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Transversals in latin squares

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Abstract

A latin square of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in each row and column. A transversal of such a square is a set of n entries such that no two entries share the same row, column or symbol. Transversals are closely related to the notions of complete mappings and orthomorphisms in (quasi)groups, and are fundamental to the concept of mutually orthogonal latin squares.

Here we provide a brief survey of the literature on transversals. We cover (1) existence and enumeration results, (2) generalisations of transversals including partial transversals and plexes, (3) the special case when the latin square is a group table, (4) a connection with covering radii of sets of permutations. The survey includes a number of conjectures and open problems.

1. Introduction

A *latin square* of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in each row and in each column. By a *diagonal* of such a square we mean a set of entries which contains exactly one representative of each row and column. A *transversal* is a diagonal in which no symbol is repeated.

Historically, interest in transversals arose from the study of orthogonal latin squares. A pair of latin squares $A = [a_{ij}]$ and $B = [b_{ij}]$ of order n are said to be *orthogonal mates* if the n^2 ordered pairs (a_{ij}, b_{ij}) are distinct. It is simple to see that if we look at all n occurrences of a given symbol in B, the corresponding positions in A must form a transversal. Indeed,

Theorem 1. *A latin square has an orthogonal mate iff it has a decomposition into disjoint transversals.* □

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For example, below there are two orthogonal latin squares of order 8. Subscripted letters are used to mark the transversals of the left hand square which correspond to the positions of each symbol in its orthogonal mate (the right hand square).

$$\begin{array}{cccccccc}
1_a & 2_b & 3_c & 4_d & 5_e & 6_f & 7_g & 8_h \\
7_b & 8_a & 5_d & 6_c & 2_f & 4_e & 1_h & 3_g \\
2_c & 1_d & 6_a & 3_b & 4_g & 5_h & 8_e & 7_f \\
8_d & 7_c & 4_b & 5_a & 6_h & 2_g & 3_f & 1_e \\
4_f & 3_e & 1_g & 2_h & 7_a & 8_b & 5_c & 6_d \\
6_e & 5_f & 7_h & 8_g & 1_b & 3_a & 2_d & 4_c \\
3_h & 6_g & 2_e & 1_f & 8_c & 7_d & 4_a & 5_b \\
5_g & 4_h & 8_f & 7_e & 3_d & 1_c & 6_b & 2_a
\end{array}
\qquad
\begin{array}{cccccccc}
a & b & c & d & e & f & g & h \\
b & a & d & c & f & e & h & g \\
c & d & a & b & g & h & e & f \\
d & c & b & a & h & g & f & e \\
f & e & g & h & a & b & c & d \\
e & f & h & g & b & a & d & c \\
h & g & e & f & c & d & a & b \\
g & h & f & e & d & c & b & a
\end{array} \tag{1}$$

More generally, there is interest in sets of *mutually orthogonal latin squares* (MOLS), that is, sets of latin squares in which each pair is orthogonal in the above sense. The literature on MOLS is vast (start with [15, 16, 37]) and provides ample justification for an interest in transversals. Subsequent investigations have ranged far beyond the initial justification of Theorem 1 and have proved that transversals are interesting objects in their own right. Despite this, a number of basic questions about their properties remain unresolved, as will become obvious in the subsequent pages.

Orthogonal latin squares exist for all orders $n \notin \{2,6\}$. For $n=6$ there is no pair of orthogonal squares, but we can get close. Finney [25] gives the following example which contains 4 disjoint transversals indicated by the subscripts a, b, c and d.

$$\begin{array}{cccccc}
1_a & 2 & 3_b & 4_c & 5 & 6_d \\
2_c & 1_d & 6 & 5_b & 4_a & 3 \\
3 & 4_b & 1 & 2_d & 6_c & 5_a \\
4 & 6_a & 5_c & 1 & 3_d & 2_b \\
5_d & 3_c & 2_a & 6 & 1_b & 4 \\
6_b & 5 & 4_d & 3_a & 2 & 1_c
\end{array}$$

Table 1 shows the squares of order n, for $4 \leqslant n \leqslant 8$, counted according to their maximum number m of disjoint transversals. The entries in the table are counts of main classes (A *main class*, or *species* is an equivalence class of latin squares each of which has essentially the same structure. See [15, 37] for the definition.)

Evidence such as that in Table 1 led van Rees [54] to conjecture that, as $n \to \infty$, a vanishingly small proportion of latin squares have orthogonal

m	$n = 4$	5	6	7	8
0	1	0	6	0	33
1	0	1	0	1	0
2	0	0	2	5	7
3	-	0	0	24	46
4	1	-	4	68	712
5	-	1	-	43	71330
6	-	-	0	-	209505
7	-	-	-	6	-
8	-	-	-	-	2024
Total	2	2	12	147	283657

Table 1: Number m of disjoint transversals in latin squares of order $n \leqslant 8$.

mates. However, the trend seems to be quite the reverse (see [57]), although no rigorous way of establishing this has yet been found.

A point that Table 1 raises is that some latin squares have no transversals at all. We now look at some results in this regard.

