e_\ell;\mathcal{N},c),$$

denoted $PWS(k,n;e_1,\ldots,e_\ell;\mathcal{N},c)$, if $\mathcal{M} = \mathcal{N} \cup \{e_1,\ldots,e_\ell\}$, and the holes are

$$\mathcal{H} = \{H_{\infty_1}, H_{\infty_2}, \ldots, H_{\infty_\ell}\} \cup \{H_{ij} : 1 \le i \le c, 1 \le j \le d\},$$

so that, for each $1 \le i \le c$, $H_{\infty_1}, H_{\infty_2}, \ldots, H_{\infty_\ell}, H_{i1}, H_{i2}, \ldots, H_{id}$ are all *disjoint*, $\mathcal{N}$ contains all hole sizes among the $\{H_{ij}\}$, and, for all $1 \le i \le c$, $\bigcup_{j=1}^{\ell} H_{\infty_j} \cup \bigcup_{j=1}^{d} H_{ij} = V$. Moreover, $H_{\infty i}$ is a hole of size e_i.

One might think about the $\{H_{ij} : 1 \le j \le d\}$, $1 \le i \le c$ as being "partial parallel classes of holes"; each, together with the special holes $\{H_{\infty_j}\}$, forms a "parallel class of holes".

In fact, we have the following equivalence:

Lemma 6.12 *A PITD*$(k,n;b_1,\ldots,b_s)$ *is equivalent to a*

$$PWS(k,n;b_1,\ldots,b_s;\emptyset,c)$$

for all c. For every $0 \le \ell \le s$*, it implies the existence of a*

$$PWS(k,n;b_1,\ldots,b_\ell;\{b_{\ell+1},\ldots,b_s\},1).$$

Lemma 6.13 *A resolvable TD*(k,n) *is equivalent to a* $PWS(k,n;\emptyset;\{1\},n)$*.*

Inflation, as in Lemma 6.11 works again, but we obtain a stronger result:

Lemma 6.14 *If a* $PWS(k,n;e_1,\ldots,e_\ell;\mathcal{M}^\star,c)$ *exists and a TD*(k,w) *exists, so also does a* $PWS(k,wn;we_1,\ldots,we_\ell;\{wm : m \in \mathcal{M}^\star\},c)$*.*

Naturally, since a PWS is a WS, one can apply Lemma 6.10 to fill the holes. However, the structure of the partitioning can be exploited to obtain a more sophisticated result:

Lemma 6.15 *Suppose that there exists a* $PWS(k,n;e_1,\ldots,e_\ell;\mathcal{M}^\star,c)$*. Let* $\gamma_1,\ldots,\gamma_c$ *be nonnegative integers, and write* $\sigma = \sum_{i=1}^{c}\gamma_i$*. Suppose that, for every* $m \in \mathcal{M}^\star$*, and every* $1 \le i \le \ell$*, there exists*

$$TD(k,m+\gamma_i) - TD(k,\gamma_i).$$

Suppose further that, for $2 \le i \le \ell$*, there exists*

$$TD(k,e_i+\sigma) - TD(k,\sigma).$$

Then there exists

$$TD(k,n+\sigma) - TD(k,e_1+\sigma).$$

Again, we have the phenomenon that in the middle of the construction, we have ℓ holes of sizes $\sigma + e_i$ for $1 \leq i \leq \ell$, but they all intersect in a hole of size σ. Lemma 6.15 gives one way to fill all but one of the holes. Here is another:

Lemma 6.16 *Suppose that there exists a $PWS(k, n; e_1, \ldots, e_\ell; \mathcal{M}^\star, c)$. Let $\gamma_1, \ldots, \gamma_c$ be nonnegative integers, and write $\sigma = \sum_{i=1}^{c} \gamma_i$. Suppose that, for every $m \in \mathcal{M}^\star$, and every $1 \leq i \leq \ell$, there exists*

$$TD(k, m + \gamma_i) - TD(k, \gamma_i).$$

Suppose further that, for $1 \leq i \leq \ell$, there exists

$$TD(k, e_i + \sigma) - TD(k, e_i).$$

Then there exists

$$TD(k, n + \sigma) - \sum_{i=1}^{\ell} TD(k, e_i).$$

When $\ell = 1$ and $\mathcal{M} = \{m\}$, we can also avoid filling one of the parallel classes of holes, to obtain:

