Rectangle Free Coloring of Grids

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Abstract

A two-dimensional *grid* is a set $G_{n,m} = [n] \times [m]$. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi_{n,m} : G_{n,m} \rightarrow [c]$ such that there are no rectangles with all four corners the same color. We address the following question: for which values of n and m is $G_{n,m}$ c-colorable? This problem can be viewed as a bipartite Ramsey problem and is related to a the Gallai-Witt theorem (also called the multidimensioanl Van Der Waerden's Theorem). We determine (1) *exactly* which grids are 2-colorable, (2) *exactly* which grids are 3-colorable, and (3) (assuming a reasonable conjecture) *exactly* which grids are 4-colorable. We use combinatorics, finite fields, and tournament graphs.

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Contents

1 Introduction

A two-dimensional *grid* is a set $G_{n,m} = [n] \times [m]$ where $[t] = \{1, \ldots, t\}$. A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a, b), (a + c_1, b), (a + c_1, b + c_2), (a, b + c_2)\}$ for some constants c_1 and c_2. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi_{n,m} : G_{n,m} \to [c]$ such that there are no rectangles with all four corners the same color. Not all grids have c-colorings. As an example, for any c clearly $G_{c+1,c^{c+1}+1}$ does not have a c-coloring by two applications of the pigeonhole principle. If a grid has a c-coloring, we say it is *c-colorable.* In this paper, we ask the following question: what are the exact values of m and n for which $G_{n,m}$ is c-colorable?

Def 1.1 Let $n, m, n', m' \in \mathsf{N}$. $G_{m,n}$ *contains* $G_{n',m'}$ if $n' \leq n$ and $m' \leq m$. $G_{m,n}$ *is contained in* $G_{n',m'}$ if $n \leq n'$ and $m \leq m'$. Proper containment means that at least one of the $\leq$ is actually $<$.

Clearly, if $G_{n,m}$ is c-colorable, then all grids that it contains are c-colorable. Likewise, if $G_{n,m}$ is not c-colorable then all grids that contain it are not not c-colorable.

Def 1.2 Fix c. then OBS_c is the set of all grids $G_{n,m}$ such that $G_{n,m}$ is not c-colorable but all grids properly contained in $G_{m,n}$ are c-colorable. OBS_c stands for *Obstruction Sets.* We also call such grids *c-minimal.*

We leave the proof of the following theorem to the reader.

Theorem 1.3 *Fix c. A grid $G_{a,b}$ is c-colorable iff it does not contain any element of* OBS_c.

By Theorem 1.3 we can rephrase the questions of finding which grids are c-colorable: find OBS_c. Note that if $G_{n,m} \in \mathrm{OBS}_c$, then $G_{n,m} \in \mathrm{OBS}_c$.

This problem arises as follows. The Gallai-Witt theorem[1] (also called the multi-dimensional Van Der Waerden theorem) has the following corollary: *For all c, there exists $W = W(c)$ such that, for all c-colorings of $[W] \times [W]$ there exists a monochromatic square.* The classical proof of the theorem gives enormous upper bounds on $W(c)$. Despite some improvements[2] the known bounds on $W(c)$ are still enormous. If we relax the problem to seeking a *monochromatic rectangle* then we can obtain far smaller bounds. In fact, we will obtain, in some cases, exact characterizations of when a grid is c-colorable.

Another motivation is the bipartite Ramsey problem: Given a, c, what is the least n such that for any c-coloring of the edges of $K_{n,n}$ there is a monochromatic $K_{a,a}$? A coloring of $G_{n,n}$ can be viewed as an edge coloring of $K_{n,n}$. A monochromatic rectangle corresponds to a monochromatic $K_{2,2}$. Beineke and Schwenk [2] study a closely related problem: what is the minimum value of b such that any two-coloring of $K_{b,b}$ results in a monochromatic $K_{n,m}$? In their work, this minimal value is denoted $R(n, m)$. Later, Hattingh and Henning [7] define $b(n, m)$ as the minimum b for which any two-coloring of $K_{b,b}$ contains a monochromatic $K_{m,m}$ or a monochromatic $K_{n,n}$.

In a related paper, Cooper, Fenner, and Purewal [3] generalize the problem to multiple dimensions and obtain upper and lower bounds on the sizes of the obstruction sets.

The remainder of this paper is organized as follows. In Sections 2 and 3 we develop tools to show grids *are not* c-colorable. In Section 4 we develop tools to show grids *are* c-colorable. In Section 5 we obtain upper and lower bounds on $|\mathrm{OBS}_c|$. In Section 6 and 7 we find OBS_2 and OBS_3 respectively. In Section 8 we obtain a small handful of possibilities for OBS_4.

[1]It was attributed to Gallai in [11] and [12]; Witt proved the theorem in [15].

[2]Both [6] and [4] can be used to obtain better bounds on $W(c)$.

We also propose a reasonable conjecture which, if true, would yield the exact elements of OBS_4. In Section 9 we apply the results to finding some new bipartite Ramsey numbers. We conclude with some open questions. The appendix contains some sizes of maximum rectangle free sets (to be defined later).

2 Lower Bounds on Uncolorability

A *rectangle-free subset* $A \subseteq G_{n,m}$ is a subset that does not contain a rectangle as defined above. A problem that is closely related to grid-colorability is that of finding a rectangle-free subset of maximum cardinality. This relationship is illustrated by the following lemma.

Theorem 2.1 *If $G_{n,m}$ is c-colorable, then it contains a rectangle-free subset of size $\lceil \frac{nm}{c} \rceil$.*

Proof: A c-coloring partitions the elements of $G_{n,m}$ into c rectangle-free subsets. By the pigeon-hole principle, one of these sets must be of size at least $\lceil \frac{nm}{c} \rceil$. ∎

Def 2.2 Let $n, m \in \mathsf{N}$. $\mathrm{maxrf}(n,m)$ is the size of the maximum rectangle-free $A \subseteq G_{n,m}$.