A latin square of order mq is said to be of *q-step type* if it can be represented by a matrix of $q \times q$ blocks A_{ij} as follows

$$\begin{matrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{matrix}$$

where each block A_{ij} is a latin subsquare of order q and two blocks A_{ij} and $A_{i'j'}$ contain the same symbols iff $i + j \equiv i' + j' \bmod m$. The following classical theorem is due to Maillet [39].

Theorem 2. *Suppose that q is odd and m is even. No q-step type latin square of order mq possesses a transversal.* □

As we will see in §4, this rules out many group tables having transversals. In particular, no cyclic group of even order has a transversal. By contrast, there is no known example of a latin square of odd order without transversals.

Conjecture 1. *Each latin square of odd order has at least one transversal.*

This conjecture is known to be true for $n \leqslant 9$ (see §3). It is attributed to Ryser [46] and has been open for forty years. In fact, Ryser's original conjecture was somewhat stronger: for every latin square of order n, the number of transversals is congruent to n mod 2. In [2], Balasubramanian proved the even case.

Theorem 3. *In any latin square of even order the number of transversals is even.* □

Despite this, it has been noted in [8] (and other places) that there are many counterexamples of odd order to Ryser's original conjecture. Hence the conjecture has now been weakened to Conjecture 1 as stated. One obstacle to proving this conjecture was recently revealed in [57].

Theorem 4. *For every $n > 3$ there exists a latin square of order n which contains an entry that is not included in any transversal.* □

Given Theorem 1, this latest theorem showed existence for all $n > 3$ of a latin square without an orthogonal mate. The same result was obtained in [24] without showing Theorem 4.

2. Partial transversals

We have seen in §1 that not all latin squares have transversals, which prompts the question of how close we can get to finding a transversal in such cases. We define a *partial transversal of length k* to be a set of k entries, each selected from different rows and columns of a latin square such that no two entries contain the same symbol. Note that in some papers (e.g. [50]) a partial transversal of length k is defined slightly differently to be a diagonal on which k different symbols appear.

Since not all squares of order n have a partial transversal of length n (i.e., a transversal), the best we can hope for is to find one of length $n-1$. The following conjecture has been attributed by Brualdi (see [15, p.103]).

Conjecture 2. *Every latin square of order n possesses a partial transversal of length $n-1$.*

A claimed proof of this conjecture by Derienko [18] contains a fatal error [8]. Recently, a paper [32] has appeared in the maths arXiv claiming a proof of Conjecture 2. However, given the history of the problem such a claim should be treated cautiously, at least until the paper has been refereed.

The best reliable result to date states that there must be a partial transversal of length at least $n - O(\log^2 n)$. This was shown by Shor [50], and the implicit constant in the 'big O' was very marginally improved by Fu et al. [26]. Subsequently Hatami and Shor [29] discovered an error in [50] (duplicated in [26]) and corrected the constant to a higher one. Nonetheless, the important thing remains that the bound is $n-O(\log^2 n)$. This improved on a number of earlier bounds including $\frac{2}{3}n+O(1)$ (Koksma [35]), $\frac{3}{4}n+O(1)$ (Drake [19]) and $n - \sqrt{n}$ (Brouwer et al. [4] and Woolbright [59]).

Erdős and Spencer [21] showed that any $n \times n$ array in which no entry occurs more than $(n-1)/16$ times has a transversal (in the sense of a diagonal with n different symbols on it). It has also been shown by Cameron and Wanless [8] that every latin square possesses a diagonal in which no symbol appears more than twice.

Conjecture 2 has been well known and open for decades. A much simpler problem is to consider the shortest possible length of a maximal partial transversal (maximal in the sense that it is contained in no partial transversal of greater length). It is easy to see that no partial transversal of length strictly less than $\frac{1}{2}n$ can be maximal, since there are not enough 'used' symbols to fill the submatrix formed by the 'unused' rows and columns. However, for all $n > 4$, maximal partial transversals of length $\lceil \frac{1}{2}n \rceil$ can easily be constructed using a square of order n which contains a subsquare S of order $\lfloor \frac{1}{2}n \rfloor$ and a partial transversal containing the symbols of S but not using any of the same rows or columns as S.

3. Number of transversals

In this section we consider the question of how many transversals a latin square can have. We define $t(n)$ and $T(n)$ to be respectively the minimum and maximum number of transversals among the latin squares of order n.

We have seen in §1 that some latin squares have no transversals but it is not settled for which orders such latin squares exist. Thus for lower bounds on $t(n)$ we cannot do any better than to observe that $t(n) \geqslant 0$, with equality occurring at least when n is even. A related question, for which no work seems to have been published, is to find an upper bound on $t(n)$ when n is odd.

Turning to the maximum number of transversals, it should be clear that $T(n) \leqslant n!$ since there are only $n!$ different diagonals. An exponential improvement on this trivial bound was obtained by McKay et al. [42]:

Theorem 5. *For $n \geqslant 5$,*

$$15^{n/5} \leqslant T(n) \leqslant c^n \sqrt{n}\, n!$$

where $c = \sqrt{\frac{3-\sqrt{3}}{6}}\, e^{\sqrt{3}/6} \approx 0.61354$. □

The lower bound in Theorem 5 is very simple and would not be too difficult to improve. The upper bound took considerably more work, although it too is probably far from the truth.