Lemma 6.17 *Suppose that a $PWS(k, n; e; \{m\}, c)$ exists. Let $\gamma_1, \ldots, \gamma_c$ be nonnegative integers, and write $\sigma = \sum_{i=1}^{c} \gamma_i$. Suppose that there exists, for every $1 \leq i \leq c$, a*

$$TD(k, m + \gamma_i) - TD(k, \gamma_i).$$

Then there exists

$$TD(k, n + \sigma) - TD(k, e + \sigma) - \sum_{i=1}^{(n-e)/m} TD(k, m).$$

The power of Wojtas structures in general is that, rather than filling them immediately, one can inflate them and then fill them. This can often yield better results than are obtained by filling them and then inflating.

The additional power of partitioned Wojtas structures is the more sophisticated manner in which they can be filled.

It appears that Wojtas structures and partitioned Wojtas structures can lead to new incomplete transversal designs, but of course we have seen no ways to construct them except via equivalences to transversal designs and incomplete transversal designs.

6.6 Making Wojtas Structures

By now, it should come as no surprise that one way to construct Wojtas structures is to use Wilson's theorem in its many disguises. But an easier way to get some Wojtas structures is by removing one group from incomplete TDs:

Lemma 6.18 *1. If a TD$(k+1,n)$ exists, then a PWS$(k,n;0;\{1\},n)$ exists.*

2. If an ITD$(k+1,n;h)$ exists, then a PWS$(k,n;h;\{1\},h)$ exists. If in addition a TD(k,h) exists, then a PWS$(k,n;0;\{1\},n-h)$ exists.

3. If an ITD$(k+1,n;b_1,\ldots,b_s)$ exists, and TD(k,b_i) exist for $2 \leq i \leq s$, then a PWS$(k,n;b_1;\{1\},b_1)$ exists.

Using Wilson–type constructions, more general Wojtas structures can be made. Here is a variant of Theorem 6.1, using the same notation:

Theorem 6.19 *Suppose that a TD$(k+\ell,t)$ exists. Let $\mathcal{D} \subset \mathcal{B}$. so that if $B \cap D \subset E_i$ for distinct $B, D \in \mathcal{D}$ and $1 \leq i \leq \ell$, then $w_i^B = 0$. Suppose that for each $B \in \mathcal{B} \setminus \mathcal{D}$, there exists*

$$TD(k, m + \sum_{i=1}^{\ell} w_i^B) - \sum_{i=1}^{\ell} TD(k, w_i^B).$$

Suppose that for each $1 \leq i \leq \ell$, there exists a

$$TD(k, \sum_{j=1}^{t} w_{ij}) - \sum_{j=1}^{t} TD(k, w_{ij}).$$

Then there exists a

$$WS(k, mt + \sum_{i=1}^{\ell} \sum_{j=1}^{t} w_{ij}, \mathcal{M})$$

where $\mathcal{M} = \{m + \sum_{i=1}^{\ell} w_i^B : B \in \mathcal{D}\}$.

When a single level is used, one can in fact make partitioned WSs:

Theorem 6.20 *Suppose that a TD$_\mu(k+1,t)$ exists. Suppose that for each $B \in \mathcal{B}$, there exists*

$$TD_\lambda(k, m + w_1^B) - TD_\lambda(k, w_1^B),$$

whenever *w_1^B is nonzero. Let ζ be the number of $\{w_{1j} : 1 \leq j \leq t\}$ which are zero. Then we obtain a Wojtas structure WS$(mt + \sum_{j=1}^{t} w_{ij}, \{1, m, \sum_{j=1}^{t} w_{ij})\}$ in which all blocks of size m form ζ holey parallel classes for the hole of size $\sum_{j=1}^{t} w_{ij}$.*

Numerous variants are possible as well, but we do not consider them here.

