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CONCERNING THE NUMBER OF MUTUALLY ORTHOGONAL LATIN SQUARES*

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Abstract. Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order n. It is shown that for large n,

$$N(n) \geq n^{1/17} - 2.$$

In addition to a known number-theoretic result, the proof uses a new combinatorial construction which also allows a quick derivation of the existence of a pair of orthogonal squares of all orders $n > 14$. In addition, it is proven that $N(n) \geq 6$ whenever $n > 90$.

1. Introduction

A *Latin square* of order n is a map $L: R \times C \to S$, where $|R| = |C| = |S| = n$ ($|X|$ denotes the cardinality of the set X), such that for fixed $i_0 \in R$ and $j_0 \in C$, and for any $x \in S$, the equation

$$L(i_0, j) = x$$

has a unique solution $j \in C$ and the equation

$$L(i, j_0) = x$$

has a unique solution $i \in R$. Elements of R are called *rows*, elements of C are *columns*, and elements of S are *symbols*. A Latin square is usually written as a square array, the cell in the ith row and jth column containing the symbol $L(i, j)$. In this context, we are requiring that in every row and column of the array, each symbol appears exactly once.

Two Latin squares $L_1: R \times C \to S_1$ and $L_2: R \times C \to S_2$ are said to

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be *orthogonal* iff for any $(x_1, x_2) \in S_1 \times S_2$, the equations

$$L_1(i,j) = x_1, \qquad L_2(i,j) = x_2$$

have a unique simultaneous solution $(i,j) \in R \times C$. k Latin squares $L_i: R \times C \to S_i$, $i = 1, 2, ..., k$, having the same row and column sets, are said to be *mutually orthogonal* iff every two of them are orthogonal. $N(n)$ will denote the largest integer k for which there exists a set of k mutually orthogonal Latin squares of order n.

The following four theorems are well known and easy to prove (see [6,14]).

Theorem 1.1. *For* $n \geq 2$, $1 \leq N(n) \leq n-1$.

Any two Latin squares of order 1 are orthogonal. There is only one Latin square of order 0 (the null square), but it is orthogonal to itself. Thus it is not unreasonable to adopt the conventions that $N(0) = N(1) = \infty$.

Theorem 1.2. $N(n) = n-1$ *if n is a prime power.*

Theorem 1.3. $N(nm) \geq \min\{N(n), N(m)\}$.

From Theorem 1.2 and 1.3 follows

Theorem 1.4. *If* $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ *is the factorization of n into powers of distinct primes p_i, then* $N(n) \geq \min_{1 \leq i \leq r} (p_i^{\alpha_i} - 1)$.

Theorem 1.4 is due to MacNeish [9] and Mann [10].

Euler conjectured that $N(n) = 1$ (i.e., no pair of orthogonal squares exists) for $n \equiv 2 \pmod 4$. MacNeish went so far as to conjecture that equality holds in Theorem 1.4. In 1901, Tarry [15] showed that in fact $N(6) = 1$ by a systematic enumeration.

Nothing else was known about $N(n)$ until the late 1950's when Parker [11] discovered three orthogonal Latin squares of order 21, disproving MacNeish's conjecture. Bose and Shrikhande [1] found the first counter-example to Euler's conjecture, a pair of orthogonal squares of order 22, and Parker [12] exhibited the first pair of order 10. More techniques were put forth by Bose and Shrikhande [2]. The work of the three authors culminated in 1960 with a joint paper [3] where it was proved that $N(n) \geq 2$ for all $n > 6$, demolishing Euler's conjecture. Their proof uses Theorem 1.4, some general construction methods using pairwise balanced designs, and sòme more special constructions using the "method of differences". One of their most significant results is the following:

Theorem 1.5. *If $m \leq N(t) + 1$ and $1 < u < t$, then*

$$N(mt + u) \geq \min\{N(m)-1, N(m + 1)-1, N(t), N(u)\}.$$

Also in 1960, Chowla, Erdös and Straus [5] observed that Theorems 1.4 and 1.5 imply $N(n) \to \infty$ as $n \to \infty$. Indeed, using a result of Brun's sieve method due to Radamacher, they proved that $N(n) > \frac{1}{3}n^{1/91}$ for sufficiently large n. With a similar argument, but using a result of Buchstab [4] in the sieve argument, Rogers [13], in 1964, showed that $N(n) > n^{1/(42+\epsilon)}$ for $n > n_\epsilon$. We shall also use Buchstab's result; it will be stated in Section 4.

Recently, Hanani [7] has shown that $N(n) \geq 3$ for $n > 51$, $N(n) \geq 5$ for $n > 62$, and $N(n) \geq 29$ for $n > 34{,}115{,}553$. His proof again uses Theorem 1.4 and 1.5, and some special constructions.

2. A construction and some inequalities

In [5], the authors remark that the numerical estimate on the lower bound for $N(n)$ could be improved if, for example, the occurrences of both $N(m)$ and $N(m + 1)$ in the inequality of Theorem 1.5, or the hypothesis $m \leq N(t) + 1$, could be eliminated. We show below (Theorem 2.3) that, indeed, the hypothesis $m \leq N(t) + 1$ can be eliminated.

Let $k \geq 2, n \geq 1$ be given. By a *transversal design* with k groups of size n, in brief a TD(k, n), we mean a triple $(X, \mathcal{G}, \mathcal{A})$, where X is a set of *kn points*, $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ is a partition of X into k subsets G_i (called *groups*), each containing n points, and $\mathcal{A}$ is a class of subsets of X (called *blocks* or *transversals*) such that each block $A \in \mathcal{A}$ contains precisely one point from each group and each pair x, y of points not contained in the same group occur together in precisely one block A.

Evidently, each block of a TD(k, n) contains k points. It is not difficult to see that each point occurs in precisely n blocks and the total number of blocks is n^2. Note that for any k, a (unique) TD(k, 1) exists. To be consistent with our convention $N(0) = \infty$, it is convenient to accept the existence of a degenerate TD(k, 0) with no points, k empty groups, and no blocks.

Transversal designs provide a compact and concise language with which to manipulate sets of orthogonal Latin squares. The following well-known lemma is due to Bose and Shrikhande [2]. For completeness, we sketch a proof here.

Lemma 2.1. *The existence of a set of $k-2$ mutually orthogonal Latin squares of order n is equivalent to the existence of a* TD(k, n).