Finding the maximum cardinality of a rectangle-free subset is equivalent to a special case of a well-known problem of Zarankiewicz [16] (see [5] or [14] for more information). The Zarankiewicz function, denoted $Z_{r,s}(n,m)$, counts the minimum number of edges in a bipartite graph with vertex sets of size n and m that guarantees a subgraph isomorphic to $K_{r,s}$. Zarankiewicz's problem was to determine $Z_{r,s}(n,m)$.

If $r = s$, the function is denoted $Z_r(n,m)$. If one views a grid as an incidence matrix for a bipartite graph with vertex sets of cardinality n and m, then a rectangle is equivalent to a subgraph isomorphic to $K_{2,2}$. Therefore the maximum cardinality of a rectangle-free set in $G_{n,m}$ is $Z_2(n,m) - 1$. We will use this lemma in its contrapositive form, i.e., we will often show that $G_{n,m}$ is not c-colorable by showing that $Z_2(n,m) \le \lceil \frac{nm}{c} \rceil$.

Reiman [13] proved the following lemma. Roman [14] later generalized it.

Lemma 2.3 *Let $m \le n \le \binom{m}{2}$. Then $Z_2(n,m) \le \left\lfloor \frac{n}{2} \left(1 + \sqrt{1 + 4m(m-1)/n}\right) \right\rfloor + 1.$*

Corollary 2.4 *Let $m \le n \le \binom{m}{2}$. Let $z_{n,m} = \left\lfloor \frac{n}{2} \left(1 + \sqrt{1 + 4m(m-1)/n}\right) \right\rfloor + 1$ be the upper-bound on $Z_2(n,m)$ in Lemma 2.3. If $z_{n,m} \le \lceil \frac{nm}{c} \rceil$ then $G_{n,m}$ is not c-colorable.*

Corollary 2.4, and some 2-colorings of grids, are sufficient to find OBS_2. To find OBS_3 and OBS_4, we need slightly more powerful tools to show grids are not colorable (along with some 3-colorings and 4-colorings of grids). This next lemma, which has a proof that is very similar to the previous lemma gives us two more uncolorability corollaries.

Def 2.5 Let $n, m, x_1, \ldots, x_m \in \mathsf{N}$. $(x_1, \ldots, x_m)$ is (n,m)-*placeable* if there exists a rectangle-free $A \subseteq G_{n,m}$ such that, for $1 \le j \le m$, there are x_j elements of A in the j^{th} column.

Lemma 2.6 *Let $n, m, x_1, \ldots, x_m \in \mathsf{N}$ be such that $(x_1, \ldots, x_m)$ is (n,m)-placeable. Then* $\sum_{i=1}^{m} \binom{x_i}{2} \le \binom{n}{2}$.

Proof: Let $A \subseteq G_{n,m}$ be a set that shows that $(x_1, \ldots, x_m)$ is (n,m)-placeable. Let $\binom{A}{2}$ be the set of pairs of elements of A. Let $2^{\binom{A}{2}}$ be the powerset of $\binom{A}{2}$.

Define the function $f : [m] \to 2^{\binom{A}{2}}$ as follows. For $1 \le j \le m$,

$$f(j) = \{\{a, b\} : (a, j), (b, j) \in A\}.$$

If $\sum_{j=1}^{m} |f(j)| > \binom{n}{2}$ then there exists $j_1 \ne j_2$ such that $f(j_1) \cap f(j_2) \ne \emptyset$. Let $\{a, b\} \in f(j_1) \cap f(j_2)$. Then

$$\{(a, j_1), (a, j_2), (b, j_1), (b, j_2)\} \in A.$$

Hence A contains a rectangle. Since this cannot happen, $\sum_{j=1}^{m} |f(j)| \le \binom{n}{2}$. Note that $|f(j)| = \binom{x_j}{2}$. Hence $\sum_{i=1}^{m} \binom{x_i}{2} \le \binom{n}{2}$. ∎

Lemma 2.7 *Let $a, n, m \in \mathsf{N}$. Let q, r be such that $a = qn + r$ with $0 \le r \le n$. Assume that there exists $A \subseteq G_{n,m}$ such that $|A| = a$ and A is rectangle-free.*

1. *If $q \ge 2$ then*
$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

2. *If $q = 1$ then*
$$r \le \frac{m(m-1)}{2}.$$

Proof: The proof for the $q \ge 2$ and the $q = 1$ case begins the same; hence we will not split into cases yet.

Assume that, for $1 \le j \le m$, the number of elements of A in the j^{th} column is x_j. Note that $\sum_{j=1}^{m} x_j = a$. $\sum_{j=1}^{m} \binom{x_j}{2} \le \binom{n}{2}$. We look at the least value that $\sum_{j=1}^{n} \binom{x_j}{2}$ can have.

Consider the following question:
Minimize $\sum_{j=1}^{n} \binom{x_j}{2}$

Constraints:

- $\sum_{j=1}^{n} x_j = a$.
- $x_1, \ldots, x_n$ are natural numbers.

One can easily show that this is minimized when, for all $1 \le j \le n$,

$$x_j \in \{\lfloor a/n \rfloor, \lceil a/n \rceil\} \subseteq \{q, q+1\}.$$

In order for $\sum_{j=1}^{n} x_j = a$ we need to have $n-r$ many q's and r many $q+1$'s. Hence we obtain

$\sum_{j=1}^{n} \binom{x_j}{2}$ is at least

$$(n-r)\binom{q}{2} + r\binom{q+1}{2}.$$

Hence we have

$$(n-r)\binom{q}{2} + r\binom{q+1}{2} \le \sum_{j=1}^{n} \binom{x_j}{2} \le \binom{m}{2}$$

$$nq(q-1) - rq(q-1) + r(q+1)q \le m(m-1)$$

$$nq(q-1) - rq^2 + rq + rq^2 + rq \le m(m-1)$$

$$nq(q-1) + 2rq \le m(m-1)$$

Case 1: $q \ge 2$.