6.7 Thwarts

Naturally, applications of Wilson's theorem depend on the presence of appropriate ingredients, and a natural question is to determine the ways in which the blocks of a TD$(k+\ell,t)$ intersect the points of nonzero weight in the ℓ "extra" groups. With this in mind, we give a definition. Let ℓ be a nonnegative integer, and let $\mathcal{I} = \{i_1,\ldots,i_s\}$ with $0 \leq i_1, i_2, \cdots, i_s \leq \ell$. Further suppose that $0, s_1, s_2, \cdots, s_\ell \leq t$. Let $(X,\mathcal{G},\mathcal{B})$ be a TD$(k+\ell,t)$ with $\mathcal{G} = \{G_1,\ldots,G_k,E_1,\ldots,E_\ell\}$. Then an $(\ell,\mathcal{I},s_1,s_2,\ldots,s_\ell)$ *-thwart* is a set $S = \bigcup_{j=1}^{\ell} S_j$, where $S_j \subseteq E_j$ with $|S_j| = s_j$ for each $1 \leq j \leq \ell$, such that for every $B \in \mathcal{B}$, $|B \cap S| \in \mathcal{I}$.

Thwarts provide a convenient notation for simpler applications of Wilson's theorem, in which it is sufficient to know the number of points of intersection of each block with the points of nonzero weight in the extra groups. When different weights are chosen, however, more detailed structural information is required. Here we consider the structure of various thwarts. Given a set $\mathcal{I}$, let $\overline{\mathcal{I}}_\ell = \{\ell - i : i \in \mathcal{I}\}$.

Lemma 6.21 *If a TD$(k+\ell,t)$ contains an $(\ell,\mathcal{I},s_1,\ldots,s_\ell)$-thwart, it also contains an $(\ell,\overline{\mathcal{I}}_\ell,t-s_1,\ldots,t-s_\ell)$-thwart.*

Levels

The simplest thwarts are found by simply truncating ℓ groups in each possible way to obtain:

Lemma 6.22 *Let ℓ be a positive integer, and let a TD$(k+\ell,t)$ exist. Then for all choices of integers $s_1,\ldots,s_\ell$ satisfying $0 \leq s_i \leq t$ for $1 \leq i \leq \ell$, the TD contains a $(\ell,\{0,1,2,\ldots,\ell-1,\ell\},s_1,\ldots,s_\ell)$–thwart.*

Often we refer to such a thwart as ℓ *levels* in the TD.

Spikes and Stairs

If all of the points of nonzero weight are on a single block, we obtain an $(\ell,\{0,1\},1,1,\ldots,1)$–thwart, which we call a *spike*. Every TD$(k+\ell,t)$ contains such a spike.

We can choose one point to be of nonzero weight on each level, so that no block intersects the points of nonzero weight in more than two points. We call the resulting $(\ell,\{0,1,2\},1,1,\ldots,1)$–thwart a *stair*. Stairs are essentially the analogue of arcs in projective planes. In fact, if the TD$(k+\ell,t)$ is the truncation of the TD$(t+1,t)$ arising from the desarguesian plane, the existence of an oval in the plane ensures that the stair is present for all choices of ℓ. If the TD$(k+\ell,t)$ arises in another way, we cannot assume to inherit the structure of a plane. Nevertheless, if $\binom{\ell-1}{2} < t$, simple counting ensures that the stair is present.

One can find an intermediate structure between spikes and stairs. Suppose that one point of nonzero weight is chosen on each extra group so that there are s blocks that are disjoint on the extra levels and intersect the extra levels in $x_1,\ldots,x_s$ positions, respectively. This is an $(\ell,\{0,1,2,x_1,\ldots,x_s\},1,1,\ldots,1)$–thwart, provided that no block intersects one of the

s chosen blocks in more than two points of nonzero weight in the extra levels. Call this a *generalized stair*, and use the notation $(\ell; x_1, \ldots, x_s)$–*stair*. The presence of such thwarts is not as easily checked, but when $s = 1$ it suffices to ensure that $\binom{\ell-1}{2} - \binom{x_1}{2} + 1 < t$. Another case is when $s = 2$ and $x_1 + x_2 = \ell$; then the thwart is always present.