Proof. Given a TD(k, n) $(X, \mathcal{G}, \mathcal{A})$, where $\mathcal{G} = \{G_1, G_2, ..., G_k\}$, define the maps $L_i : G_1 \times G_2 \to G_i$, $i = 3, 4, ..., k$, as follows: $L_i(x, y)$ is to be that element $z \in G_i$ for which $A \cap G_i = \{z\}$, where A is the unique block of $\mathcal{A}$ containing $\{x, y\}$. From the properties of the transversal design, it readily follows that $L_3, L_4, ..., L_k$ are Latin squares and are mutually orthogonal.

Conversely, let $L_i : R \times C \to S_i$, $i = 3,4, ..., k$, be $k-2$ mutually orthogonal Latin squares of order n. We may assume that the sets R, C, S_3, $S_4, ..., S_k$ are pairwise disjoint. With this understanding, put

$$X = R \cup C \cup S_3 \cup ... \cup S_k,$$
$$\mathcal{G} = \{R, C, S_3, ..., S_k\},$$
$$\mathcal{A} = \{\{i, j, L_3(i, j), L_4(i, j), ..., L_k(i, j)\} : i \in R, j \in C\}.$$

Then $(X, \mathcal{G}, \mathcal{A})$ is a TD(k, n).

Evidently, the existence of a TD(k, n) is equivalent to the statement $N(n) \geq k-2$.

Through each point of a block A of a TD(k, n), $n \geq 2$, there pass $n-1$ other blocks. Thus $k(n-1)$ blocks meet A in one point; the other $n^2-k(n-1)-1$ blocks are disjoint from A. If $k > n + 1$, we have a contradiction (proving Theorem 1.1); if $k = n + 1$, every block must meet A; if $k < n + 1$, there exists a block disjoint from A.

Theorem 2.2. *Let $(X, \mathcal{G}, \mathcal{A})$ be a* TD($k + l$, t), *where $\mathcal{G} = \{G_1, G_2, ..., G_k, H_1, H_2, ..., H_l\}$. Let S be any subset of $H_1 \cup H_2 \cup ... \cup H_l$. Let $m \geq 0$ be given and assume*:

(i) *for each* $i = 1, 2, ..., l$, *there exists a* TD(k, h_i), *where* $h_i = |S \cap H_i|$;
(ii) *for each block* $A \in \mathcal{A}$, *there exists a* TD($k, m + u_A$) *in which there may be found* $u_A = |S \cap A|$ *disjoint blocks.*
Then there exists a TD($k, mt + s$), *where* $s = |S|$.

Proof. Let $X_0 = G_1 \cup G_2 \cup ... \cup G_k$. For each block $A \in \mathcal{A}$, we write $A_0 = A \cap X_0$, $A' = A \cap S$.

We construct a TD($k, mt + s$) on the set of $k(mt + s)$ points $X^* = (X_0 \times M) \cup (I_k \times S)$, where M is a set of m elements and $I_k = \{1,2, ..., k\}$. As groups, we take $\mathcal{G}^* = \{G_1^*, G_2^*, ..., G_k^*\}$, where $G_i^* = (G_i \times M) \cup (\{i\} \times S)$, $i = 1,2, ..., k$. The blocks are obtained as follows:

For each block $A \in \mathcal{A}$, construct a TD($k, m + u_A$) with point set $(A_0 \times M) \cup (I_k \times A')$, groups $(A_0 \cap G_i) \times M \cup (\{i\} \times A')$, $i = 1,2, ..., k$, and blocks $\mathcal{B}_A$. Under our hypothesis that such a transversal design exists with u_A disjoint blocks, we may effect the construction so that $I_k \times \{z\}$, $z \in A'$, are blocks of $\mathcal{B}_A$. With this understanding, we denote by $\mathcal{B}'_A$ the remaining $(m + u_A)^2 - u_A$ blocks of $\mathcal{B}_A$ and put $\mathcal{B} = \bigcup_{A \in \mathcal{A}} \mathcal{B}'_A$
For each $j = 1,2, ..., l$, construct a TD(k, h_j) on the set of points $I_k \times (S \cap H_j)$ with groups $\{i\} \times (S \cap H_j)$, $i = 1,2, ..., k$, and blocks $\mathcal{C}_j$. Put $\mathcal{A}^* = \mathcal{B} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_l$. We claim that $(X^*, \mathcal{G}^*, \mathcal{A}^*)$ is a TD($k, mt + s$).

Most verifications are trivial. We check below the condition that two points of X^* which belong to different groups of $\mathcal{G}^*$ occur in precisely one block of $\mathcal{A}^*$.

The points of X^* are of the form (x, μ), $x \in X_0$, $\mu \in M$, or (i, z), $i \in I_k$, $z \in S$.

Two points $\{(x_1, \mu_1), (x_2, \mu_2)\}$ lie in different groups of $\mathcal{G}^*$ iff x_1, x_2 lie in different groups of $\mathcal{G}$. Two points $\{(i_1, z_1), (i_2, z_2)\}$ lie in different groups of $\mathcal{G}^*$ iff $i_1 \neq i_2$. Two points $\{(x, \mu), (i, z)\}$ lie in different groups of $\mathcal{G}^*$ iff $x \notin G_i$.

The pairs of points of X^* occurring in one (and only one) block of $\mathcal{C}_j$ are $\{(i_1, z_1), (i_2, z_2)\}$, where $i_1 \neq i_2$, $\{z_1, z_2\} \subseteq H_j$. The pairs of points occurring in one (and only one) block of $\mathcal{B}'_A$ are $\{(x_1, \mu_1), (x_2, \mu_2)\}$, where $x_1 \neq x_2$, $\{x_1, x_2\} \subseteq A$; $\{(x, \mu), (i, z)\}$, where $x \notin G_i$, $\{x, z\} \subseteq A$; and $\{(i_1, z_1), (i_2, z_2)\}$, where $i_1 \neq i_2$, $z_1 \neq z_2$, $\{z_1, z_2\} \subseteq A$.

With this enumeration, the properties of the original TD($k + l, t$) establish our claim.

We derive a number of corollaries of Theorem 2.2. We shall use only

Theorem 2.3 in Section 3 (two squares) and Section 4 ($n^{1/17}$ squares). Theorems 2.4 and 2.5 will be applied in our discussion of the existence of six squares in Section 5.