In the same paper the authors reported the results of an exhaustive computation of the transversals in latin squares of orders up to and including 9. Table 2 lists the minimum and maximum number of transversals over all latin squares of order n for $n \leqslant 9$, and the mean and standard deviation to 2 decimal places.

n	$t(n)$	Mean	Std Dev	$T(n)$
2	0	0	0	0
3	3	3	0	3
4	0	2	3.46	8
5	3	4.29	3.71	15
6	0	6.86	5.19	32
7	3	20.41	6.00	133
8	0	61.05	8.66	384
9	68	214.11	15.79	2241

Table 2: Transversals in latin squares of order $n \leqslant 9$.

Table 2 confirms Conjecture 1 for $n \leqslant 9$. The following semisymmetric squares (see [15] for a definition of semisymmetric) are representatives of the unique main class with $t(n)$ transversals for $n \in \{5, 7, 9\}$. In each case the largest subsquares are shown in **bold**.

$$\begin{array}{ccccc}
\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\
\mathbf{2} & \mathbf{1} & 4 & 5 & 3 \\
\mathbf{3} & 5 & \mathbf{1} & 2 & 4 \\
\mathbf{4} & 3 & 5 & \mathbf{1} & 2 \\
\mathbf{5} & 4 & 2 & 3 & \mathbf{1}
\end{array}
\qquad
\begin{array}{ccccccc}
\mathbf{3} & \mathbf{2} & \mathbf{1} & 5 & 4 & 7 & 6 \\
\mathbf{2} & \mathbf{1} & \mathbf{3} & 6 & 7 & 4 & 5 \\
\mathbf{1} & \mathbf{3} & \mathbf{2} & 7 & 6 & 5 & 4 \\
5 & 6 & 7 & 4 & 1 & 2 & 3 \\
4 & 7 & 6 & 1 & 5 & 3 & 2 \\
7 & 4 & 5 & 2 & 3 & 6 & 1 \\
6 & 5 & 4 & 3 & 2 & 1 & 7
\end{array}
\qquad
\begin{array}{ccccccccc}
\mathbf{2} & \mathbf{1} & \mathbf{3} & 6 & 7 & 8 & 9 & 5 & 4 \\
\mathbf{1} & \mathbf{3} & \mathbf{2} & 5 & 4 & 9 & \mathbf{6} & \mathbf{7} & \mathbf{8} \\
\mathbf{3} & \mathbf{2} & \mathbf{1} & 4 & 9 & 5 & \mathbf{7} & \mathbf{8} & \mathbf{6} \\
9 & 5 & 4 & 3 & 2 & 1 & \mathbf{8} & \mathbf{6} & \mathbf{7} \\
8 & 4 & 6 & 2 & 5 & 7 & 1 & 9 & 3 \\
4 & \mathbf{7} & \mathbf{9} & \mathbf{8} & 3 & 6 & 5 & 1 & 2 \\
5 & \mathbf{8} & \mathbf{7} & \mathbf{9} & 6 & \mathbf{2} & \mathbf{3} & \mathbf{4} & 1 \\
6 & \mathbf{9} & \mathbf{8} & \mathbf{7} & 1 & \mathbf{4} & \mathbf{2} & \mathbf{3} & 5 \\
7 & 6 & 5 & 1 & 8 & \mathbf{3} & \mathbf{4} & \mathbf{2} & 9
\end{array}$$

n	Lower Bound	Upper Bound
10	5504	75000
11	37851	528647
12	198144	3965268
13	1030367	32837805
14	3477504	300019037
15	36362925	2762962210
16	244744192	28218998328
17	1606008513	300502249052
18	6434611200	3410036886841
19	87656896891	41327486367018
20	697292390400	512073756609248
21	5778121715415	6803898881738477

Table 3: Bounds on $T(n)$ for $10 \leqslant n \leqslant 21$.

In Table 3 we reproduce from [42] bounds on $T(n)$ for $10 \leqslant n \leqslant 21$. The upper bound is somewhat sharper than that given by Theorem 5, though proved by the same methods. The lower bound in each case is constructive and likely to be very close to the true value. When $n \not\equiv 2 \bmod 4$ the lower bound comes from the group with the highest number of transversals (see Table 4). When $n \equiv 2 \bmod 4$ the lower bound comes from a so-called turn-square, many of which were analysed in [42]. A *turn-square* is obtained by starting with the Cayley table of a group (typically a group of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_m$ for some m) and "turning" some of the intercalates (that is, replacing a subsquare of order 2 by the other possible subsquare on the same symbols). For example,

$$\begin{array}{ccccc|ccccc}
\mathbf{5} & \mathbf{6} & 2 & 3 & 4 & \mathbf{0} & \mathbf{1} & 7 & 8 & 9 \\
\mathbf{6} & 2 & 3 & 4 & 0 & \mathbf{1} & 7 & 8 & 9 & 5 \\
2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6 \\
3 & 4 & 0 & 1 & 2 & 8 & 9 & 5 & 6 & 7 \\
4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 \\
\hline
\mathbf{0} & \mathbf{1} & 7 & 8 & 9 & \mathbf{5} & \mathbf{6} & 2 & 3 & 4 \\
\mathbf{1} & 7 & 8 & 9 & 5 & \mathbf{6} & 2 & 3 & 4 & 0 \\
7 & 8 & 9 & 5 & 6 & 2 & 3 & 4 & 0 & 1 \\
8 & 9 & 5 & 6 & 7 & 3 & 4 & 0 & 1 & 2 \\
9 & 5 & 6 & 7 & 8 & 4 & 0 & 1 & 2 & 3
\end{array} \qquad (2)$$

achieves 5504 transversals. The 'turned' entries have been marked in **bold**.