Subtract $2rq$ from both sides to obtain

$$nq(q-1) \le m(m-1) - 2rq.$$

Since $q-1 \ne 0$ we can divide by $q(q-1)$ to obtain

$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

Case 2: $q = 1$. Since $q-1 = 0$ we get

$$2r \le m(m-1)$$

$$r \le \frac{m(m-1)}{2}.$$

∎

Corollary 2.8 *Let $m, n \in \mathsf{N}$. If there exists an r where $\frac{m(m-1)}{2} < r \le n$ and $\left\lceil \frac{mn}{c} \right\rceil = n + r$, then $G_{m,n}$ is not c-colorable.*

Corollary 2.9 *Let $1 \le c' \le c$. $G_{c+c',m}$ is not c-colorable for any $m > \frac{c}{c'}\binom{c+c'}{2}$.*

Proof: Let $n = c + c'$ and $q = 1$ in Lemma 2.7. Then we have

$$\begin{aligned} \left\lceil \frac{(c+c')m}{c} \right\rceil &= m + r \\ m + \left\lceil \frac{c'm}{c} \right\rceil &= m + r \\ \left\lceil \frac{c'm}{c} \right\rceil &= r \end{aligned}$$

Note that $m \geq r$. Hence

$$\begin{aligned} m \geq \left\lceil \frac{c'm}{c} \right\rceil &= r \\ \frac{c'm}{c} &\leq r \\ m &\leq \frac{cr}{c'} \end{aligned}$$

If $m > \frac{c}{c'}\binom{c+c'}{2}$, then $r > \binom{c+c'}{2}$, and so $G_{c+c',m}$ is not c-colorable by Corollary 2.8. ∎

Corollary 2.10 *Let $n, m \in \mathsf{N}$. Let $\lceil \frac{nm}{c} \rceil = qn + r$ for some $0 \leq r \leq n$ and $q \geq 2$. If $\frac{m(m-1)-2qr}{q(q-1)} < n$ then $G_{n,m}$ is not c-colorable.*

Note 2.11 In the Appendix we use the results of this section to find the sizes of maximum rectangle free sets.

3 Tools to Show Sets Contain Rectangles

3.1 Conventions

Throughout this section we will have the following notations and conventions.

Notation 3.1 If $n, m \in \mathsf{N}$ and $A \subseteq G_{n,m}$ then we assume the following.

1. The top row of a grid is row 1.
2. We will denote that $(a, b) \in A$ by putting an R in the (a, b) position.
3. For $1 \leq j \leq m$, x_j is the number of elements of A in column j.
4. The rows and columns are reordered so that the following holds (unless we explicitly say otherwise):

(a) $x_1 \geq x_2 \geq \cdots \geq x_m$.

(b) The first column has x_1 contiguous elements of A starting at row 1.

(c) The second column has x_2 contiguous elements of A (unless we say otherwise).

5. For $1 \leq j \leq m$, C_j is the set of rows r such that A has an element in the r^{th} row of column j. Formally
$$C_j = \{r : (r, j) \in A\}.$$

6. For $1 \leq i \leq k$ let
$$I_i = \sum_{1 \leq j_1 < \cdots < j_i \leq m} |C_{j_1} \cap \cdots \cap C_{j_i}|.$$

Example 3.2

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17
1		R		R										R		R	R
2	R	R								R	R		R				
3	R								R						R	R	
4						R			R			R	R	R			
5		R	R			R											

$$C_1 = \{2,3\}, C_2 = \{1,2,5\}, C_3 = \{5\}, C_4 = \{1\}, C_5 = \{\},$$

$$C_6 = \{4,5\}, C_7 = \{\}, C_8 = \{\}, C_9 = \{3,4\}, C_{10} = \{2\},$$

$$C_{11} = \{2\}, C_{12} = \{4\}, C_{13} = \{2,4\}, C_{14} = \{1,4\}, C_{15} = \{3\},$$

$$C_{16} = \{1,3\}, C_{17} = \{1\},$$

In this example A is rectangle free. Hence, for all $i < j$ $C_i \cap C_j| \leq 1$. Hence we have the following observations.

1. I_1 is the number of R's in the grid which is 22.

2. I_2 is the number of pairs of columns that intersect. We list all of the intersecting pairs that are nonempty by listing what C_j intersects with $C_{j'}$ where $j' > j$.

   C_1 intersects $C_2, C_9, C_{10}, C_{11}, C_{13}, C_{15}, C_{16}$;

   C_2 intersects $C_4, C_6, C_{10}, C_{11}, C_{13}, C_{14}, C_{16}, C_{17}$;

   C_3 intersects C_6;

   C_4 intersects C_{14}, C_{16}, C_{17};

C_6 intersects $C_9, C_{12}, C_{13}, C_{14}$;

C_9 intersects $C_{12}, C_{13}, C_{14}, C_{15}, C_{16}$;

C_{10} intersects C_{11}, C_{13},

C_{11} intersects C_{13};

C_{12} intersects C_{13}, C_{14};

C_{14} intersects C_{16}, C_{17};

C_{15} intersects C_{16};

C_{16} intersects C_{17};

Therefore $I_2 = 37$.

3. I_3 is the number of triples of columns that intersect. We list all of the intersecting triplets that are nonempty:

   (C_1, C_2, C_{10}), (C_1, C_2, C_{11}), (C_1, C_2, C_{13}), (C_1, C_9, C_{15}), (C_1, C_9, C_{16}), (C_1, C_{10}, C_{11}), (C_1, C_{10}, C_{13}), (C_1, C_{11}, C_{13}), (C_1, C_{15}, C_{16}),

   (C_2, C_4, C_{14}), (C_2, C_4, C_{16}), (C_2, C_4, C_{17}), (C_2, C_{10}, C_{11}), (C_2, C_{10}, C_{13}), (C_2, C_{11}, C_{13}), (C_2, C_{14}, C_{16}), (C_2, C_{14}, C_{17}), (C_2, C_{16}, C_{17});

   (C_4, C_{14}, C_{16}), (C_4, C_{14}, C_{17}), (C_4, C_{16}, C_{17});

   (C_6, C_9, C_{12}), (C_6, C_9, C_{13}), (C_6, C_9, C_{14}), (C_6, C_{12}, C_{13}), (C_6, C_{12}, C_{14}), (C_6, C_{13}, C_{14});

   (C_9, C_{15}, C_{16}), (C_9, C_{12}, C_{13}), (C_9, C_{12}, C_{14}), (C_9, C_{13}, C_{14});

   (C_{10}, C_{11}, C_{13}),

   (C_{12}, C_{13}, C_{14}),

   (C_{14}, C_{15}, C_{16}),

   (C_{15}, C_{16}, C_{17}).