Theorem 2.3. *If* $0 \le u \le t$, *then*

$$N(mt+u) \ge \min\{N(m), N(m+1), N(t)-1, N(u)\}.$$

Proof. Let $k = 2 + \min\{N(m), N(m+1), N(t)-1, N(u)\}$. Then by Lemma 2.1, transversal designs TD(k, m), TD(k, $m+1$), TD($k+1$, t) and TD(k, u) exist. In the notation of Theorem 2.2, we take $l = 1$ and let S be any subset of H_1 containing u points. For each block $A \in \mathcal{A}$, $u_A = 0$ or 1. Theorem 2.2 then asserts the existence of a TD(k, $mt+u$); hence $N(mt+u) \ge k-2$.

When $l = 0$, $S = \emptyset$ in Theorem 2.2, we obtain Theorem 1.3.

Theorem 2.4. *If* $0 \le u, v \le t$, *then*

$$N(mt+u+v) \ge \min\{N(m), N(m+1), N(m+2), N(t)-2, N(u), N(v)\}.$$

Proof. Set $k-2$ equal to the indicated minimum. A TD($k+2$, t) exists. In Theorem 2.2, let $l = 2$ and choose S such that $|S \cap H_1| = u$, $|S \cap H_2| = v$. Transversal designs TD(k, u) and TD(k, v) exist by our choice of k. For any block A of the TD($k+2$, t), $u_A = 0$, 1 or 2. But transversal designs TD(k, $m+i$), $i = 0, 1, 2$, exist. Moreover, since $k \le N(m) + 2 \le m+1$, the TD(k, $m+2$) contains two disjoint blocks by an earlier remark. Theorem 2.2 asserts the existence of a TD(k, $mt+u+v$).

Theorem 2.5. *If* $t > \frac{1}{2}(l-1)(l-2)$, *then*

$$N(mt+l) \ge \min\{N(m), N(m+1), N(m+2), N(t)-l\}.$$

Proof. Let $k-2$ be the indicated minimum. A TD($k+l$, t) exists.

In the notation of Theorem 2.2, we form the set $S = \{z_1, z_2, \ldots, z_l\}$ by selecting one point z_i from each group H_i, $1 \le i \le l$, in such a way that no block A contains three elements of S. Under our hypothesis $t > \frac{1}{2}(l-1)(l-2)$, this can always be done: Inductively, if $z_1, z_2, \ldots, z_r$, $r < l$, have been chosen with no three in a common block, consider the

$\frac{1}{2}r(r-1)$ blocks A_{ij}, $1 \leq i < j \leq r$, such that $\{z_i, z_j\} \subseteq A_{ij}$. There must be at least one point $z_{r+1} \in H_{r+1}$ not contained in any of the blocks A_{ij}; then no three of $z_1, z_2, \ldots, z_{r+1}$ lie in a common block.

With this choice of S, $u_A = 0$, 1 or 2 for each block A of the TD$(k+l, t)$. Again, transversal designs TD$(k, m+i)$, $i = 0, 1, 2$, exist and the TD$(k, m+2)$ has two disjoint blocks. By Theorem 2.2, a TD$(k, mt+l)$ exists.

3. Two squares

We pause here to give a proof of the theorem of Bose, Shrikhande and Parker [3].

Theorem 3.1. *For $n \neq 2, 6, N(n) \geq 2$.*

Proof. Pairs of orthogonal Latin squares of orders 10 and 14 are constructed in [3]. (They are also exhibited in [6].) In view of this and Theorem 1.4, it remains to show $N(n) \geq 2$ for $n \equiv 2 \pmod 4$, $n \geq 18$.

Given $n \equiv 2 \pmod 4$, $n \geq 18$, define t and u as in Table 1, depending on the residue of n modulo 18.

Table 1

n	t	u
$18s$	$6s-1$	3
$18s+2$	$6s-1$	5
$18s+4$	$6s+1$	1
$18s+6$	$6s+1$	3
$18s+8$	$6s+1$	5
$18s+10$	$6s+1$	7
$18s+12$	$6s+1$	9
$18s+14$	$6s+1$	11
$18s+16$	$6s+5$	1

By Theorem 1.4, $N(t) \geq 4$, $N(u) \geq 2$. With the exception of $n = 30$, we have $0 \leq u \leq t$, so taking $m = 3$ in Theorem 2.3,

$$N(n) = N(mt+u) \geq \min\{N(m), N(m+1), N(t)-1, N(u)\} \geq \min\{2,3,3,2\} = 2$$

0a	X	0b	1a	Y	1b	2a	Z	2b	3a	3b	3c	4a	4b	4c	0c	1c	2c
0c	0b	X	1c	1b	Y	2c	2b	Z	3b	3c	3a	4b	4c	4a	0a	1a	2a
X	0a	0c	Y	1a	1c	Z	2a	2c	3c	3a	3b	4c	4a	4b	0b	1b	2b
1a	1b	1c	2a	X	2b	3a	Y	3b	4a	Z	4b	0a	0b	0c	2c	3c	4c
1b	1c	1a	2c	2b	X	3c	3b	Y	4c	4b	Z	0b	0c	0a	2a	3a	4a
1c	1a	1b	X	2a	2c	Y	3a	3c	Z	4a	4c	0c	0a	0b	2b	3b	4b
2a	2b	2c	3a	3b	3c	4a	X	4b	0a	Y	0b	1a	Z	1b	4c	0c	1c
2b	2c	2a	3b	3c	3a	4c	4b	X	0c	0b	Y	1c	1b	Z	4a	0a	1a
2c	2a	2b	3c	3a	3b	X	4a	4c	Y	0a	0c	Z	1a	1c	4b	0b	1b
3a	Z	3b	4a	4b	4c	0a	0b	0c	1a	X	1b	2a	Y	2b	1c	2c	3c
3c	3b	Z	4b	4c	4a	0b	0c	0a	1c	1b	X	2c	2b	Y	1a	2a	3a
Z	3a	3c	4c	4a	4b	0c	0a	0b	X	1a	1c	Y	2a	2c	1b	2b	3b
4a	Y	4b	0a	Z	0b	1a	1b	1c	2a	2b	2c	3a	X	3b	3c	4c	0c
4c	4b	Y	0c	0b	Z	1b	1c	1a	2b	2c	2a	3c	3b	X	3a	4a	0a
Y	4a	4c	Z	0a	0c	1c	1a	1b	2c	2a	2b	X	3a	3c	3b	4b	0b
0b	0c	0a	2b	2c	2a	4b	4c	4a	1b	1c	1a	3b	3c	3a	X	Y	Z
4b	4c	4a	1b	1c	1a	3b	3c	3a	0b	0c	0a	2b	2c	2a	Y	Z	X
3b	3c	3a	0b	0c	0a	2b	2c	2a	4b	4c	4a	1b	1c	1a	Z	X	Y