The study of turn-squares was pioneered by Parker (see [5] and the references therein) in his unsuccessful quest for a triple of MOLS of order 10. He noticed that turn-squares often have many more transversals than is typical for squares of their order, and used this as a heuristic in the search for MOLS.

It is has long been suspected that $T(10)$ is achieved by (2). This suspicion was strengthened by McKay et al. [41] who examined several billion squares of order 10, including every square with a non-trivial symmetry, and found none had more than 5504 transversals. Parker was indeed right that the square (2) is rich in orthogonal mates (it has 12265168 of them [38], which is an order of magnitude greater than he estimated). However, using the number of transversals as a heuristic in searching for MOLS is not fail-safe. For example, the turn-square of order 14 with the most transversals (namely, 3477504) does not have any orthogonal mates [42]. Meanwhile there are squares of order n with orthogonal mates but which possess only the bare minimum of n transversals (the left hand square in (1) is one such).

Nevertheless, the number of transversals does provide a useful invariant for squares of small orders where this number can be computed in reasonable time (see, for example, [34] and [55]). It is straightforward to write a backtracking algorithm to count transversals in latin squares of small order, though this method currently becomes impractical if the order is much over 20. See [30], [31] for some algorithms and complexity theory results on the problem of counting transversals.

It seems very difficult to find theoretical estimates for the number of transversals (unless, of course, that number is zero). This difficulty is so acute that there are not even good estimates for z_n, the number of transversals of the cyclic group of order n. Vardi [52] makes the following prediction:

Conjecture 3. *There exist real constants* $0 < c_1 < c_2 < 1$ *such that*

$$c_1^n n! \leqslant z_n \leqslant c_2^n n!$$

for all odd $n \geqslant 3$.

Vardi makes this conjecture while considering a variation on the toroidal n-queens problem. The toroidal n-queens problem is that of determining in how many different ways n non-attacking queens can be placed on a toroidal $n \times n$ chessboard. Vardi considered the same problem using semiqueens in place of queens, where a semiqueen is a piece which moves like a toroidal queen except that it cannot travel on right-to-left diagonals. The solution to

Vardi's problem provides an upper bound on the toroidal n-queens problem. The problem can be translated into one concerning latin squares by noting that every configuration of n non-attacking semiqueens on a toroidal $n \times n$ chessboard corresponds to a transversal in a cyclic latin square L of order n, where $L_{ij} \equiv i - j \bmod n$. Note that the toroidal n-queens problem is equivalent to counting diagonals which simultaneously yield transversals in L and L', where $L'_{ij} = i + j \bmod n$.

As a corollary of Theorem 5 we can infer that the upper bound in Conjecture 3 is true (asymptotically) with $c_2 = 0.614$. This also yields an upper bound for the number of solutions to the toroidal n-queens problem. Theorem 5 is valid for all latin squares, but Conjecture 3 has also been attacked by methods which are specific to the cyclic square. Cooper and Kovalenko [12] first showed that Vardi's upper bound is asymptotically true with $c_2 = 0.9153$, and this was then improved to $c = 1/\sqrt{2} \approx 0.7071$ in [36]. Finding a lower bound of the form given in Conjecture 3 is still an open problem. However, [10] and [45] do give some lower bounds, each of which applies only for some n. Cooper et al. [11] estimated that perhaps the correct rate of growth for z_n is around $0.39^n\, n!$.

4. Finite Groups

By using the symbols of a latin square to index its rows and columns, each latin square can be interpreted as the Cayley table of a quasigroup. In this section we consider the important special case when that quasigroup is associative; in other words, it is a group.

Much of the study of transversals in groups has been phrased in terms of the equivalent concepts of complete mapping and orthomorphisms. Mann [40] introduced complete mappings for groups, but their definition works just as well for quasigroups. It is this: a permutation θ of the elements of a quasigroup $(Q, \oplus)$ is a *complete mapping* if $\eta : Q \mapsto Q$ defined by $\eta(x) = x \oplus \theta(x)$ is also a permutation. The permutation η is known as an *orthomorphism* of $(Q, \oplus)$, following terminology introduced in [33]. All of the results of this paper could be rephrased in terms of complete mappings and/or orthomorphisms because of our next observation.