   Hence $I_3 = 35$.

4. I_4 is the number of 4-tuples of columns that intersect. We list all of the intersecting 4-sets that are nonempty:

   $(C_1, C_2, C_{10}, C_{11})$, $(C_1, C_2, C_{10}, C_{13})$, $(C_1, C_9, C_{15}, C_{16})$, $(C_1, C_{10}, C_{11}, C_{13})$;

   $(C_2, C_4, C_{14}, C_{16})$, $(C_2, C_4, C_{14}, C_{17})$, $(C_2, C_4, C_{16}, C_{17})$, $(C_2, C_{10}, C_{11}, C_{13})$, $(C_2, C_{14}, C_{16}, C_{17})$,

   $(C_4, C_{14}, C_{16}, C_{17})$,

   $(C_6, C_9, C_{12}, C_{13})$, $(C_6, C_9, C_{12}, C_{14})$, $(C_6, C_9, C_{13}, C_{14})$,

   $(C_6, C_{12}, C_{13}, C_{14})$,

$(C_9, C_{12}, C_{13}, C_{14})$.

Hence $I_4 = 15$.

5. I_5 is the number of 5-tuples of columns that intersect. We list all of the intersecting 5-sets that are nonempty:

   $(C_1, C_2, C_{10}, C_{11}, C_{13})$,

   $(C_2, C_4, C_{14}, C_{16}, C_{17})$,

   $(C_6, C_9, C_{12}, C_{13}, C_{14})$,

   Hence $I_5 = 3$.

6. I_6 is the number of 6-tuples of columns that intersect. There are none of these, so $I_6 = 0$.

Def 3.3 Let $n, m \in \mathsf{N}$ and $A \subseteq G_{n,m}$. Let $1 \le i_1 < i_2 \le n$. C_{i_1} and C_{i_2} *intersect* if $C_{i_1} \cap C_{i_2} \ne \emptyset$. The following picture portrays this happening with C_1 and C_2.

	1	2	$\ldots$
1	R		$\ldots$
2	R		$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_1 - 1$	R		$\ldots$
x_1	R	R	$\ldots$
$x_1 + 1$		R	$\ldots$
$x_1 + 2$		R	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_1 + x_2 - 1$		R	$\ldots$
$x_1 + x_2$			$\ldots$
$x_1 + x_2 + 1$			$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n			$\ldots$

3.2 $\sum_{j=1}^{k} x_j$ and $|\bigcup_{j=1}^{k} C_j|$

Lemma 3.4 *Let $n, m \in \mathsf{N}$. Let A be a rectangle free subset of $G_{n,m}$. Let $1 \le j_1 < j_2 \le n$. Then $|C_{j_1} \cap C_{j_2}| \le 1$.*

Proof:

As the following picture shows what happens if $|C_1 \cap C_2| \geq 2$. Note that a rectangle is formed. We leave it to the reader to make this into a formal argument.

	1	2	$\ldots$
1	R		$\ldots$
2	R		$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_1 - 1$	R	R	$\ldots$
x_1	R	R	$\ldots$
$x_1 + 1$		R	$\ldots$
$x_1 + 2$		R	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_1 + x_2 - 2$		R	$\ldots$
$x + x_2 + 1 - 1$			$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n			$\ldots$

∎

Lemma 3.5 *Let $n, m \in \mathsf{N}$. Let $1 \leq k \leq m$. Let $x_1, \ldots, x_k \in \mathsf{N}$. Assume $(x_1, \ldots, x_m)$ is (n,m)-placeable via A. (We need not assume that $x_1 \geq \cdots \geq x_m$ and hence can use this for any set of columns.)*

1. $x_1 + \cdots + x_k \leq n + \binom{k}{2}$.
2. *If* $x_1 + \cdots + x_k = n + \binom{k}{2}$ *then*
   - *for all* $1 \leq j_1 < j_2 \leq k$, $|C_{j_1} \cap C_{j_2}| = 1$, *and*
   - *for all* $1 \leq j_1 < j_2 < j_3 \leq k$, $|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 0$.
3. *If* $\bigcap_{j=1}^{k} C_j \neq \emptyset$ *then* $\sum_{j=1}^{k} x_j \leq n + \sum_{j=2}^{k} (-1)^j \binom{k}{j}$.
4. $|\bigcup_{j=1}^{k} C_j| \geq \sum_{j=1}^{k} x_i - \binom{k}{2}$.

Proof:

We begin with facts that are useful for all four parts.

By the law of inclusion-exclusion

$$|\bigcup_{j=1}^{k} C_j| = \sum_{j=1}^{k} |C_j| - I_2 + I_3 - I_4 + \cdots + (-1)^{k+1} I_k.$$

Since $|C_j| = x_j$ we have

$$|\bigcup_{j=1}^{k} C_j| = \sum_{j=1}^{k} x_j - I_2 + I_3 - I_4 + \cdots + (-1)^{k+1} I_k.$$

$$\sum_{j=1}^{k} x_j = |\bigcup_{j=1}^{k} C_j| + I_2 - I_3 + I_4 + \cdots + (-1)^{k} I_k.$$

Since $|\bigcup_{j=1}^{k} C_j| \le n$ we have

$$\sum_{j=1}^{k} x_j \le n + I_2 - I_3 + I_4 + \cdots + (-1)^{k} I_k.$$

1) Assume k is odd (the case of k even is similar).

$$\sum_{j=1}^{k} x_j \le n + I_2 + (I_4 - I_3) + \cdots + (I_{k-1} - I_{k-2}) - I_k.$$

Since $I_2 \le \binom{k}{2}$ and, for $3 \le j \le k-2$, $(I_{j+1} - I_j) \le 0$ and $-I_k \le 0$ we have

$$\sum_{j=1}^{k} x_j \le n + \binom{k}{2}.$$

2) We assume k is even. The k odd case is similar. We always have

$$\sum_{j=1}^{k} x_j = |\bigcup_{j=1}^{k} C_j| + I_2 - I_3 + I_4 + \cdots + (-1)^{k} I_k.$$