0a	0c	X	1a	1c	Y	2a	2c	Z	3a	3b	3c	4a	4b	4c	0b	1b	2b
X	0b	0a	Y	1b	1a	Z	2b	2a	3c	3a	3b	4c	4a	4b	0c	1c	2c
0b	X	0c	1b	Y	1c	2b	Z	2c	3b	3c	3a	4b	4c	4a	0a	1a	2a
2a	2b	2c	3a	3c	X	4a	4c	Y	0a	0c	Z	1a	1b	1c	3b	4b	0b
2c	2a	2b	X	3b	3a	Y	4b	4a	Z	0b	0a	1c	1a	1b	3c	4c	0c
2b	2c	2a	3b	X	3c	4b	Y	4c	0b	Z	0c	1b	1c	1a	3a	4a	0a
4a	4b	4c	0a	0b	0c	1a	1c	X	2a	2c	Y	3a	3c	Z	1b	2b	3b
4c	4a	4b	0c	0a	0b	X	1b	1a	Y	2b	2a	Z	3b	3a	1c	2c	3c
4b	4c	4a	0b	0c	0a	1b	X	1c	2b	Y	2c	3b	Z	3c	1a	2a	3a
1a	1c	Z	2a	2b	2c	3a	3b	3c	4a	4c	X	0a	0c	Y	4b	0b	1b
Z	1b	1a	2c	2a	2b	3c	3a	3b	X	4b	4a	Y	0b	0a	4c	0c	1c
1b	Z	1c	2b	2c	2a	3b	3c	3a	4b	X	4c	0b	Y	0c	4a	0a	1a
3a	3c	Y	4a	4c	Z	0a	0b	0c	1a	1b	1c	2a	2c	X	2b	3b	4b
Y	3b	3a	Z	4b	4a	0c	0a	0b	1c	1a	1b	X	2b	2a	2c	3c	4c
3b	Y	3c	4b	Z	4c	0b	0c	0a	1b	1c	1a	2b	X	2c	2a	3a	4a
0c	0a	0b	3c	3a	3b	1c	1a	1b	4c	4a	4b	2c	2a	2b	X	Y	Z
3c	3a	3b	1c	1a	1b	4c	4a	4b	2c	2a	2b	0c	0a	0b	Z	X	Y
1c	1a	1b	4c	4a	4b	2c	2a	2b	0c	0a	0b	3c	3a	3b	Y	Z	X

Fig. 1.

0a	X	0b	1a	1b	1c	2a	2b	2c	3a	3b	3c	4a	4b	4c	5a	5b	5c	6a	6b	6c	0c
0c	0b	X	1b	1c	1a	2b	2c	2a	3b	3c	3a	4b	4c	4a	5b	5c	5a	6b	6c	6a	0a
X	0a	0c	1c	1a	1b	2c	2a	2b	3c	3a	3b	4c	4a	4b	5c	5a	5b	6c	6a	6b	0b
1a	1b	1c	2a	X	2b	3a	3b	3c	4a	4b	4c	5a	5b	5c	6a	6b	6c	0a	0b	0c	2c
1b	1c	1a	2c	2b	X	3b	3c	3a	4b	4c	4a	5b	5c	5a	6b	6c	6a	0b	0c	0a	2a
1c	1a	1b	X	2a	2c	3c	3a	3b	4c	4a	4b	5c	5a	5b	6c	6a	6b	0c	0a	0b	2b
2a	2b	2c	3a	3b	3c	4a	X	4b	5a	5b	5c	6a	6b	6c	0a	0b	0c	1a	1b	1c	4c
2b	2c	2a	3b	3c	3a	4c	4b	X	5b	5c	5a	6b	6c	6a	0b	0c	0a	1b	1c	1a	4a
2c	2a	2b	3c	3a	3b	X	4a	4c	5c	5a	5b	6c	6a	6b	0c	0a	0b	1c	1a	1b	4b
3a	3b	3c	4a	4b	4c	5a	5b	5c	6a	X	6b	0a	0b	0c	1a	1b	1c	2a	2b	2c	6c
3b	3c	3a	4b	4c	4a	5b	5c	5a	6c	6b	X	0b	0c	0a	1b	1c	1a	2b	2c	2a	6a
3c	3a	3b	4c	4a	4b	5c	5a	5b	X	6a	6c	0c	0a	0b	1c	1a	1b	2c	2a	2b	6b
4a	4b	4c	5a	5b	5c	6a	6b	6c	0a	0b	0c	1a	X	1b	2a	2b	2c	3a	3b	3c	1c
4b	4c	4a	5b	5c	5a	6b	6c	6a	0b	0c	0a	1c	1b	X	2b	2c	2a	3b	3c	3a	1a
4c	4a	4b	5c	5a	5b	6c	6a	6b	0c	0a	0b	X	1a	1c	2c	2a	2b	3c	3a	3b	1b
5a	5b	5c	6a	6b	6c	0a	0b	0c	1a	1b	1c	2a	2b	2c	3a	X	3b	4a	4b	4c	3c
5b	5c	5a	6b	6c	6a	0b	0c	0a	1b	1c	1a	2b	2c	2a	3c	3b	X	4b	4c	4a	3a
5c	5a	5b	6c	6a	6b	0c	0a	0b	1c	1a	1b	2c	2a	2b	X	3a	3c	4c	4a	4b	3b
6a	6b	6c	0a	0b	0c	1a	1b	1c	2a	2b	2c	3a	3b	3c	4a	4b	4c	5a	X	5b	5c
6b	6c	6a	0b	0c	0a	1b	1c	1a	2b	2c	2a	3b	3c	3a	4b	4c	4a	5c	5b	X	5a
6c	6a	6b	0c	0a	0b	1c	1a	1b	2c	2a	2b	3c	3a	3b	4c	4a	4b	X	5a	5c	5b
0b	0c	0a	2b	2c	2a	4b	4c	4a	6b	6c	6a	1b	1c	1a	3b	3c	3a	5b	5c	5a	X

Fig. 2.

for $n \neq 30$. Finally, with $m = 3, t = 9, u = 3$, Theorem 2.3 implies $N(30) \geq 2$.