Stairs, Spikes and Levels

One can take a stair, spike or generalized stair on some levels, and truncate the remaining levels. For example, a spike involving u levels and a v disjoint levels truncated to $s_1, \ldots, s_v$ points leads to a $(u+v, \{0, 1, \ldots, v+1, v+u\}, 1^u s_1 s_2 \cdots s_v)$–thwart, in which only one block of size $v+u$ is present if $u > 1$.

Similarly, one can take a stair or generalized stair together with some truncated levels. For example, to an $(\ell; x)$–stair, we can append a truncated level on $s \leq t - \binom{\ell}{2} + \binom{x}{2} + 1$ points in a $\mathrm{TD}(k,t)$ to obtain a $(\ell+1, \{0,1,2\}, 1^\ell s^1)$–thwart (in other words, adding a truncated level does not introduce a new intersection, provided that the truncated level is short enough).

Numerous variations are possible; the stair or spike can meet the added level(s) in one (or more) point(s). An $(\ell; x_1, x_2)$–stair could also have a further truncated level, and the blocks of size x_1 and x_2 could each meet or miss the truncated level.

Subplanes

When a projective plane of order t contains a subplane of order s, the corresponding $\mathrm{TD}(t+1,t)$ contains an $(s+1, \{0, 1, s+1\}, s, s, \cdots, s)$–thwart. If the subplane is a Baer subplane, then the "0" can be omitted.

Instead deleting a point outside the subplane from the projective plane, we obtain an $(s^2+s+1, \{0, 1, s+1\}; 1^{s^2+s+1})$–thwart.

Subsquares

When a $\mathrm{TD}(k,t)$ contains a $\mathrm{TD}(3,t)$ having a "subsquare", i.e. a sub-$\mathrm{TD}(3,s)$, then the $\mathrm{TD}(k,t)$ contains a $(3, \{0,1,3\}, s, s, s)$–thwart. When $t = 2s$, the "0" can be omitted.

Affine Subplane

When a projective plane of order t contains an affine subplane of order s, the $\mathrm{TD}(t+1,t)$ contains a $(s+1, \{0,1,s\}, s-1, s-1, \cdots, s-1)$–thwart.

Trinity

Wojtas [115] observed that one can truncate three levels but obtain a thwart with blocks intersecting in 1, 2 and 3 points only — none in 0 points. The precise condition under which one can obtain such a $(3, \{1,2,3\}, s_1, s_2, s_3)$–thwart is *open* for $\mathrm{TD}(k,t)$ in general, although some bounds on s_1, s_2, s_3 are given in [37]. When the $\mathrm{TD}(3,t)$ involved arises from a cyclic latin square (which can be assumed if we are free to choose three groups of the TD$(t+1,t)$ from the desarguesian plane of prime order t), a sufficient condition is that $s_1+s_2+s_3 \geq 2t-1$. The conditions for arbitrary TD(k,n) seem very difficult; see [37] for some other observations in this regard.

7 Direct Constructions

Until this point, we have concentrated on recursive methods, and despite a large collection of constructions being introduced, we have failed to construct any examples. Let us remedy that situation. Lemma 5.1, together with the well–known fact that projective planes exist (at least) for all prime power orders, gives the main set of basic ingredients:

Theorem 7.1 *If t is a prime power, a TD$(t+1,t)$ exists.*

Surprisingly little else in the way of *general* direct constructions is known, although much is known from clever hand and machine computations in specific cases. The main device used is to assume that the TD has a "reasonably large" automorphism group acting on it, and to use the structure of the automorphisms to reduce the computational search.

We require some basic definitions. Let $(?, \odot)$ be a group of order g. A $(g,k;\lambda)$-*difference matrix* is a $k \times g\lambda$ matrix $D=(d_{ij})$ with entries from ?, so that for each $1 \leq i < j \leq k$, the multiset

$$\{d_{i\ell} \odot d_{j\ell}^{-1} : 1 \leq \ell \leq g\lambda\}$$

contains every element of ? λ times. When ? is abelian, typically additive notation is used, so that differences $d_{i\ell} - d_{j\ell}$ are employed.