If this sum equals $n + \binom{k}{2}$ then we obtain

$$n + \binom{k}{2} = |\bigcup_{j=1}^{k} C_j| + I_2 - I_3 + I_4 + \cdots + (-1)^{k} I_k = |\bigcup_{j=1}^{k} C_j| + I_2 + (I_4 - I_3) + \cdots + (I_k - I_{k-1}).$$

Since

$$|\bigcup_{j=1}^{k} C_j| \le n,$$

$$I_2 \le \binom{k}{2},$$

and

$$(\forall j \ge 3)[I_{j+1} - I_j \le 0]$$

the only way that equality can hold is if

$$|\bigcup_{j=1}^{k} C_j| = n,$$

$$I_2 = \binom{k}{2},$$

and

$$(\forall j \ge 3)[I_{j+1} - I_j = 0].$$

By Lemma 3.4 for all $1 \le j_1 < j_2 \le k$, $|C_{j_1} \cap C_{j_2}| \le 1$. Since $I_2 = \binom{k}{2}$ we have that, for all $1 \le j_1 < j_2 \le k$, $|C_{j_1} \cap C_{j_2}| = 1$.

Since $(\forall j \ge 3)[I_{j+1} - I_j = 0]$ for all $3 \le j \le k$, $I_k = 0$. Hence for all $1 \le j_1 < j_2 < j_3 \le k$, $|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 0$.

3) Since $C_1 \cap \cdots \cap C_k \ne \emptyset$, for all j, $I_j = \binom{k}{j}$.

Hence

$$\sum_{j=1}^{k} x_j \le n + \sum_{j=2}^{k} (-1)^j I_j = n + \sum_{j=2}^{k} (-1)^j \binom{k}{j}.$$

4) We will assume k is odd. The k even case is similar.

$$\begin{array}{ll} |\bigcup_{j=1}^{k} C_j| & = \sum_{j=1}^{k} |C_j| - I_2 + I_3 - I_4 + \cdots + I_{k-1} - I_k \\ & = \sum_{j=1}^{k} |C_j| - I_2 + (I_3 - I_4) + \cdots + (I_{k-1} - I_{k-2}) - I_k \end{array}$$

Since $|I_2| \le \binom{k}{2}$ and, for all $3 \le j \le k-1$, $(I_j - I_{j+1}) \ge 0$, we have

$$|\bigcup_{j=1}^{k} C_j| \ge \sum_{j=1}^{k} |C_j| - \binom{k}{2} = \sum_{j=1}^{k} x_j - \binom{k}{2}.$$

∎

It will be convenient to specify the $k = 2$ case of Lemma 3.5.

Lemma 3.6 *Let $n, m \in \mathsf{N}$. Assume $(x_1, \ldots, x_m)$ is (n,m)-placeable via A. Then $x_1 + x_2 \le n + 1$.*

3.3 Using maxrf

Lemma 3.7 *Let $n, m \in \mathsf{N}$. Let $x \le x_1 \le n$. Assume $(x_1, \ldots, x_m)$ is (n,m)-placeable via A. Then*

$$|A| \le x + m - 1 + \mathrm{maxrf}(n - x, m - 1).$$

Proof: The following picture portrays what might happen in the case of $n = 12$, $x_1 = 8$. We use double lines to partition the grid in a way that will be helpful later.

	1	2	3	4	5	...	j	$\cdots$	m
1	R	R				$\cdots$		$\cdots$	
2	R		R			$\cdots$		$\cdots$	
3	R			R		$\cdots$		$\cdots$	
4	R				R	$\cdots$		$\cdots$	
5	R					$\cdots$		$\cdots$	
6	R					$\cdots$		$\cdots$	
7	R					$\cdots$		$\cdots$	R
8	R					$\cdots$	R	$\cdots$	
9		R	R			$\cdots$		$\cdots$	
10		R		R		$\cdots$		$\cdots$	
11		R				$\cdots$		$\cdots$	
12		R				$\cdots$		$\cdots$	

We view this grid in three parts.

Part 1: The first column. This has x_1 elements of A in it.

Part 2: Consider the grid consisting of rows $1, \ldots, x_1$ and columns $2, \ldots, m$. Look at the j^{th} column, $2 \le j \le m$ in this grid. For each such j, this column has at most one element in A (else there would be a rectangle using the first column). Hence the total number of elements of A from this part of the grid is $m - 1$.

Part 3: The bottom most $n - x_1$ elements of the right most $m - 1$ columns. This clearly has $\le \text{maxrf}(n - x_1, m - 1)$ elements in it.

Taking all the parts into account we obtain

$$|A| \le x_1 + (m - 1) + \text{maxrf}(n - x_1, m - 1).$$

We leave it as an exercise to show that, if $x \le x_1$, then

$$x_1 + (m - 1) + \text{maxrf}(n - x_1, m - 1) \le x + (m - 1) + \text{maxrf}(n - x, m - 1).$$

∎

3.4 Disjoint Columns

In this section we prove a very general theorem about what happens if the first k columns are disjoint. In order to actually use this theorem we will need some lemmas.

Def 3.8 Let $n, m, q, u \in \mathsf{N}$.

1. $\mathrm{maxrf}_q(m, n)$ is the size of the maximum rectangle-free $A \subseteq G_{n,m}$ where every column has at least q elements in it.

2. $\mathrm{colsrf}_q(n)$ is the largest m such that there is a rectangle-free $A \subseteq G_{n,m}$ where every column has at least q elements in it.

3. $\mathrm{colsint}_q(n)$ is the largest m such that there is set $A \subseteq G_{n,m}$ where no two columns intersect and there are at least q elements in each column.

Lemma 3.9 *Let $p, q \in \mathsf{N}$ such that $p \geq q$. Then*

1.

$$\mathrm{colsrf}_q(p) \leq \left\lfloor \frac{p(p-1)}{q(q-1)} \right\rfloor$$

(We allow the case of $q = 1$ though it gives the trivial result that $\mathrm{colsrf}_q(p) \leq \infty$*.)*

2.

$$\mathrm{colsrf}_q(p) \cdot q \leq p + \binom{\mathrm{colsrf}_q(p)}{2}.$$

Proof:

Let $A \subseteq G_{p,u}$ be a rectangle-free set such that every column has at least q elements in it. Let $C_1, \ldots, C_u$ be the columns. We show conditions that u satisfies, and hence $\mathrm{colsrf}_q(p)$ satisfies.