The actual construction of Latin squares by Theorem 2.3, especially in the cases in Table 1, is very easy. We illustrate by explicitly constructing a pair of orthogonal squares of order $n = 3t + u$, when $0 \leq u \leq t$, $N(t) \geq 3, N(u) \geq 2$. (Such pairs of orders 18 and 22 are shown in Figs. 1 and 2; here $t = 5, u = 3$, and $t = 7, u = 1$, respectively.) We lapse into a more informal language.

Start with three orthogonal squares L_1, L_2, L_{-1} of order t. (When $t = \pm 1 \pmod 6$, we may take the squares with row, column, and symbol set $\mathbf{Z}_t$ (the integers modulo t) defined by $L_r(i,j) = ri + j$, $r = 1, 2, -1$.)

0a	0c	X	1a	1b	1c	2a	2b	2c	3a	3b	3c	4a	4b	4c	5a	5b	5c	6a	6b	6c	0b
X	0b	0a	1c	1a	1b	2c	2a	2b	3c	3a	3b	4c	4a	4b	5c	5a	5b	6c	6a	6b	0c
0b	X	0c	1b	1c	1a	2b	2c	2a	3b	3c	3a	4b	4c	4a	5b	5c	5a	6b	6c	6a	0a
2a	2b	2c	3a	3c	X	4a	4b	4c	5a	5b	5c	6a	6b	6c	0a	0b	0c	1a	1b	1c	3b
2c	2a	2b	X	3b	3a	4c	4a	4b	5c	5a	5b	6c	6a	6b	0c	0a	0b	1c	1a	1b	3c
2b	2c	2a	3b	X	3c	4b	4c	4a	5b	5c	5a	6b	6c	6a	0b	0c	0a	1b	1c	1a	3a
4a	4b	4c	5a	5b	5c	6a	6c	X	0a	0b	0c	1a	1b	1c	2a	2b	2c	3a	3b	3c	6b
4c	4a	4b	5c	5a	5b	X	6b	6a	0c	0a	0b	1c	1a	1b	2c	2a	2b	3c	3a	3b	6c
4b	4c	4a	5b	5c	5a	6b	X	6c	0b	0c	0a	1b	1c	1a	2b	2c	2a	3b	3c	3a	6a
6a	6b	6c	0a	0b	0c	1a	1b	1c	2a	2c	X	3a	3b	3c	4a	4b	4c	5a	5b	5c	2b
6c	6a	6b	0c	0a	0b	1c	1a	1b	X	2b	2a	3c	3a	3b	4c	4a	4b	5c	5a	5b	2c
6b	6c	6a	0b	0c	0a	1b	1c	1a	2b	X	2c	3b	3c	3a	4b	4c	4a	5b	5c	5a	2a
1a	1b	1c	2a	2b	2c	3a	3b	3c	4a	4b	4c	5a	5c	X	6a	6b	6c	0a	0b	0c	5b
1c	1a	1b	2c	2a	2b	3c	3a	3b	4c	4a	4b	X	5b	5a	6c	6a	6b	0c	0a	0b	5c
1b	1c	1a	2b	2c	2a	3b	3c	3a	4b	4c	4a	5b	X	5c	6b	6c	6a	0b	0c	0a	5a
3a	3b	3c	4a	4b	4c	5a	5b	5c	6a	6b	6c	0a	0b	0c	1a	1c	X	2a	2b	2c	1b
3c	3a	3b	4c	4a	4b	5c	5a	5b	6c	6a	6b	0c	0a	0b	X	1b	1a	2c	2a	2b	1c
3b	3c	3a	4b	4c	4a	5b	5c	5a	6b	6c	6a	0b	0c	0a	1b	X	1c	2b	2c	2a	1a
5a	5b	5c	6a	6b	6c	0a	0b	0c	1a	1b	1c	2a	2b	2c	3a	3b	3c	4a	4c	X	4b
5c	5a	5b	6c	6a	6b	0c	0a	0b	1c	1a	1b	2c	2a	2b	3c	3a	3b	X	4b	4a	4c
5b	5c	5a	6b	6c	6a	0b	0c	0a	1b	1c	1a	2b	2c	2a	3b	3c	3a	4b	X	4c	4a
0c	0a	0b	3c	3a	3b	6c	6a	6b	2c	2a	2b	5c	5a	5b	1c	1a	1b	4c	4a	4b	X

Fig. 2 (continued).

For $t = 5$,

$$L_1 = \begin{array}{|ccccc|} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{array}, \quad L_2 = \begin{array}{|ccccc|} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{array}, \quad L_{-1} = \begin{array}{|ccccc|} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{array}$$

We construct a pair L_1^*, L_2^* of orthogonal Latin squares of order n. The symbols are to be Ia, Ib, Ic, where I runs through the t symbols of L_1 and L_2, and u additional symbols $X_1, X_2, \ldots, X_u$. Form L'_{-1} by replacing u of the symbols of L_{-1} with $X_1, \ldots, X_u$ and leaving the other cells blank. For $t = 5$, $u = 3$,

$$L'_{-1} = \begin{array}{|ccccc|} X_1 & X_2 & X_3 & & \\ & X_1 & X_2 & X_3 & \\ & & X_1 & X_2 & X_3 \\ X_3 & & & X_1 & X_2 \\ X_2 & X_3 & & & X_1 \end{array}$$

To effect the construction, we require the following "ingredients": Let U_1 and U_2 be two orthogonal Latin squares of order u on the symbols $X_1, ..., X_u$. Take two orthogonal squares of order 3 on the symbols a, b, c, say

$$A_1 = \begin{array}{|ccc|} a & b & c \\ b & c & a \\ c & a & b \end{array} \quad \text{and} \quad A_2 = \begin{array}{|ccc|} a & b & c \\ c & a & b \\ b & c & a \end{array}$$

Further, we need two orthogonal squares of order 4 on the symbols a, b, c, X, say

$$B_1 = \begin{array}{|cccc|} a & X & b & c \\ c & b & X & a \\ X & a & c & b \\ b & c & a & X \end{array} \quad \text{and} \quad B_2 = \begin{array}{|cccc|} a & c & X & b \\ X & b & a & c \\ b & X & c & a \\ c & a & b & X \end{array}$$

It is important here that the symbol X occurs in the lower right-hand corner of both squares.