A $(g,k;\lambda,\mu;u)$-*quasi-difference matrix* (QDM) is a matrix $Q=(q_{ij})$ with k rows, with each entry either empty (usually denoted by $-$) or containing a single element of ?. Each row contains exactly λu empty entries, and each column contains at most one empty entry. Furthermore, for each $1 \leq i < j \leq k$, the multiset

$$\{q_{i\ell} - q_{j\ell} : 1 \leq \ell \leq \lambda(g-1+2u)+\mu, \text{ with } q_{i\ell} \text{ and } q_{j\ell} \text{ not empty}\}$$

contains every nonzero element of ? λ times, and contains 0 μ times.

The essential connections with transversal designs follow:

Lemma 7.2 *1. A $(g,k;\lambda)$-difference matrix gives a TD$_\lambda(k+1,g)$.*

2. A $(g,k;\lambda,\mu;u)$–quasi–difference matrix with $\mu \leq \lambda$ gives a TD$_\lambda(k,g+u) - TD_\lambda(k,u)$.

3. A $(g,k;1,0;u)$–quasi–difference matrix gives an ITD$^{g-u(k-2)}(k,g+u;u)$.

The latter statement gives means to construct master designs for Du's variation, Theorem 6.7. We can now give the constructions for four new difference matrices. Each of these yield improvements to the lower bound for $N(n)$ given in [4].

Theorem 7.3 *(Abel [2]) There exists a (36,7,1)-difference matrix. Hence there exists a TD$(8,36)$, implying that $N(36) \geq 6$.*

Proof: To construct a $(36,7,1)$ difference matrix over $? = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Consider the following 7×12 array over $?$:

$$\begin{pmatrix}
0000 & 0000 & 0000 & 0100 & 0100 & 0100 & 1000 & 1000 & 1000 & 1100 & 1100 & 1100 \\
0100 & 1020 & 1120 & 0100 & 1010 & 1110 & 0100 & 1010 & 1110 & 0100 & 1020 & 1120 \\
0100 & 1000 & 1122 & 1122 & 0102 & 1020 & 1020 & 1112 & 0101 & 1021 & 1111 & 0101 \\
1000 & 1102 & 0102 & 0100 & 1001 & 1102 & 1101 & 0121 & 1010 & 1012 & 1111 & 0110 \\
1000 & 1122 & 0112 & 1010 & 1102 & 0110 & 1020 & 1110 & 0112 & 0101 & 1022 & 1111 \\
0100 & 1022 & 1110 & 1012 & 1110 & 0102 & 1101 & 0101 & 1001 & 1110 & 0101 & 1002 \\
1100 & 0110 & 1010 & 1002 & 1122 & 0112 & 0120 & 1011 & 1112 & 1011 & 1121 & 0101
\end{pmatrix}$$

Use each column $[(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2),...,(a_7,b_7,c_7,d_7)]^T$ to generate two others:

$$[(a_1b_1c_1d_1)(a_2b_2c_2d_2{+}1)(a_3b_3c_3{+}1d_1)(a_4b_4c_4{+}2d_4{+}1)(a_5b_5c_5{+}2d_5{+}2)(a_6b_6c_6{+}2d_6)(a_7b_7c_7{+}1d_7{+}1)]^T$$

and

$$[(a_1b_1c_1d_1)(a_2b_2c_2d_2{+}2)(a_3b_3c_3{+}2d_1)(a_4b_4c_4{+}1d_4{+}2)(a_5b_5c_5{+}1d_5{+}1)(a_6b_6c_6{+}1d_6)(a_7b_7c_7{+}2d_7{+}2)]^T.$$

The resulting 36 columns form a (36,7,1) difference matrix.