Theorem 6. *Let $(Q, \oplus)$ be a quasigroup and L_Q its Cayley table. Then $\theta : Q \mapsto Q$ is a complete mapping iff we can locate a transversal of L_Q by selecting, in each row x, the entry in column $\theta(x)$. Similarly, $\eta : Q \mapsto Q$*

is an orthomorphism iff we can locate a transversal of L_Q by selecting, in each row x, the entry containing symbol $\eta(x)$. □

Having noted that transversals, complete mappings and orthomorphisms are essentially the same thing, we will adopt the practice of expressing our results in terms of transversals even when the original authors used one of the other notions.

As mentioned, this section is devoted to the case when our latin square is L_G, the Cayley table of a finite group G. The extra structure in this case allows for much stronger results. For example, suppose we know of a transversal of L_G that comprises a choice from each row i of an element g_i. Let g be any fixed element of G. Then if we select from each row i the element $g_i g$ this will give a new transversal and as g ranges over G the transversals so produced will be mutually disjoint. Hence

Theorem 7. *If L_G has a single transversal then it has a decomposition into disjoint transversals.* □

We saw in §1 that the question of which latin squares have transversals has not been settled. The same is true for group tables, but we are getting much closer to answering the question, building on the pioneering work of Hall and Paige.

Consider the following five propositions:

(i) L_G has a transversal.

(ii) L_G can be decomposed into disjoint transversals.

(iii) There exists a latin square orthogonal to L_G.

(iv) There is some ordering of the elements of G, say $a_1, a_2, \ldots, a_n$, such that $a_1 a_2 \cdots a_n = \varepsilon$, where ε denotes the identity element of G.

(v) The Sylow 2-subgroups of G are trivial or non-cyclic.

The fact that (i), (ii) and (iii) are equivalent comes directly from Theorem 1 and Theorem 7. Paige [43] showed that (i) implies (iv). Hall and Paige [28] then showed that (iv) implies (v). They also showed that (v) implies (i) if G is a soluble, symmetric or alternating group. They conjectured that (v) is equivalent to (i) for all groups.

It was subsequently noted in [17] that both (iv) and (v) hold for all non-soluble groups, which proved that (iv) and (v) are equivalent. A much more direct and elementary proof of this fact was given in [53].

To summarise:

Theorem 8. $(i) \iff (ii) \iff (iii) \implies (iv) \iff (v)$. □

Conjecture 4. $(i) \iff (ii) \iff (iii) \iff (iv) \iff (v)$.

As mentioned above, Conjecture 4 is known to be true for all soluble, symmetric and alternating groups. It has also been shown for many other groups including the linear groups $GL(2,q)$, $SL(2,q)$, $PGL(2,q)$ and $PSL(2,q)$ (see [23] and the references therein).

After decades of incremental progress on Conjecture 4 there has recently been what would appear to be a very significant breakthrough. In a preprint Wilcox [58] has claimed to reduce the problem to showing it for the sporadic simple groups (of which the Mathieu groups have already been handled in [13]). See [15], [22] or [58] for further reading and references on the Hall-Paige conjecture.

An immediate corollary of the proof of Theorem 7 is that for any G the number of transversals through a given entry of L_G is independent of the entry chosen. Hence (see Theorem 3.5 of [16]) we get:

Theorem 9. *The number of transversals in L_G is divisible by $|G|$, the order of G.* □

McKay et al. [42] also showed the following simple results, in the spirit of Theorem 3:

Theorem 10. *The number of transversals in any symmetric latin square of order n is congruent to n modulo 2.* □

Corollary 1. *Let G be a group of order n. If G is abelian or n is even then the number of transversals in G is congruent to n modulo 2.* □

Corollary 1 cannot be generalised to non-abelian groups of odd order, given that the non-abelian group of order 21 has 826814671200 transversals.

Theorem 11. *If G is a group of order $n \not\equiv 1 \bmod 3$ then the number of transversals in G is divisible by 3.* □

We will see below that the cyclic groups of small orders $n \equiv 1 \bmod 3$ have a number of transversals which is not a multiple of three.

The semiqueens problem in §3 led to an investigation of z_n, the number of transversals in the cyclic group of order n. Let $z'_n = z_n/n$ denote the number of transversals through any given entry of the cyclic square of order n. Since $z_n = z'_n = 0$ for all even n by Theorem 8 we shall assume for the following discussion that n is odd.

The initial values of z'_n are known from [47] and [48]. They are $z'_1 = z'_3 = 1$, $z'_5 = 3$, $z'_7 = 19$, $z'_9 = 225$, $z'_{11} = 3441$, $z'_{13} = 79259$, $z'_{15} = 2424195$, $z'_{17} = 94471089$, $z'_{19} = 4613520889$, $z'_{21} = 275148653115$, $z'_{23} = 19686730313955$ and $z'_{25} = 1664382756757625$. Interestingly, if we take these numbers modulo 8 we find that this sequence begins 1,1,3,3,1,1,3,3,1,1,3,3,1. We know from Theorem 10 that z'_n is always odd for odd n, but it is an open question whether there is any deeper pattern modulo 4 or 8. We also know from Theorem 11 that z'_n is divisible by 3 when $n \equiv 2 \bmod 3$. The initial terms of $\{z'_n \bmod 3\}$ are 1,1,0,1,0,0,2,0,0,1,0,0,2.