L_1^* and L_2^* are now obtained as follows: The n (= $3t + u$) rows and columns are labelled by (i, k), $i = 1, 2, ..., t$, $k = 1, 2, 3$, and $(1), (2), ..., (u)$. Place U_i in the $(u \times u)$-subsquare of L_i^* consisting of rows $(1), ..., (u)$ and columns $(1), ..., (u)$, $i = 1, 2$. (Cf. Fig. 1, where we have used symbols X, Y, Z instead of X_1, X_2, X_3.) To complete L_1^*, for each (i, j), $i, j = 1, 2, ..., t$, find the symbol I occurring in the (i, j)th cell of L_1. If the (i, j)-cell of L'_{-1} is unoccupied, fill in the (3×3)-subsquare of L_1^* consisting of rows $(i, 1)$, $(i, 2)$, $(i, 3)$ and columns $(j, 1)$, $(j, 2)$, $(j, 3)$ with symbols Ia, Ib, Ic using A_1 as a model, viz.

row \ col.	$(j, 1)$	$(j, 2)$	$(j, 3)$
$(i, 1)$	Ia	Ib	Ic
$(i, 2)$	Ib	Ic	Ia
$(i, 3)$	Ic	Ia	Ib

If, however, the (i, j)-cell of L'_{-1} contains a symbol X_l, fill in the (4×4)-subsquare of L_1^* consisting of rows $(i, 1)$, $(i, 2)$, $(i, 3)$, (l) and columns $(j, 1)$, $(j, 2)$, $(j, 3)$, (l) (except for the cell in the (l)th row and (l)th column, which is already filled) with symbols Ia, Ib, Ic, X_l using B_1 as a model, viz.

row \ col.	$(j, 1)$	$(j, 2)$	$(j, 3)$		(l)
$(i, 1)$	Ia	X_l	Ib		Ic
$(i, 2)$	Ic	Ib	X_l		Ia
$(i, 3)$	X_l	Ia	Ic		Ib
(l)	Ib	Ic	Ia		

L_2^* is completed similarly, the only differences being that the symbol I is to be found by referring to L_2 and A_2, B_2 are to be used as the models.

It is easily checked directly that the resulting squares are orthogonal. The reader will be able to generalize to the construction analogous to the full statement of Theorem 2.3.

Fig. 3 shows a pair of orthogonal Latin squares of order 22 obtained from the construction of Theorem 2.4 with $m = 3$, $t = 5$, $u = 3$, $v = 4$. Compare Figs. 2 and 3 with the original constructions of [1, 2] using Kirkman designs, a general construction for orders congruent to 10 modulo 12 (see [3]), and two pairs of order 22 obtained by Hedayat and Seiden [8].

4. $n^{1/17}$ squares

We introduce the notation of Buchstab [4]. Let x, y be positive real numbers. Let $p_0 = 2, p_1 = 3, p_2 = 5, p_3, \ldots, p_r$ be all the primes less than

0a 0b 0c 0b 0c 0a 0c 0a 0b	1a 1c A A 1b Y Y 1a 1c	2a B 2b 2c 2b B B 2a 2c	3a 3c C C 3b Z Z 3a 3c	4a 4c D D 4b X X 4a 4c	4b 1b 3b 4a 1a 3a D A C	Y 2c Z X 1c 2a 3c 4c 1b 2b 3b 4b
2a 2c D D 2b Y Y 2a 2c	3a 3b 3c 3b 3c 3a 3c 3a 3b	4a 4c A A 4b Z Z 4a 4c	0a 0c B B 0b X X 0a 0c	1a C 1b 1c 1b C C 1a 1c	0b 2b 4b 0a 2a 4a B D A	Z X 1c Y 4c 0c 1a 2c 4b 0b 1b 2b
4a C 4b 4c 4b C C 4a 4c	0a 0c D D 0b Z Z 0a 0c	1a X 1b 1c 1b X X 1a 1c	2a A 2b 2c 2b A A 2a 2c	3a 3c B B 3b Y Y 3a 3c	1c 3b 0b 1a 3a 0a 1b B D	2c Y 4c Z 2a 3c 4a 0c 2b 3b 4b 0b
1a 1c B B 1b Z Z 1a 1c	2a 2c C C 2b X X 2a 2c	3a D 3b 3c 3b D D 3a 3c	4a Y 4b 4c 4b Y Y 4a 4c	0a A 0b 0c 0b A A 0a 0c	2b 4c 1b 2a 4a 1a C 4b B	0c Z X 3c 0a 1c 2c 3a 0b 1b 2b 3b
3a 3c A A 3b X X 3a 3c	4a B 4b 4c 4b B B 4a 4c	0a 0c C C 0b Y Y 0a 0c	1a D 1b 1c 1b D D 1a 1c	2a Z 2b 2c 2b Z Z 2a 2c	3b 0b 2c 3a 0a 2a A C 2b	X 4c Y 1c 3c 4a 0c 1a 3b 4b 0b 1b
3c A 3b 2c D 2b 1c B 1b	2c C 2b 1c A 1b 0c D 0b	1b 1c 1a 0c C 0b 4c A 4b	0c B 0b 4b 4c 4a 3c C 3b	4c D 4b 3c B 3b 2b 2c 2a	X Y Z Y Z X Z X Y	3a 0a 2a 4a 1a 3a 0a 2a 4a 1a 3a 0a
3b X 3a 1b Z 1a 4b 4c 4a 2b Y 2a	1b Y 1a 4b 4c 4a 2b X 2a 0b Z 0a	4b Z 4a 2b 2c 2a 0b Y 0a 3b 3c 3a	2b 2c 2a 0b X 0a 3b Z 3a 1b 1c 1a	0b 0c 0a 3b Y 3a 1b 1c 1a 4b X 4a	3c 1c 4c 0c 3c 1c 2c 0c 3c 4c 2c 0c	A D B C C B D A D A C B B C A D

Fig. 3.

y and let ω denote the choice of integers $a_0, a_1, ..., a_r, b_1, ..., b_r$. Then $P_\omega(x; y)$ is to denote the number of non-negative integers not exceeding x which do *not* lie in any of the arithmetic progressions $a_0 \pmod{p_0}$, $a_i \pmod{p_i}$, $b_i \pmod{p_i}$. Buchstab proves

$$P_\omega(x; x^{1/5}) > \lambda(5)\, cx/(\log x)^2 + O(x/(\log x)^3),$$

independent of the choice of ω. Here c is a constant 0.4161 ... and $\lambda(5) > 0.96$.