Theorem 7.4 *(Abel [2]) There exists a $(39,6,1)$-difference matrix. Hence there exists a TD$(7,39)$, implying that $N(39) \geq 5$.*

Proof: To construct a $(39,6,1)$-difference matrix over $? = \mathbb{Z}_{39}$, let

$$A_1 = \begin{pmatrix} 1 \\ 16 \\ 22 \\ 17 \\ 38 \\ 23 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 23 & 13 & 5 & 12 & 11 \\ 25 & 11 & 22 & 34 & 23 & 6 \\ 13 & 4 & 20 & 17 & 15 & 29 \\ 27 & 21 & 8 & 16 & 19 & 26 \\ 16 & 19 & 34 & 38 & 26 & 21 \end{pmatrix}.$$

Define automorphisms α, β (acting on the columns of A_1 and A_2) by $\alpha(a,b,c,d,e,f)^T = (16c, 16a, 16b, 16f, 16d, 16e)^T$ and $\beta(a,b,c,d,e,f)^T = (-a,-b,-c,-d,-e,-f)^T$.

Apply the group of order 2 generated by β to the column of A_1 and the group order 6 generated by α and β to the columns of A_2. Finally, append a column of zeros to obtain a (39,6,1)-difference matrix.

Theorem 7.5 *(Wotjas [122]) There exists a $(48,8,1)$-difference matrix. Hence there exists a TD$(9,48)$, implying that $N(48) \geq 7$.*

$$
\begin{pmatrix}
0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 \\
0000 & 0001 & 0010 & 0011 & 3000 & 3001 & 3010 & 3011 \\
0000 & 0010 & 0011 & 0111 & 0001 & 3011 & 0101 & 3110 \\
0000 & 0011 & 0001 & 0100 & 3001 & 0010 & 3111 & 0101 \\
0000 & 0100 & 1000 & 0010 & 0011 & 2111 & 5101 & 4010 \\
0000 & 0101 & 1010 & 0001 & 3011 & 5110 & 2111 & 1001 \\
0000 & 0110 & 1011 & 0101 & 0010 & 5100 & 5000 & 1100 \\
0000 & 0111 & 1001 & 0110 & 3010 & 2101 & 2010 & 4111 \\
0000 & 1000 & 0100 & 2000 & 1010 & 5001 & 3000 & 5111 \\
0000 & 1001 & 0110 & 2011 & 4010 & 2000 & 0010 & 2100 \\
0000 & 1010 & 0111 & 2111 & 1011 & 2010 & 3101 & 2001 \\
0000 & 1011 & 0101 & 2100 & 4011 & 5011 & 0111 & 5010 \\
0000 & 1100 & 2100 & 4010 & 5000 & 2001 & 4101 & 1000 \\
0000 & 1101 & 2110 & 4001 & 2000 & 5000 & 1111 & 4011 \\
0000 & 1110 & 2111 & 4101 & 5001 & 5010 & 4000 & 4110 \\
0000 & 1111 & 2101 & 4110 & 2001 & 2011 & 1010 & 1101 \\
0000 & 2000 & 4000 & 1100 & 3100 & 4111 & 3011 & 5011 \\
0000 & 2001 & 4010 & 1111 & 0100 & 1110 & 0001 & 2000 \\
0000 & 2010 & 4011 & 1011 & 3101 & 1100 & 3110 & 2101 \\
0000 & 2011 & 4001 & 1000 & 0101 & 4101 & 0100 & 5110 \\
0000 & 2100 & 5000 & 4100 & 0111 & 0011 & 1110 & 3100 \\
0000 & 2101 & 5010 & 4111 & 3111 & 3010 & 4100 & 0111 \\
0000 & 2110 & 5011 & 4011 & 0110 & 3000 & 1011 & 0010 \\
0000 & 2111 & 5001 & 4000 & 3110 & 0001 & 4001 & 3001 \\
0000 & 3000 & 5100 & 2010 & 5011 & 3110 & 3100 & 4001 \\
0000 & 3001 & 5110 & 2001 & 2011 & 0111 & 0110 & 1010 \\
0000 & 3010 & 5111 & 2101 & 5010 & 0101 & 3001 & 1111 \\
0000 & 3011 & 5101 & 2110 & 2010 & 3100 & 0011 & 4100 \\
0000 & 3100 & 1100 & 5010 & 2100 & 4110 & 4011 & 2110 \\
0000 & 3101 & 1110 & 5001 & 5100 & 1111 & 1001 & 5101 \\
0000 & 3110 & 1111 & 5101 & 2101 & 1101 & 4110 & 5000 \\
0000 & 3111 & 1101 & 5110 & 5101 & 4100 & 1100 & 2011 \\
0000 & 4000 & 4110 & 5100 & 4101 & 0100 & 2001 & 3000 \\
0000 & 4001 & 4100 & 5111 & 1101 & 3101 & 5011 & 0011 \\
0000 & 4010 & 4101 & 5011 & 4100 & 3111 & 2100 & 0110 \\
0000 & 4011 & 4111 & 5000 & 1100 & 0110 & 5110 & 3101 \\
0000 & 4100 & 3110 & 1110 & 2110 & 4000 & 5010 & 0001 \\
0000 & 4101 & 3100 & 1101 & 5110 & 1001 & 2000 & 3010 \\
0000 & 4110 & 3101 & 1001 & 2111 & 1011 & 5111 & 3111 \\
0000 & 4111 & 3111 & 1010 & 5111 & 4010 & 2101 & 0100 \\
0000 & 5000 & 2010 & 3010 & 4111 & 4001 & 5001 & 2111 \\
0000 & 5001 & 2000 & 3001 & 1111 & 1000 & 2011 & 5100 \\
0000 & 5010 & 2001 & 3101 & 4110 & 1010 & 5100 & 5001 \\
0000 & 5011 & 2011 & 3110 & 1110 & 4011 & 2110 & 2010 \\
0000 & 5100 & 3010 & 3000 & 4001 & 2110 & 1000 & 1011 \\
0000 & 5101 & 3000 & 3011 & 1001 & 5111 & 4010 & 4000 \\
0000 & 5110 & 3001 & 3111 & 4000 & 5101 & 1101 & 4101 \\
0000 & 5111 & 3011 & 3100 & 1000 & 2100 & 4111 & 1110
\end{pmatrix}
$$

