© DISCRETE MATHEMATICS 5 (1973) 79–86. North-Holland Publishing Company

CONSTRUCTION OF DOUBLY DIAGONALIZED ORTHOGONAL LATIN SQUARES

Charles C. LINDNER
Auburn University, Auburn, Ala., USA

Received 22 June 1972*

Abstract. A *transversal* T of a latin square is a collection of n cells no two in the same row or column and such that each of the integers $1, 2, ..., n$ appears in exactly one of the cells of T. A latin square is *doubly diagonalized* provided that both its main diagonal and off-diagonal are transversals. Although it is known that a doubly diagonalized latin square of every order $n \geq 4$ exists and that a pair of orthogonal latin squares of order n exists for every $n \neq 2$ or 6, it is still an open question as to what the spectrum is for pairs of doubly diagonalized orthogonal latin squares. The best general result seems to be that pairs of orthogonal doubly diagonalized latin squares of order n exist whenever n is odd or a multiple of 4, except possibly when n is a multiple of 3 but not of 9. In this paper we give a new construction for doubly diagonalized latin squares which is used to enlarge the known class for doubly diagonalized orthogonal squares. The construction is based on Sade's *singular direct product of quasigroups.*

1. Introduction

A *latin* square is an $n \times n$ array such that in each row and column each of the integers $1, 2, ..., n$ occurs exactly once. Algebraically, a latin square is a quasigroup. A *transversal* T of a latin square is a collection of n cells no two in the same row or column and such that each of the integers $1, 2, ..., n$ appears in exactly one of the cells of T. A latin square is said to be *diagonalized* if the main diagonal (cells $(1, 1), (2, 2), ..., (n, n)$) is a transversal and *off-diagonalized* if the off-diagonal (cells $(1, n), (2, n-1), ..., (n, 1)$) is a transversal. A latin square is *doubly diagonalized* provided that it is both diagonalized and off-diagonalized. If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ latin squares, then by (A, B) is meant the set $\{(a_{ij}, b_{ij}) | \, i, j = 1, 2, ..., n\}$. If $|(A, B)| = n^2$, then the squares A and B are said to be orthogonal.

* Original version received 4 January 1972.

Although it is known that a doubly diagonalized latin square of every order $n \geq 4$ exists [3] and that a pair of orthogonal latin squares of order n exists for every $n \neq 2$ or 6 (see [1]), it is still an open question as to what the spectrum is for pairs of doubly diagonalized orthogonal latin squares. The best general result seems to be that pairs of orthogonal doubly diagonalized latin squares of order n exist whenever n is odd or a multiple of 4, except possibly when n is a multiple of 3 but not of 9 (see [2]). The proof of this result is obtained by the use of direct product type constructions. In this paper, we give a new construction for doubly diagonalized latin squares which is used to enlarge the known class for doubly diagonalized orthogonal latin squares. The construction is based on Sade's *singular direct product of quasigroups* [4].

2. The singular direct product for latin squares

We will give here Sade's singular direct product of quasigroups and then restate the construction in terms of latin squares.

Let $(V, \odot)$ be an idempotent quasigroup, $(Q, \circ)$ a quasigroup containing the subquasigroup $(P, \circ)$ and $\bar{P} = (Q \backslash P, \otimes)$ any quasigroup. On the set $P \cup (\bar{P} \times V)$ define a binary operation $\oplus$ as follows:

(1) $p \oplus q = p \circ q$, if $p, q \in P$;

(2) $p \oplus (q, v) = (p \circ q, v)$, if $p \in P$, $q \in \bar{P}$, $v \in V$;

(3) $(q, v) \oplus p = (q \circ p, v)$, if $p \in P$, $q \in \bar{P}$, $v \in V$;

(4) $(p, v) \oplus (q, v) = \begin{cases} p \circ q, & \text{if } p \circ q \in P; \\ (p \circ q, v), & \text{if } p \circ q \notin P; \end{cases}$

(5) $(p, v) \oplus (q, w) = (p \otimes q, v \odot w)$, $v \neq w$.