An interesting fact about z_n is that it is the number of *diagonally cyclic latin squares* of order n (in other words, the number of quasigroups on the set $\{1, 2, \ldots, n\}$ which have the transitive automorphism $(123\cdots n)$). See [56] for a survey on such objects.

We now discuss the number of transversals in general groups of small order. For groups of order $n \equiv 2 \bmod 4$ there can be no transversals, by Theorem 8. For each other order $n \leqslant 23$ the number of transversals in each group is given in Table 4. The groups are ordered according to the catalogue of Thomas and Wood [51]. The numbers of transversals in abelian groups of order at most 16 and cyclic groups of order at most 21 were obtained by Shieh et al [49]. The remaining values in Table 4 were computed by Shieh [47]. McKay et al. [42] then independently confirmed all counts except those for cyclic groups of order $\geqslant 21$, correcting one misprint in Shieh [47].

Bedford and Whitaker [3] offer an explanation for why all the non-cyclic groups of order 8 have 384 transversals. The groups of order 4, 9 and 16 with the most transversals are the elementary abelian groups of those orders. Similarly, for orders 12, 20 and 21 the group with the most transversals is the direct sum of cyclic groups of prime order. It is an open question whether such a statement generalises to all n.

By Corollary 1 we know that in each case covered by Table 4 (except the non-abelian group of order 21), the number of transversals must have

n	Number of transversals in groups of order n
3	3
4	0, 8
5	15
7	133
8	0, 384, 384, 384, 384
9	2025, 2241
11	37851
12	0, 198144, 76032, 46080, 0
13	1030367
15	36362925
16	0, 235765760, 237010944, 238190592, 244744192, 125599744, 121143296, 123371520, 123895808, 122191872, 121733120, 62881792, 62619648, 62357504
17	1606008513
19	87656896891
20	0, 697292390400, 140866560000, 0, 0
21	5778121715415, 826814671200
23	452794797220965

Table 4: Transversals in groups of order $n \leqslant 23$.

the same parity as the order of the square. It is remarkable though, that the groups of even order have a number of transversals which is divisible by a high power of 2. Indeed, any 2-group of order $n \leqslant 16$ has a number of transversals which is divisible by 2^{n-1}. It would be interesting to know if this is true for general n.

5. Generalised transversals

There are several ways to generalise the notion of a transversal. We have already seen one of them, namely the partial transversals in §2. In this section we collect results on another generalisation, namely plexes.

A *k-plex* in a latin square of order n is a set of kn entries which includes k representatives from each row and each column and of each symbol. A transversal is a 1-plex. The marked entries form a 3-plex in the following square:

$$\begin{array}{cccccc}
1^* & 2 & 3 & 4^* & 5 & 6^* \\
2^* & 1 & 4 & 3^* & 6^* & 5 \\
3 & 5^* & 1 & 6 & 2^* & 4^* \\
4 & 6 & 2^* & 5 & 3^* & 1^* \\
5^* & 4^* & 6^* & 2 & 1 & 3 \\
6 & 3^* & 5^* & 1^* & 4 & 2
\end{array} \tag{3}$$

The name k-plex was coined in [55] only recently. It is a natural extension of the names duplex, triplex, and quadruplex which have been in use for many years (principally in the statistical literature, such as [25]) for 2, 3 and 4-plexes.

The entries not included in a k-plex of a latin square L of order n form an $(n-k)$-plex of L. Together the k-plex and its complementary $(n-k)$-plex are an example of what is called an *orthogonal partition* of L. For discussion of orthogonal partitions in a general setting see Gilliland [27] and Bailey [1]. For our purposes, if L is decomposed into disjoint parts K_1, $K_2, \ldots, K_d$ where K_i is a k_i-plex then we call this a $(k_1, k_2, \ldots, k_d)$-partition of L. A case of particular interest is when all parts are the same size, k. We call such a partition a *k-partition*. For example, the marked 3-plex and its complement form a 3-partition of the square in (3). By Theorem 1, finding a 1-partition of a square is equivalent to finding an orthogonal mate.

Some results about transversals generalise directly to other plexes, while others seem to have no analogue. Theorem 3 and Theorem 7 seem to be in the latter class, as observed in [42] and [55] respectively. However, Theorems 2 and 8 showed that not every square has a transversal, and exactly the same arguments work for any k-plex where k is odd [55].

Theorem 12. *Suppose that q and k are odd integers and m is even. No q-step type latin square of order mq possesses a k-plex.* □

Theorem 13. *Let G be a group of finite order n with a non-trivial cyclic Sylow 2-subgroup. The Cayley table of G contains no k-plex for any odd k but has a 2-partition and hence contains a k-plex for every even k in the range $0 \leqslant k \leqslant n$.* □

The situation for even k is quite different to the odd case. Rodney [9, p.105] conjectures that every latin square has a duplex. This conjecture was strengthened in [55] to the following:

Conjecture 5. *Every latin square has the maximum possible number of disjoint duplexes. In particular, every latin square of even order has a* 2*-partition and every latin square of odd order has a* $(2, 2, 2, \ldots, 2, 1)$*-partition.*

Note that this conjecture also strengthens Conjecture 1. It also implies that every latin square has k-plexes for every even value of k up to the order of the square.