Table 2: $(48, 8, 1)$-difference matrix

Proof: Let ? $= \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Transpose the array in Table 2 to obtain a $(48, 8, 1)$-difference matrix.

Theorem 7.6 *(Abel [2]) There exists a $(45, 7; 1, 1; 9)$-quasi–difference matrix. Hence there exists a TD$(7, 54)$, implying that $N(54) \geq 5$.*

Proof: Consider the matrices

$$A_1 = \begin{pmatrix} - & - & - & - \\ 0 & 0 & 0 & 0 \\ 1 & 27 & 16 & 7 \\ 24 & 40 & 1 & 35 \\ 10 & 30 & 22 & 44 \\ 5 & 18 & 14 & 33 \\ 30 & 16 & 33 & 27 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} - \\ 0 \\ 3 \\ 7 \\ 7 \\ 3 \\ 0 \end{pmatrix}.$$

As in Theorem II.2.54 of [4] replace each column of $[A_1| - A_1|A_2]$ by its 7 cyclic shifts. This gives a $(45, 7; 1, 1; 9)$-quasi–difference matrix.

One important device for constructing quasi–difference matrices is the use of $V(m, t)$ vectors. See [33] for a definition, and for the following:

Theorem 7.7 *A $V(m, t)$ vector gives a $(mt+1, m+2; 1, 0; t)$-QDM. A $V(m, t)$ vector exists if m and t are not both even,*

1. *whenever $mt + 1 \leq 5000$, $m - 1 \leq t$, $m \leq 10$ and $mt + 1$ is prime, except when $m = 9$ and $t = 8$, as no $V(9, 8)$ exists.*
2. *whenever $mt + 1 \leq 5000$, $m - 1 \leq t$, $m \leq 6$ and $mt + 1$ is a prime power, except when $m = 3$ and $t = 5$, as no $V(3, 5)$ exists.*

Related computational constructions for $(mt + 1, m + 2; 1, 0; t)$-QDMs are reported in [34]. Building on these, $V(m, t)$s have now been shown to exist whenever $mt + 1$ is a prime power, $m \leq 6$, and $m \geq t - 1$, except when $(m, t) = (3, 5)$ [66]. Abel [2] found another useful family:

Theorem 7.8 *For $11t + 1$ a prime, there exists a $(11t + 1, k; 1, 0; t)$-QDM exists for $k = 11$ if $198 < 11t + 1 < 600$ and for $k = 12$ if $600 < 11t + 1 < 992$.*

He also found $(9 \cdot 4 + 1, 9; 1, 0; 4)-$ and $(9 \cdot 8 + 1, 9; 1, 0; 8)$-QDMs.