As an immediate consequence, $P_\omega(x; x^{1/5}) \geq 2$ for sufficiently large x, or equivalently,

Lemma 4.1. *There is a constant n_0 such that for $n > n_0$, we have*

$$P_\omega(n^{5/17}; n^{1/17}) \geq 2.$$

0a 0b 0c 0c 0a 0b 0b 0c 0a	1a Y 1b Y 1b A 1b A 1c	2a 2c B B 2b 2a 2b B 2c	3a Z 3b Z 3b C 3b C 3c	4a X 4b X 4b D 4b D 4c	D A C 4c 1c 3c 4a 1a 3a	1c 2b 3c 4c 1a 2c 3a 4a Y 2a Z X
3a Y 3b Y 3b D 3b D 3c	4a 4b 4c 4c 4a 4b 4b 4c 4a	0a Z 0b Z 0b A 0b A 0c	1a X 1b X 1b B 1b B 1c	2a 2c C C 2b 2a 2b C 2c	B D A 1c 3c 0c 1a 3a 0a	0c 1c 2b 3c 0a 1a 2c 3a Z X 2a Y
1a 1c C C 1b 1a 1b C 1c	2a Z 2b Z 2b D 2b D 2c	3a 3c X X 3b 3a 3b X 3c	4a 4c A A 4b 4a 4b A 4c	0a Y 0b Y 0b B 0b B 0c	3b B D 3c 0c 2c 3a 0a 2a	4b 0c 1b 2c 4c 0a 1c 2a 4a Y 1a Z
4a Z 4b Z 4b B 4b B 4c	0a X 0b X 0b C 0b C 0c	1a 1c D D 1b 1a 1b D 1c	2a 2c Y Y 2b 2a 2b Y 2c	3a 3c A A 3b 3a 3b A 3c	C 2b B 0c 2c 4c 0a 2a 4a	3b 4c 0c 1b 3c 4a 0a 1c 3a Z X 1a
2a X 2b X 2b A 2b A 2c	3a 3c B B 3b 3a 3b B 3c	4a Y 4b Y 4b C 4b C 4c	0a 0c D D 0b 0a 0b D 0c	1a 1c Z Z 1b 1a 1b Z 1c	A C 1b 2c 4c 1c 2a 4a 1a	2c 3b 4c 0b 2a 3c 4a 0c X 3a Y 0a
A 2c 2a D 3c 3a B 4c 4a	C 0c 0a A 1c 1a D 2c 2a	3c 3a 3b C 4c 4a A 0c 0a	B 1c 1a 2c 2a 2b C 3c 3a	D 4c 4a B 0c 0a 1c 1a 1b	X Y Z Z X Y Y Z X	2b 1b 0b 4b 1b 0b 4b 3b 0b 4b 3b 2b
2c 2a X 4c 4a Z 1c 1a 1b 3c 3a Y	1c 1a Y 3c 3a 3b 0c 0a X 2c 2a Z	0c 0a Z 2c 2a 2b 4c 4a Y 1c 1a 1b	4c 4a 4b 1c 1a X 3c 3a Z 0c 0a 0b	3c 3a 3b 0c 0a Y 2c 2a 2b 4c 4a X	2b 1b 0b 1b 0b 4b 0b 4b 3b 4b 3b 2b	A C D B D B A C B D C A C A B D

Fig. 3 (continued).

Theorem 4.2. *For $n > n_0$, $N(n) \geq n^{1/17} - 2$.*

Proof. Let $n > n_0$ be given. Choose an integer l such that $2^l < n^{1/17} - 1 \leq 2^{l+1}$. By Lemma 4.1, we may select an integer s satisfying

$$0 \leq s \leq n^{5/17} ,$$
$$s \equiv 0 \pmod 2 ,$$
$$s \not\equiv 0 \pmod{p_i} , \qquad 1 \leq i \leq r ,$$
$$2^l s \not\equiv (-1)^n \pmod{p_i}, \quad 1 \leq i \leq r ,$$

where $p_1, p_2, ..., p_r$ are all the odd primes less than $n^{1/17}$. Define m by

$$m = \begin{cases} 2^l s - 1 & \text{if } n \text{ is even,} \\ 2^l s & \text{if } n \text{ is odd.} \end{cases}$$

No prime divisors of $2^l s - (-1)^n$ are less than $n^{1/17}$; $2^l s$ is divisible by 2^{l+1} and no odd primes less than $n^{1/17}$. Thus by Theorem 1.4,

(1) $\quad N(m) \geq n^{1/17} - 2$,

(2) $\quad N(m+1) \geq n^{1/17} - 2$.

Note that

(3) $\quad m + 1 \leq n^{6/17}$.

Again by Lemma 4.1, select an integer t', $0 < t' \leq n^{5/17}$, such that, with $t = [n/(m+1)] + t'$, we have

$$t \equiv 1 \pmod 2,$$
$$t \not\equiv 0 \pmod{p_i}, \qquad 1 \leq i \leq r,$$
$$mt \not\equiv n \pmod{p_i}, \qquad 1 \leq i \leq r.$$

(No prime p_i divides m, so this last incongruence is equivalent to an incongruence of the form $t \not\equiv n' \pmod{p_i}$.) Note that $m \not\equiv n \pmod 2$, so we also have $mt \not\equiv n \pmod 2$. Put $u = n - mt$, so that $n = mt + u$. By Theorem 1.4,

(4) $\quad N(t) \geq n^{1/17} - 1$,

(5) $\quad N(u) \geq n^{1/17} - 1$.

By our choice of t', $t > n/(m+1)$ and $t \leq t' + n/(m+1) \leq n^{5/17} + n/(m+1) = n/m + (n^{5/17} - n/m(m+1)) < n/m$ because of the inequality (3). Equivalently,

(6) $\quad 0 < u < t$.

In view of the inequalities (1), (2), (4), (5), (6), and Theorem 2.3, we have $N(n) = N(mt+u) \geq n^{1/17} - 2$.