We remark that the operations $\odot$, $\circ$ and $\otimes$ are not necessarily related. The quasigroup so constructed is called the *singular direct product* of the quasigroups $(V, \odot)$, $(Q, \circ)$, $(P, \circ)$, and $(\bar{P}, \otimes)$ and is denoted by

$$V(\odot) \times Q(\circ, P, \bar{P}, \otimes) .$$

We now restate the singular direct product for quasigroups in terms of latin squares: Let V be an idempotent latin square of order v based on $1, 2, ..., v$; Q a latin square of order q based on $1, 2, ..., q$ containing the subsquare P of order p based on $1, 2, ..., p$ in the upper left-hand corner; and

$A(0,0)$	$A(0,1)$	$A(0,2)$	...	$A(0,v)$
$A(1,0)$	$A(1,1)$	$A(1,2)$	...	$A(1,v)$
$A(2,0)$	$A(2,1)$	$A(2,2)$	...	$A(2,v)$
⋮	⋮	⋮	⋱	⋮
$A(v,0)$	$A(v,1)$	$A(v,2)$	...	$A(v,v)$

Diagram 1.

$\bar{P}$ a latin square of order $q-p$ based on $p+1, p+2, ..., q$. The singular direct product of the latin squares V, Q, P and $\bar{P}$ denoted by $V \times Q(P, \bar{P})$ is the latin square of order $v(q-p)+p$ based on $P \cup (\bar{P} \times V)$ defined in Diagram 1. In the cell $A(0,0)$ is the latin square P. In the four cells $A(0,0)$, $A(0,1)$, $A(1,0)$ and $A(1,1)$ is the latin square Q with each symbol $x \notin P$ replaced by $(x, 1)$. Similarly, in the four cells $A(0,0)$, $A(0,k)$, $A(k,0)$ and $A(k,k)$ is the latin square Q with each $x \notin P$ replaced by (x, k). Finally, in the remaining cells $A(i,j)$, $i \neq j$, $i \neq 0 \neq j$, is the latin square $\bar{P}$ with every x replaced by (x, v_{ij}), where v_{ij} is the entry in cell (i,j) of V.

Example 2.1.

1	3	2
3	2	1
2	1	3

V

1	2	3	4
4	3	1	2
2	1	4	3
3	4	2	1

Q

$\cup|$

$P = \boxed{1}$

4	2	3
3	4	2
2	3	4

$\bar{P}$

1	(2,1) (3,1) (4,1)	(2,2) (3,2) (4,2)	(3,2) (3,3) (4,3)
(4,1) (2,1) (3,1)	(3,1) 1 (2,1) 1 (4,1) (3,1) (4,1) (2,1) 1	(4,3) (2,3) (3,3) (3,3) (4,3) (2,3) (2,3) (3,3) (4,3)	(4,2) (2,2) (3,2) (3,2) (4,2) (2,2) (2,2) (3,2) (4,2)
(4,2) (2,2) (3,2)	(4,3) (2,3) (3,3) (3,3) (4,3) (2,3) (2,3) (3,3) (4,3)	(3,2) 1 (2,2) 1 (4,2) (3,2) (4,2) (2,2) 1	(4,1) (2,1) (3,1) (3,1) (4,1) (2,1) (2,1) (3,1) (4,1)
(4,3) (2,3) (3,2)	(4,2) (2,2) (3,2) (3,2) (4,2) (2,2) (2,2) (3,2) (4,2)	(4,1) (2,1) (3,1) (3,1) (4,1) (2,1) (2,1) (3,1) (4,1)	(3,3) 1 (2,3) 1 (4,3) (3,3) (4,3) (2,3) 1

3. Construction of doubly diagonalized latin squares

In this section, we give a new construction for doubly diagonalized latin squares. This construction will be used in Section 4 to construct doubly diagonalized orthogonal latin squares.

Let V be a doubly diagonalized orthogonal latin square of even order based on $1, 2, ..., v = 2m$. Let Q be a diagonalized latin square of order q based on $1, 2, ..., q$ containing the doubly diagonalized latin square P based on $1, 2, ..., n$ in the upper left-hand corner. Finally, let $\bar{P}$ be an off-diagonalized latin square of order $q-p$ based on $p+1, p+2, ..., q$.

Theorem 3.1. *If there are latin squares V, Q, P and $\bar{P}$ satisfying the above conditions, where $|V| = v$, $|Q| = q$, $|P| = p$, $|\bar{P}| = q-p$, then there is a doubly diagonalized latin square of order $v(q-p) + p$.*

Proof. We can assume that V is idempotent and consider the singular direct product $V \times Q(P, \bar{P})$ as shown in Diagram 2.