Despite these few more general computational results, most direct constructions are one–of–a–kind. For TD, we summarize in Table 3 known direct constructions, not obtained by one of the three previous constructions of QDMs.

In some cases, a direct construction yields an idempotent TD(k, t) , which is a PITD$(k, t; 1^t)$. In Table 4, direct constructions of idempotent TDs having the same blocksize as the largest

Order	Blocksize	Reference(s)
6	3	[104]
10	4	[23]
12	7	[61]
14	5	[105]
15	6	[91]
18	5	[109]
20	6	[106]
21	7	[79]
22	5	[109]
24	7	[5, 87, 119]
26	6	[32]
28	6	[1, 88]
30	6	[9]
33	7	[2]
34	6	[2]
35	7	[120]
36	8	[2]
38	6	[9]
39	7	[2]
40	9	[5]

Order	Blocksize	Reference(s)
42	5	[109]
44	6	[9]
48	9	[122]
51	7	[2]
52	6	[1]
54	7	[2]
55	7	[72]
56	9	[72]
80	11	[5]
112	15	[5]
160	11	[5]
176	16	[5]
208	16	[5]
224	15	[5]
352	20	[5]
416	20	[5]
544	20	[5]
640	11	[5]
896	15	[5]

Table 3: Direct Constructions for TDs

Order	Blocksize	Reference(s)	Order	Blocksize	Reference(s)
6	3	–	34	6	[2]
10	4	[48]	38	6	[2]
14	5	[16]	42	5	[16]
18	5	[124]	44	6	[2]
20	6	[2]	52	6	[2]
22	5	[11]	55	7	[2]

Table 4: Direct Constructions for Idempotent TDs

$\mathrm{ITD}(4,6;2)$	[47]	$\mathrm{ITD}(5,8;2)$	[99]
$\mathrm{ITD}(6,10;2)$	[28]	$\mathrm{PITD}(5,9;2^1 1^7)$	[126]
$\mathrm{PITD}(5,11;2^1 1^9)$	[100]	$\mathrm{ITD}(5,12;3)$	[99]
$\mathrm{PITD}(5,12;2^6)$	[15]	$\mathrm{ITD}(6,15;3)$	[34]
$\mathrm{PITD}(5,14;3^1 1^{11})$	[7]		

Table 5: Some Incomplete TDs

known TD on the same parameters are reported. Ganter, Mathon and Rosa [48] actually construct a TD$(4, 10)$ having four disjoint parallel classes, the maximum known to date.

A large number of *ad hoc* constructions for incomplete TDs appear in the literature; we do not attempt to catalogue them all here. In Table 5, we report some of the small incomplete TDs that have been constructed directly.

Of course, many more incomplete TDs have been constructed directly. Some constructions of TDs have proceeded by making an ITD with a hole and filling the hole; see [1, 9, 32, 109]. For ITD$(4, t; h)$s, see [57] and references therein. For ITD$(5, t; h)$s, see [7, 45, 46] and references therein for a number of direct constructions. For ITD$(4, hn; h^n)$s, see [43, 101]. For ITD$(5, hn; h^n)$s, see [15, 17, 64, 78, 100]; Dinitz and Stinson [43] also give some ITD$(6, 2n; 2^n)$s and ITD$(8, 2n; 2^n)$s.

Colbourn [33] gives a number of constructions for QDMs leading to ITDs, and some sporadic examples appear in [75]. Sporadic designs that find uses include the elliptic semiplane of Baker [13], which is a $\{7\}$-GDD of type 3^{15}; and the $\{9\}$-GDD of type 3^{33} by Mathon [70].

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