Conjecture 5 is true for all latin squares of orders $\leqslant 8$ and for all soluble groups (see [53, 55]). Depending on whether a soluble group has a non-trivial cyclic Sylow 2-subgroup, it either has a k-plex for all possible k, or has them for all possible even k but no odd k. If the Hall-Paige conjecture could be proved it would completely resolve the existence question of plexes in groups, and these would remain the only two possibilities. It is worth noting that other scenarios occur for latin squares which are not based on groups. For example, the square in (3) has no transversal but clearly does have a 3-plex. It is conjectured in [55] that there exist arbitrarily large latin squares of this type.

Conjecture 6. *For all even* $n > 4$ *there exists a latin square of order* n *which has no transversal but does contain a* 3*-plex.*

Another possibility was shown by a family of squares constructed in [20].

Theorem 14. *For all even* n *there exists a latin square of order* n *which has* k*-plexes for every odd value of* k *between* $\frac{1}{4}n - \frac{1}{2}$ *and* $\frac{3}{4}n + \frac{1}{2}$*, but not for any odd value of* k *outside this range.* $\square$

Interestingly, there is no known example of odd integers $a < b < c$ and a latin square which has an a-plex and a c-plex but no b-plex.

The union of an a-plex and a disjoint b-plex of a latin square L is an $(a+b)$-plex of L. However, it is not always possible to split an $(a+b)$-plex into an a-plex and a disjoint b-plex. Consider a duplex which consists of $\frac{1}{2}n$ disjoint intercalates (latin subsquares of order 2). Such a duplex does not contain a partial transversal of length more than $\frac{1}{2}n$, so it is a long way from containing a 1-plex.

We say that a k-plex is *indivisible* if it contains no c-plex for $0 < c < k$. The duplex just described is indivisible. Indeed, for every k there is a indivisible k-plex in some sufficiently large latin square. This was first shown in [55], but "sufficiently large" in that case meant quadratic in k. This was improved to linear in [6] as a corollary of the following result.

Theorem 15. *For every $k \geqslant 2$ there exists a latin square of order $2k$ which contains two disjoint indivisible k-plexes.* □

Theorem 15 means that some squares can be split in "half" in a way that makes no further division possible. Experience with latin squares suggests that they generally have a vast multitude of partitions into various plexes, which in one sense means that latin squares tend to be a long way from being indivisible. This makes Theorem 15 slightly surprising.

It is a wide open question for what values of k and n there is a latin square of order n containing an indivisible k-plex. However, Bryant et al. [6] found the answer when k is small relative to n.

Theorem 16. *Let n and k be positive integers satisfying $5k \leqslant n$. Then there exists a latin square of order n containing an indivisible k-plex.* □

So far we have essentially looked at questions where we start with a latin square and ask what sort of plexes it might have. To complete the section we consider the reverse question. We want to start with a plex and ask what latin squares it might be contained in. Strictly speaking this is a silly question, since we defined a plex in terms of its host latin square, which therefore is the only possible answer. However, suppose we define a *k-homogeneous partial latin square* of order n to be an $n \times n$ array in which each cell is either blank or filled (the latter meaning that it contains one of $\{1, 2, \dots, n\}$), and which has the properties that (i) no symbol occurs twice within any row or column, (ii) each symbol occurs k times in the array, (iii) each row and column contains exactly k filled cells. (The standard definition of a homogeneous partial latin square is slightly more general. However, once empty rows and columns have been deleted, it agrees with ours.) We can then sensibly ask whether this k-homogeneous partial latin square is a k-plex. If it is then we say the partial latin square is *completable* because the blank entries can be filled in to produce a latin square.

Theorem 17. *If $1 < k < n$ and $k > \frac{1}{4}n$ then there exists a k-homogeneous partial latin square of order n which is not completable.* □

Burton [7], and Daykin and Häggkvist [14] independently conjecture that if $k \leqslant \frac{1}{4}n$ then every k-plex is completable. It seems certain that for k sufficiently small relative to n, every k-plex is completable. This has already been proved when $n \equiv 0 \bmod 16$ in [14]. The following partial extension result due to Burton [7] also seems relevant.

Theorem 18. *For $k \leqslant \frac{1}{4}n$ every k-homogeneous partial latin square of order n is contained in a $(k+1)$-homogeneous partial latin square of order n.* □

6. Covering radii for sets of permutations

A novel approach to Conjecture 1 and Conjecture 2 has recently been opened up by Andre Kézdy and Hunter Snevily. To explain this interesting new approach, we need to introduce some terminology.

Consider the *symmetric group* S_n as a metric space equipped with *Hamming distance*. That is, the distance between two permutations $g, h \in S_n$ is the number of points at which they disagree (n minus the number of fixed points of gh^{-1}). Let P be a subset of S_n. The *covering radius* $\mathrm{cr}(P)$ of P is the smallest r such that the balls of radius r with centres at the elements of P cover the whole of S_n. In other words every permutation is within distance r of some member of P, and r is chosen to be minimal with this property.