Remark 4.3. Using more of the power of Buchstab's result, the unsightly "-2" can be eliminated in the statement of Theorem 4.2.

Table 2

w	u_w	v_w	w	u_w	v_w	w	u_w	v_w
7	7	0	31	23	8	54	27	27
8	8	0	32	16	16	55	32	23
9	9	0	33	17	16	56	29	27
10	9	1	34	17	17	57	32	25
11	11	0	35	19	16	58	29	29
12	11	1	36	19	17	59	32	27
13	13	0	37	29	8	60	31	29
14	7	7	38	19	19	61	32	29
15	8	7	39	23	16	62	31	31
16	8	8	40	23	17	63	32	31
17	9	8	41	25	16	64	32	32
18	9	9	42	23	19	65	49	16
19	11	8	43	27	16	66	37	29
20	11	9	44	25	19	67	56	11
21	13	8	45	29	16	68	37	31
22	11	11	46	23	23	69	37	32
23	16	7	47	31	16	70	41	29
24	13	11	48	25	23	71	63	8
25	16	9	49	32	17	72	41	31
26	13	13	50	25	25	73	41	32
27	16	11	51	32	19	74	37	37
28	17	11	52	27	25	75	43	32
29	16	13	53	37	16	76	47	29
30	17	13						

5. Six squares

The three consecutive integers 7, 8, 9 are prime powers. We exploit this fact here by applying Theorems 2.4 and 2.5 with $m = 7$.

Theorem 5.1. *$N(n) \geq 6$ whenever $n > 90$.*

We begin with two lemmas.

Lemma 5.2. *For any integer r, at least one of the numbers $r, r+1, r+2, \ldots, r+9$ is a unit modulo $210 = 2 \cdot 3 \cdot 5 \cdot 7$.*

Proof. Let r' be the odd element of $\{r, r+1\}$. Of the integers $r', r'+2, r'+4, r'+6, r'+8$, at most two are divisible by 3, at most one by 5, and at most one by 7; hence at least one is divisible by neither 3, 5, 7, nor, of course, 2.

Table 3

$455 \le n \le 516$	Lemma 5.3:	$t = 64,\ 7 \le w \le 68.$
$420 \le n \le 454$	Lemma 5.3:	$t = 59,\ 7 \le w \le 41.$
$378 \le n \le 419$	Lemma 5.3:	$t = 53,\ 7 \le w \le 48.$
$350 \le n \le 377$	Lemma 5.3:	$t = 49,\ 7 \le w \le 34.$
$308 \le n \le 349$	Lemma 5.3:	$t = 43,\ 7 \le w \le 48.$
$266 \le n \le 307$	Lemma 5.3:	$t = 37,\ 7 \le w \le 48.$
$224 \le n \le 265$	Lemma 5.3:	$t = 31,\ 7 \le w \le 48.$
$196 \le n \le 223$	Lemma 5.3:	$t = 27,\ 7 \le w \le 34.$
$168 \le n \le 195$	Lemma 5.3:	$t = 23,\ 7 \le w \le 34.$
$164 \le n \le 167$	Theorem 2.5:	$m = 7,\ t = 23,\ 3 \le l \le 6.$
$140 \le n \le 163$	Lemma 5.3:	$t = 19,\ 7 \le w \le 30.$
$119 \le n \le 139$	Lemma 5.3:	$t = 16,\ 7 \le w \le 27.$
$114 \le n \le 118$	Theorem 2.5:	$m = 7,\ t = 16,\ 2 \le l \le 6.$
$98 \le n \le 113$	Lemma 5.3:	$t = 13,\ 7 \le w \le 22.$
$91 \le n \le 97$	Lemma 5.3:	$t = 11,\ 14 \le w \le 20.$
$84 \le n \le 89$	Lemma 5.3:	$t = 11,\ 7 \le w \le 12.$
$n = 83$	Theorem 1.4.	
$77 \le n \le 81$	Theorem 2.5:	$t = 11,\ m = 7,\ 0 \le l \le 4.$

For integers w, $7 \le w \le 76$, define u_w and v_w as in Table 2. Note that for each w, $7 \le w \le 76$, we have $w = u_w + v_w$, $0 \le u_w, v_w \le 63$, and $N(u_w), N(v_w) \ge 6$ by Theorem 1.4.

Lemma 5.3. *If* $7 \le w \le 76$ *and* $u_w, v_w \le t$, *then*

$$N(7t + w) \ge \min\{6, N(t) - 2\}\ .$$

Proof. Apply Theorem 2.4 with $m = 7$, $u = u_w$, $v = v_w$.

Proof of Theorem 5.1. We first show that $N(n) \ge 6$ for $n \ge 517$. Given $n \ge 517$, by Lemma 5.2, there may be found an integer t, relatively prime to 210, such that $[\frac{1}{7}n] - 10 \le t \le [\frac{1}{7}n] - 1$. Then $n-76 \le 7t \le n-7$, so with $w = n-7t$, we have $7 \le w \le 76$. Also, $t \ge \frac{1}{7}(n-76) \ge 63$. $N(t) \ge 10$ by Theorem 1.4, so Lemma 5.3 gives $N(n) = N(7t + w) \ge 6$.

We complete the proof that $N(n) \ge 6$ for $90 < n < 517$ with Table 3. The table extends far enough to prove $N(n) \ge 6$ for $n > 76$, $n \ne 82, 90$.

Hanani [7] denotes by n_r the smallest integer such that $N(n) \ge r$ for every $n > n_r$. We have proven $n_6 \le 90$.

Hanani shows that $n_5 \le 62$. In view of this, we can say $n_4 \le 60$ since $N(62) \ge 4$ by Theorem 2.4, with $t = 8$, $m = 7$, $u = 5$, $v = 1$ and $N(61) =$

60. Hanani's result that $n_3 \leq 51$ can be improved to $n_3 \leq 46$ since $N(51) \geq 3$ (Theorem 2.3: $m = 4$, $t = 11$, $u = 7$), $N(50) \geq 5$ (Theorem 2.3: $m = t = 7$, $u = 1$), $N(49) = 48$, $N(48) \geq 3$ (Theorem 2.3: $m = 4$, $t = 11$, $u = 4$) and $N(47) = 46$. Of course, $n_2 = 6$ (see [3, 15]).

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