$A(0,0)$	$A(0,1)$	$A(0,2)$	...	$A(0,v)$
$A(1,0)$	$A(1,1)$	$A(1,2)$	...	$A(1,v)$
$A(2,0)$	$A(2,1)$	$A(2,2)$	...	$A(2,v)$
⋮	⋮	⋮	⋱	⋮
$A(v,0)$	$A(v,1)$	$A(v,2)$	...	$A(v,v)$

Diagram 2. $V \times Q(P, \bar{P})$.

<table>
<tr><td>$A(0,0)$</td><td>$A(0,1)$</td><td>...</td><td>$A(0,m)$</td><td>$A(0,m+1)$</td><td>...</td><td>$A(0,v)$</td></tr>
<tr><td>$A(1,0)$</td><td colspan="3" rowspan="3">A_1</td><td colspan="3" rowspan="3">A_2</td></tr>
<tr><td>⋮</td></tr>
<tr><td>$A(m,0)$</td></tr>
<tr><td>$A(m+1,0)$</td><td colspan="3" rowspan="3">A_3</td><td colspan="3" rowspan="3">A_4</td></tr>
<tr><td>⋮</td></tr>
<tr><td>$A(v,0)$</td></tr>
</table>

Diagram 3. $V \times Q(P, \bar{P})$.

<table>
<tr><td colspan="3" rowspan="3">A_4</td><td>$A(m+1,0)$</td><td colspan="3" rowspan="3">A_3</td></tr>
<tr><td>⋮</td></tr>
<tr><td>$A(v,0)$</td></tr>
<tr><td>$A(0,m+1)$</td><td>...</td><td>$A(0,v)$</td><td>$A(0,0)$</td><td>$A(0,1)$</td><td>...</td><td>$A(0,m)$</td></tr>
<tr><td colspan="3" rowspan="3">A_2</td><td>$A(1,0)$</td><td colspan="3" rowspan="3">A_1</td></tr>
<tr><td>⋮</td></tr>
<tr><td>$A(m,0)$</td></tr>
</table>

Diagram 4. $D(V \times Q(P, \bar{P}))$.

Now partition $V \times Q(P, \bar{P})$ as shown in Diagram 3. Note that $A(0,0)$ is doubly diagonalized and based on $P = \{1, 2, ..., p\}$ and that the square consisting of the blocks A_1, A_2, A_3 and A_4 contains each element of $\bar{P} \times V$ exactly once on the main diagonal and exactly once on the off-diagonal. Now rearrange $V \times Q(P, \bar{P})$ to form the latin square $D(V \times Q(P, \bar{P}))$ given in Diagram 4. The fact that $D(V \times Q(P, \bar{P}))$ is doubly diagonalized is a consequence of the fact that $A(0,0)$ is doubly diagonalized and based on $P = \{1, 2, ..., p\}$, while the square consisting of the blocks A_4, A_3, A_2 and A_1 contains each element of $\bar{P} \times V$ exactly once on the main and off-diagonal.

Example 3.2.