Theorem 19. *Let $P \subseteq S_n$ be a set of permutations. If $|P| \leqslant n/2$, then $\mathrm{cr}(P) = n$. However, there exists P with $|P| = \lfloor n/2 \rfloor + 1$ and $\mathrm{cr}(P) < n$.* □

This result raises an obvious question. Given n and s, what is the smallest m such that there is a set S of permutations with $|S| = m$ and $\mathrm{cr}(S) \leqslant n - s$? We let $f(n,s)$ denote this minimum value m. This problem can also be interpreted in graph-theoretic language. Define the graph $G_{n,s}$ on the vertex set S_n, with two permutations being adjacent if they agree in at least s places. Now the size of the smallest dominating set in $G_{n,s}$ is $f(n,s)$.

Theorem 19 shows that $f(n,1) = \lfloor n/2 \rfloor + 1$. Since any two distinct permutations have distance at least 2, we see that $f(n, n-1) = n!$ for $n \geq 2$. Moreover, $f(n,s)$ is a monotonic increasing function of s (by definition).

The next case to consider is $f(n,2)$. Kézdy and Snevily made the following conjecture in unpublished notes.

Conjecture 7. *If n is even, then $f(n,2) = n$; if n is odd, then $f(n,2) > n$.*

The Kézdy–Snevily conjecture has several connections with transversals. The rows of a latin square of order n form a *sharply transitive set* of permutations (that is, exactly one permutation carries i to j, for any i and j); and every sharply transitive set is the set of rows of a latin square.

Theorem 20. *Let S be a sharply transitive subset of S_n. Then S has covering radius at most $n-1$, with equality if and only if the corresponding latin square has a transversal.* □

Corollary 2. *If there exists a latin square of order n with no transversal, then $f(n,2) \leqslant n$. In particular, this holds for n even.* □

Hence Conjecture 7 implies Conjecture 1, as Kézdy and Snevily observed. In fact a stronger result holds:

Theorem 21. *If S is the set of rows of a latin square L of order n with no transversal, then S has covering radius $n-2$.* □

The following result is due to Kézdy and Snevily. See [8] for a proof.

Theorem 22. *Conjecture 7 implies Conjecture 2.* □

In other words, to solve the longstanding Ryser and Brualdi conjectures it may suffice to answer this: How small can we make a subset $S \subset S_n$ which has the property that every permutation in S_n agrees with some member of S in at least two places?

In Corollary 2 we used latin squares to find an upper bound for $f(n,2)$ when n is even. For odd n we can also find upper bounds based on latin squares. The idea is to choose a latin square with few transversals, or whose transversals have a particular structure, and add a small set of permutations meeting each transversal twice. For $n = 5, 7, 9$, we now give a latin square for which a single extra permutation suffices, showing that $f(n,2) \leqslant n+1$ in these cases.

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 1 & 4 & 5 & 3 \\
3 & 5 & 1 & 2 & 4 \\
4 & 3 & 5 & 1 & 2 \\
5 & 4 & 2 & 3 & 1 \\
\\
1 & 3 & 4 & 2 & 5
\end{array}
\qquad
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 1 & 5 & 4 & 7 & 6 \\
3 & 1 & 2 & 6 & 7 & 4 & 5 \\
4 & 5 & 6 & 7 & 1 & 2 & 3 \\
5 & 4 & 7 & 1 & 6 & 3 & 2 \\
6 & 7 & 4 & 2 & 3 & 5 & 1 \\
7 & 6 & 5 & 3 & 2 & 1 & 4 \\
\\
3 & 2 & 1 & 7 & 6 & 5 & 4
\end{array}
\qquad
\begin{array}{ccccccccc}
1 & 3 & 2 & 4 & 6 & 5 & 7 & 9 & 8 \\
2 & 1 & 3 & 5 & 4 & 6 & 8 & 7 & 9 \\
3 & 2 & 1 & 7 & 9 & 8 & 4 & 6 & 5 \\
4 & 6 & 5 & 9 & 8 & 7 & 1 & 3 & 2 \\
5 & 4 & 6 & 8 & 7 & 9 & 3 & 2 & 1 \\
6 & 5 & 4 & 2 & 1 & 3 & 9 & 8 & 7 \\
7 & 9 & 8 & 1 & 3 & 2 & 5 & 4 & 6 \\
8 & 7 & 9 & 3 & 2 & 1 & 6 & 5 & 4 \\
9 & 8 & 7 & 6 & 5 & 4 & 2 & 1 & 3 \\
\\
5 & 4 & 6 & 1 & 3 & 2 & 9 & 8 & 7
\end{array}$$

In general, we have the following:

Theorem 23. $f(n,2) \leqslant \frac{4}{3}n + O(1)$ *for all n.* □

The reader is encouraged to seek out [8] and the survey by Quistorff [44] for more information on covering radii for sets of permutations.

7. Concluding Remarks

We have only been able to give the briefest of overviews of the fascinating subject of transversals in this survey. Space constraints have forced the omission of much worthy material, including proofs of the theorems quoted. However, even this brief skim across the surface has shown that many basic questions remain unanswered and much work remains to be done.

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