1) If $\{a, b\}$ is a distinct pair of elements in (say) C_1 and $\{c, d\}$ is a distinct pair of elements in (say) C_2 we have $\{a, b\} \neq \{c, d\}$.

$$\left| \bigcup_{j=1}^{u} \binom{C_j}{2} \right| \geq \binom{q}{2} \cdot u$$

Since each $\binom{C_j}{2}$ is a subset of $\binom{[p]}{2}$ we also have

$$\left| \bigcup_{j=1}^{u} \binom{C_j}{2} \right| \leq \binom{p}{2}.$$

Hence we have

$$\binom{q}{2} \cdot u \leq \binom{p}{2}.$$

$$u \leq \frac{\binom{p}{2}}{\binom{q}{2}} = \frac{p(p-1)}{q(q-1)}.$$

Since u is an integer

$$u \leq \left\lfloor \frac{p(p-1)}{q(q-1)} \right\rfloor.$$

2) By Lemma 3.5

$$|C_1| + \cdots + |C_u| \leq p + \binom{u}{2}.$$

Since each $|C_i|$ is $\geq q$ we have

$$u \cdot q \leq |C_1| + \cdots + |C_u| \leq p + \binom{u}{2}.$$

▮

The following easy lemma we leave to the reader.

Lemma 3.10

$$\mathrm{colsint}_q(p) = \left\lfloor \frac{p}{q} \right\rfloor$$

Theorem 3.11 *Let $k, n, m, p, q, u \in \mathsf{N}$. Assume the following.*

- $q \leq p \leq n$ *and* $1 \leq k \leq m$.
- u *is the largest number such that* $u \leq \left\lfloor \frac{p(p-1)}{q(q-1)} \right\rfloor$ *and* $uq \leq p + \binom{u}{2}$.
- $(x_1, \ldots, x_m)$ *is* (n, m)*-placeable via* A.
- *In this placement, for all* $1 \leq j_1 < j_2 \leq k$, $|C_{j_1} \cap C_{j_2}| = \emptyset$.
- $x_1 \geq \cdots \geq x_k$. *We make no other assumptions about the orderings of the* x_j*'s.*
- $x_1 + \cdots + x_k = n - p$.

Let $N = N_{k+q}$ be the number of j, $k+1 \leq j \leq m$, such that $|C_j| = k + q$. Then N satisfies the following conditions.

1. $N \leq \mathrm{colsrf}_q(p) \leq u$.
2. $N \leq (\mathrm{colsint}_q(p))x_{k-\mathrm{maxrf}_q(p,N)+qN} \leq \left\lfloor \frac{p}{q} \right\rfloor x_{k-\mathrm{maxrf}_q(p,N)+qN}$
3. *If* $q = 1$ *and* $p = 1$ *then* $N \leq x_k$.
4. *If* $q = 1$, $p = 2$, *and* $k \geq 2$, *then* $N \leq 2x_{k-1}$.
5. *If* $q = 1$, $p = 3$, *and* $k \geq 4$, *then* $N \leq 3x_{k-3}$.

(The last three items follow directly from item 2 and Lemmas 3.10 and 12.1. Hence we will not prove them.)

Proof:

The proofs (1) and (2) of this theorem begin the same way.

The following picture portrays an example where $k = 4$, $x_1 = 8$, $x_2 = 5$, $x_3 = 4$, $x_4 = 2$, and $p = 5$.

	1	2	3	4	5	$\ldots$	j	$\cdots$	m
1	R					$\cdots$		$\cdots$	
2	R					$\cdots$		$\cdots$	
3	R					$\cdots$		$\cdots$	
4	R					$\cdots$		$\cdots$	
5	R					$\cdots$		$\cdots$	
6	R					$\cdots$		$\cdots$	
7	R					$\cdots$		$\cdots$	
8	R					$\cdots$		$\cdots$	
9		R				$\cdots$		$\cdots$	
10		R				$\cdots$		$\cdots$	
11		R				$\cdots$		$\cdots$	
12		R				$\cdots$		$\cdots$	
13		R				$\cdots$		$\cdots$	
14			R			$\cdots$		$\cdots$	
15			R			$\cdots$		$\cdots$	
16			R			$\cdots$		$\cdots$	
17			R			$\cdots$		$\cdots$	
18				R		$\cdots$		$\cdots$	
19				R		$\cdots$		$\cdots$	
20						$\cdots$		$\cdots$	
21						$\cdots$		$\cdots$	
22						$\cdots$		$\cdots$	
23						$\cdots$		$\cdots$	
24						$\cdots$		$\cdots$	

By renumbering we can assume that our general grid looks like the one above. In particular

$$
\begin{aligned}
C_1 &= \{1, \ldots, x_1\} \\
C_2 &= \{x_1 + 1, \ldots, x_1 + x_2\} \\
C_3 &= \{x_1 + x_2 + 1, \ldots, x_1 + x_2 + x_3\} \\
&\vdots \\
C_k &= \{x_1 + \cdots + x_{k-1} + 1, \ldots, x_1 + \cdots + x_{k-1} + x_k\}
\end{aligned}
$$

Let $k+1 \le j \le k+N$. Let $C_j = \{i_1 < \cdots < i_{k+q}\}$. Note that

$$\begin{array}{ll} |C_j \cap C_1| & \le 1 \\ |C_j \cap C_2| & \le 1 \\ & \vdots \\ |C_j \cap C_k| & \le 1. \end{array}$$

hence

$$|C_j \cap \{1, \ldots, x_1 + \cdots + x_k\}| \le k.$$

Therefore

$$|C_j \cap \{x_1 + \cdots + x_k + 1, \ldots, n\}| \ge q.$$

Let $r_j \ge 0$ be such that

$$|C_j \cap \{x_1 + \cdots + x_k + 1, \ldots, x_1 + \cdots + x_k + p\}| = q + r_j.$$

Let *SETS* be the following set of sets:

- $\{1, \ldots, x_1\}$,
- $\{x_1 + 1, \ldots, x_1 + x_2\}$,
- $\vdots$
- $\{x_1 + \cdots + x_{k-1} + 1, \ldots, x_1 + \cdots + x_{k-1} + x_k\}$.