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

V

1	3	4	2
2	4	3	1
3	1	2	4
4	2	1	3

Q

$\cup|$

$P = \boxed{1}$

4	3	2
2	4	3
3	2	4

$\bar{P}$

1	(3,1) (4,1) (2,1)	(3,2) (4,2) (2,2)	(3,3) (4,3) (2,3)	(3,4) (4,4) (2,4)
(2,1)	(4,1) (3,1) 1	(4,4) (3,4) (2,4)	(4,2) (3,2) (2,2)	(4,3) (3,3) (2,3)
(3,1)	1 (2,1) (4,1)	(2,4) (4,4) (3,4)	(2,2) (4,2) (3,2)	(2,3) (4,3) (3,3)
(4,1)	(2,1) 1 (3,1)	(3,4) (2,4) (4,4)	(3,2) (2,2) (4,2)	(3,3) (2,3) (4,3)
(2,2)	(4,3) (3,3) (2,3)	(4,2) (3,2) 1	(4,4) (3,4) (2,4)	(4,1) (3,1) (2,1)
(3,2)	(2,3) (4,3) (3,3)	1 (2,2) (4,2)	(2,4) (4,4) (3,4)	(2,1) (4,1) (3,1)
(4,2)	(3,3) (2,3) (4,3)	(2,2) 1 (3,2)	(3,4) (2,4) (4,4)	(3,1) (2,1) (4,1)
(2,3)	(4,4) (3,4) (2,4)	(4,1) (3,1) (2,1)	(4,3) (3,3) 1	(4,2) (3,2) (2,2)
(3,3)	(3,4) (4,4) (3,4)	(2,1) (4,1) (3,1)	1 (2,3) (4,3)	(2,2) (4,2) (3,2)
(4,3)	(3,4) (2,4) (4,4)	(3,1) (2,1) (4,1)	(2,3) 1 (3,3)	(3,2) (2,2) (4,2)
(2,4)	(4,2) (3,2) (2,2)	(4,3) (3,3) (2,3)	(4,1) (3,1) (2,1)	(4,4) (3,4) 1
(3,4)	(2,2) (4,2) (3,2)	(2,3) (4,3) (3,3)	(2,1) (4,1) (3,1)	1 (2,4) (4,4)
(4,4)	(3,2) (2,2) (4,2)	(3,3) (2,3) (4,3)	(3,1) (2,1) (4,1)	(2,4) 1 (3,4)

(4,3) (3,3) 1	(4,2) (3,2) (2,2)	(2,3)	(4,4) (3,4) (2,4)	(4,1) (3,1) (2,1)
1 (2,3) (4,3)	(2,2) (4,2) (3,2)	(3,3)	(2,4) (4,4) (3,4)	(2,1) (4,1) (3,1)
(2,3) 1 (3,3)	(3,2) (2,2) (4,2)	(4,3)	(3,4) (2,4) (4,4)	(3,1) (2,1) (4,1)
(4,1) (3,1) (2,1)	(4,4) (3,4) 1	(2,4)	(4,2) (3,2) (2,2)	(4,3) (3,3) (2,3)
(2,1) (4,1) (3,1)	1 (2,4) (4,4)	(3,4)	(2,2) (4,2) (3,2)	(2,3) (4,3) (3,3)
(3,1) (2,1) (4,1)	(2,4) 1 (3,4)	(4,4)	(3,2) (2,2) (4,2)	(3,3) (2,3) (4,3)
(3,3) (4,3) (2,3)	(3,4) (4,4) (2,4)	1	(3,1) (4,1) (2,1)	(3,2) (4,2) (2,2)
(4,2) (3,2) (2,2)	(4,3) (3,3) (2,3)	(2,1)	(4,1) (3,1) 1	(4,4) (3,4) (2,4)
(2,2) (4,2) (3,2)	(2,3) (4,3) (3,3)	(3,1)	1 (2,1) (4,1)	(2,4) (4,4) (3,4)
(3,2) (2,2) (4,2)	(3,3) (2,3) (4,3)	(4,1)	(2,1) 1 (3,1)	(3,4) (2,4) (4,4)
(4,4) (3,4) (2,4)	(4,1) (3,1) (2,1)	(2,2)	(4,3) (3,3) (2,3)	(4,2) (3,2) 1
(2,4) (4,4) (3,4)	(2,1) (4,1) (3,1)	(3,2)	(2,3) (4,3) (3,3)	1 (2,2) (4,2)
(3,4) (2,4) (4,4)	(3,1) (2,1) (4,1)	(4,2)	(3,3) (2,3) (4,3)	(2,2) 1 (3,2)

Diagram 5. $V \times Q(P, \bar{P}); D(V \times Q(P, \bar{P}))$.

4. Construction of doubly diagonalized orthogonal latin squares

Let V_1 and V_2 be a pair of doubly diagonalized orthogonal latin squares of even order based on $1, 2, ..., v = 2m$. Let Q_1 and Q_2 be a pair of orthogonal diagonalized latin squares of order q based on 1, 2, ..., q and containing the doubly diagonalized latin squares P_1 and P_2 based on $1, 2, ..., p$ in their upper left-hand corners. Finally, let $\bar{P}_1$ and $\bar{P}_2$ be a pair of orthogonal off-diagonalized latin squares of order $q-p$ based on $p+1, p+2, ..., q$.