C_j has to intersect at least $(k+q) - (q+r_j) = k - r_j$ of the sets in *SETS*. Let $MISS_j$ be the sets in *SETS* that C_j does not intersect. Note that $|MISS_j| = k - (k - r_j) = r_j$.

Look at the grid formed by rows $x_1 + \cdots + x_k + 1, \ldots, x_1 + \cdots + x_k + p$ and columns $k+1, \ldots, k+N$. Let B be the restriction of A to this set. Note the following:

- Every column of this grid has $\ge q$ elements in it.
- $|B| \le \mathrm{maxrf}_q(p, N)$.
- $|B| = (q + r_1) + \cdots + (q + r_N) = qN + r_1 + \cdots + r_N$.
- Combining the last two items we obtain

$$r_1 + \cdots + r_N = |B| - qN \le \mathrm{maxrf}_q(p, N) - qN.$$

1)

Note that B is a rectangle-free subset of $G_{p,N}$ with at least q elements in each column. Hence, by the definition of $\mathrm{colsrf}_q(p)$, $N \le \mathrm{colsrf}_q(p)$. By Lemma 3.9 $\mathrm{colsrf}_q(p) \le u$. Hence

$$N \le \mathrm{colsrf}_q(p) \le u.$$

2)

Assume, by way of contradiction, that

$$N \ge (\mathrm{colsint}_q(p)) x_{k-\mathrm{maxrf}_q(p,N)+qN} + 1.$$

Let $1 \le j_0 \le k$ be the largest number (hence the smallest x_{j_0} value) such that, for all $k+1 \le j \le N$,

$$C_j \cap \{x_1 + \cdots + x_{j_0-1} + 1, \ldots, x_1 + \cdots + x_{j_0-1} + x_{j_0}\} \ne \emptyset.$$

(We define $x_0 = 0$ for this notation.) We want a lower bound on j_0. For each $k+1 \le j \le N$ there are at most $|MISS_j| = r_j$ sets $Z \in SETS$ such that $C_j \cap Z = \emptyset$. Hence there are at most $r_1 + \cdots + r_N$ elements of *SETS* such that there is a $k+1 \le j \le N$ such that C_j does not intersect. Hence $j_0 \ge k - (r_1 + \cdots + r_N) \ge k - \mathrm{maxrf}_q(p,N) + qN$. Therefore

$$|\{x_1 + \cdots + x_{j_0-1} + 1, \ldots, x_1 + \cdots + x_{j_0-1} + x_{j_0}\}| = x_{j_0} \ge x_{k-\mathrm{maxrf}_q(p,N)+qN}.$$

Map each $k+1 \le j \le k+N$ to the following ordered pair:

$$(C_j \cap \{x_1 + \cdots + x_k + 1, \ldots, x_1 + \cdots + x_k + p\}, C_j \cap \{x_1 + \cdots + x_{j_0-1} + 1, \ldots, x_1 + \cdots + x_{j_0-1} + x_{j_0}\}.)$$

Note that there will be exactly one element in the second component of this ordered pair. Since

$$N \ge (\mathrm{colsint}_q(p)) x_{k-\mathrm{maxrf}(p,N)+qN} + 1 \ge (\mathrm{colsint}_q(p)) x_{j_0} + 1$$

there are $\mathrm{colsint}_q(p) + 1$ values of j in $k+1 \le j \le N$ that all map to the same second coordinate. We can renumber so that these are columns $k+1 \le j \le k + \mathrm{colsint}_q(p)$. The following picture portrays what is happening (though we have so far left out R's in the last p rows).

	$k+1$	$k+2$	$\cdots$	$k+\mathrm{colsint}_q(p)+1$
$x_1+\cdots+x_{j_0-1}+1$	R	R	$\cdots$	R
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$x_1+\cdots+x_{j_0-1}+x_k$			$\cdots$	
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$x_1+\cdots+x_{k-1}+x_k+1$			$\cdots$	
$x_1+\cdots+x_{k-1}+x_k+2$			$\cdots$	
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$x_1+\cdots+x_{k-1}+x_k+p$			$\cdots$	

Look at the grid formed by rows $x_1 + \cdots + x_k + 1, \ldots, x_1 + \cdots + x_k + p$ and columns $k+1, \ldots, k + \text{colsint}_q(p) + 1$. Let B be the restriction of A to this set. Note that there are at least q elements in each column. By the definition of $\text{colsint}_q(p)$ there are two columns that intersect. By renumbering we can assume they are columns $k+1$ and $k+2$. The following picture portrays what happens and shows that a rectangle is formed. Hence we have a contradiction. Hence

$$N \le (\text{colsint}_q(p)) x_{k-\text{maxrf}_q(p,N)+qN}.$$

	$k+1$	$k+2$	$\cdots$	$k + \text{colsint}_q(p) + 1$
$x_1 + \cdots + x_{j_0-1} + 1$	R	R	$\cdots$	R
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$x_1 + \cdots + x_{j_0-1} + x_k$			$\cdots$	
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$x_1 + \cdots + x_{k-1} + x_k + 1$	R		$\cdots$	
$x_1 + \cdots + x_{k-1} + x_k + 2$	R	R	$\cdots$	
$x_1 + \cdots + x_{k-1} + x_k + 3$		R	$\cdots$	
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$x_1 + \cdots + x_{k-1} + x_k + p$			$\cdots$	

By Lemma 3.10 $\text{colsint}_q(p) = \left\lfloor \frac{p}{q} \right\rfloor$. Combining this with

$$N \le (\text{colsint}_q(p)) x_{k-\text{maxrf}_q(p,N)+qN}$$

we obtain

$$N \le (\text{colsint}_q(p)) x_{k-\text{maxrf}_q(p,N)+qN} \le \left\lfloor \frac{p}{q} \right\rfloor x_{k-\text{maxrf}_q(p,N)+qN}.$$

∎

4 Tools for Finding Proper c-Colorings

4.1 Strong c-colorings and Strong (c, c')-colorings

Def 4.1 Let $c, c', n, m \in \mathsf{N}$ and let $\chi : G_{n,m} \to [c]$. χ is a *strong* (c, c')*-coloring* if the following holds: For all rectangles where (1) the two right most corners are the same color, say c_1, and (2) the two left most corners are the same color, say c_2, we have $c_1 \ne c_2$ and $c_1, c_2 \in [c']$.