Theorem 4.1. *If there are latin squares V_1, V_2, Q_1, Q_2, P_1, P_2, and $\bar{P}_1$, $\bar{P}_2$ satisfying the above conditions, where $|V_1| = |V_2| = v$, $|Q_1| = |Q_2| = q$, $|P_1| = |P_2| = p$ and $|\bar{P}_1| = |\bar{P}_2| = q-p$, then there is a pair of doubly diagonalized orthogonal latin squares of order $v(q-p) + p$.*

Proof. We can assume that V_1 and V_2 are idempotent. Sade [4] has shown that the singular direct products $V_1 \times Q_1(P_1, \bar{P}_1)$ and $V_2 \times Q_2(P_2, \bar{P}_2)$ are orthogonal. Now form the doubly diagonalized latin squares $D_1(V_1 \times Q_1(P_1, \bar{P}_1))$ and $D_2(V_2 \times Q_2(P_2, \bar{P}_2))$ as in Section 3. Since corresponding cells in $D_1(V_1 \times Q_1(P_1, \bar{P}_1))$ and $D_2(V_2 \times Q_2(P_2, \bar{P}_2))$ are in the same relative position as in $V_1 \times Q_1(P_1, \bar{P}_1)$ and $V_2 \times Q_2(P_2, \bar{P}_2)$, it follows that $D_1(V_1 \times Q_1(P_1, \bar{P}_1))$ and $D_2(V_2 \times Q_2(P_2, \bar{P}_2))$ are orthogonal. This completes the proof.

Corollary 4.2. *If there are t mutually orthogonal doubly diagonalized latin squares of order $v = 2m$, t mutually orthogonal diagonalized latin squares of order q containing t doubly diagonalized latin squares of order p in their upper left-hand corners, and t mutually orthogonal off-diagonalized latin squares of order $q-p$, then there are t mutually orthogonal doubly diagonalized latin squares of order $v(q-p)+p$.*

5. Extension of known results

As mentioned in Section 1, it is known that orthogonal doubly diagonalized latin squares of order n exist whenever n is odd or a multiple of 4, except possibly when n is a multiple of 3 but not of 9. We indicate here some extensions of this result.

(i) Since there is a pair of doubly diagonalized orthogonal latin squares of order 8, a pair of orthogonal diagonalized latin squares of order 5 containing a pair of doubly diagonalized latin squares of order 1 in their upper left-hand corners, and a pair of off-diagonalized orthogonal latin squares of order 4, there is a pair of doubly diagonalized orthogonal latin squares of order $8(5-1)+1 = 33$. Taking direct products produces an infinite number of pairs of doubly diagonalized orthogonal latin squares of odd order n, where n is divisible by 3 but not by 9.

(ii) Since there are 6 mutually orthogonal doubly diagonalized latin squares of order 8, at least 6 mutually orthogonal diagonalized latin squares of order 17 each containing a subsquare of order 1 in the upper left-hand corner (a latin square of order 1 is doubly diagonalized), and at least 6 mutually orthogonal off-diagonalized latin squares of order 16, there are 6 mutually orthogonal doubly diagonalized latin squares of order $8(17-1)+1 = 129$, a number that is divisible by 3 but not by 9.

6. Remarks

The problem of constructing a pair of orthogonal doubly diagonalized latin squares of order $n \equiv 2 \pmod 4$ is still open. It is doubtful if the techniques in this paper can be used for such a construction for the following reasons. The only known doubly diagonalized orthogonal latin squares of even order are divisible by 4 so that $v(q-p)+p \equiv 2 \pmod 4$ for appropriate v, q and p would have to have $v \equiv 0 \pmod 4$. But then $v(q-p)+p \equiv 2 \pmod 4$ would give $p \equiv 2 \pmod 4$ with p the order of a pair of doubly diagonalized orthogonal latin squares.

References

[1] R.C. Bose, S.S. Shrikhande and E.T. Parker, Further results on the construction of mutually orthogonal latin squares and the falsity of Euler's conjecture, Can. J. Math. 12 (1960) 189–203.

[2] J. Dénes and A.D. Keedwell, Latin squares and their applications (book in preparation).

[3] A.J.W. Hilton, On double diagonal and cross latin squares, to appear.

[4] A. Sade, Produit direct-singulier de quasigroupes orthogonaux et anti-abéliens, Ann. Soc. Sci. Bruxelles Ser. I, 74 (1960) 91–99.