Def 4.2 Let $c, c', n, m \in \mathsf{N}$. $G_{n,m}$ is *strongly* (c, c')*-colorable* if it has a strong (c, c')-coloring.

Note 4.3 Let $c, n, m \in \mathsf{N}$ and let $\chi : G_{n,m} \to [c]$. If there are *no* rectangles such that (1) the two right most corners are the same color and (2) the two left most corners are the same color, then, for all c', χ is a strong (c, c')-coloring. However, we will in this case take $c' = 1$. We call such colorings *strong c-colorings*

Example 4.4

1. The following is a strong 4-coloring of $G_{5,8}$.

1	1	1	4	1	1	4	4
2	2	4	1	2	4	1	4
3	4	2	2	4	2	4	1
4	3	3	3	4	4	2	2
4	4	4	4	3	3	3	3

2. The following is a strong 3-coloring of $G_{4,6}$.

1	1	3	1	3	3
2	3	1	3	1	3
3	2	2	3	3	1
3	3	3	2	2	2

3. The following is a strong $(4, 2)$-coloring of $G_{6,15}$.

1	1	1	1	1	3	3	3	2	3	3	2	2	2	2
1	2	2	2	2	1	1	1	1	4	4	3	3	3	2
2	1	3	3	2	1	2	2	2	1	1	1	4	4	3
2	2	1	4	3	2	1	4	3	1	2	2	1	1	4
3	3	2	1	4	2	2	1	4	2	1	4	1	2	1
4	4	4	2	1	4	4	2	1	2	2	1	2	1	1

4. The following is a strong $(6, 2)$-coloring of $G_{8,6}$.

1	1	2	2	3	6
1	2	1	2	4	5
2	1	2	1	5	4
2	2	1	1	6	3
3	4	5	6	1	1
4	5	6	4	1	1
5	6	3	3	1	2
6	3	4	5	1	2

5. The following is a strong $(5,3)$-coloring of $G_{8,28}$.

1	1	1	1	1	1	1	5	5	5	5	3	2	4	3	4	3	2	3	4	3	2	3	3	2	2	2	2
1	2	2	2	2	2	2	1	1	1	1	1	1	5	4	5	4	3	4	3	4	3	3	4	3	3	3	2
2	1	3	3	3	3	2	1	2	2	2	2	2	1	1	1	1	1	5	5	5	4	4	3	4	3	4	3
2	2	1	4	4	4	3	2	1	3	3	3	3	1	2	2	2	2	1	1	1	1	5	5	5	4	3	3
3	3	2	1	5	3	3	2	2	1	4	4	4	2	1	3	3	3	1	2	2	2	1	1	1	5	5	4
3	4	3	2	1	5	4	3	3	2	1	5	3	2	2	1	5	4	2	1	3	3	1	2	2	1	1	5
4	3	4	3	2	1	5	3	4	3	2	1	5	3	3	2	1	5	2	2	1	5	2	1	3	1	2	1
5	5	5	5	3	2	1	4	3	4	3	2	1	3	5	3	2	1	3	3	2	1	2	2	1	2	1	1

Lemma 4.5 *Let $c, c', n, m \in \mathsf{N}$. Let $x = \lfloor c/c' \rfloor$. If $G_{n,m}$ is strongly (c,c')-colorable then $G_{n,xm}$ is c-colorable.*

Proof:

Let χ be a strong (c,c')-coloring of $G_{n,m}$. Let the colors be $\{1,\ldots,c\}$. Let χ^i be the coloring

$$\chi^i(a,b) = \chi(a,b) + i \pmod{c}.$$

(During calculations mod c we use $\{1,\ldots,c\}$ instead of the more conventional $\{0,\ldots,c-1\}$.)

Take $G_{n,m}$ with coloring χ. Place next to it $G_{n,m}$ with coloring $\chi^{c'}$. Then place next to that $G_{n,m}$ with coloring $\chi^{2c'}$ Keep doing this until you have $\chi^{(x-1)c'}$ placed. The following is an example using the strong $(6,2)$-coloring of $G_{8,6}$ in Example 4.4.4. Since $c' = 2$ and $x = 3$ we will be shifting the colors first by 2 then by 4.

1	1	2	2	3	6	3	3	4	4	5	2	5	5	6	6	1	4
1	2	1	2	4	5	3	4	3	4	6	2	5	6	5	5	2	4
2	1	2	1	5	4	4	3	4	3	1	6	6	5	6	5	3	2
2	2	1	1	6	3	4	4	3	3	2	5	6	6	5	5	4	2
3	4	5	6	1	2	5	6	2	2	3	4	1	2	4	4	5	6
4	5	6	4	1	1	6	1	2	6	3	3	2	3	4	2	5	5
5	6	3	3	1	2	1	2	5	5	3	4	3	4	1	1	5	6
6	3	4	5	1	2	2	5	6	1	3	4	4	1	2	3	5	6

We claim that the construction always creates a c-coloring of $G_{m,xn}$.

We show that there is no rectangle with the two leftmost points from the first $G_{n,m}$. From this, to show that there are no rectangles at all is just a matter of notation.

Assume that in column i_1 there are two points colored R (in this proof $1 \le R, B, G \le c$.) We call these *the i_1-points*. The points cannot form a rectangle with any other points in $G_{n,m}$ since χ is a c-coloring of $G_{n,m}$. The i_1-points cannot form a rectangle with points in columns $i_1 + m$, $i_1 + 2m$, …, $i_1 + (c-1)m$ since the colors of those points are $R + c' \pmod{c}$, $R + 2c'$

(mod c), ..., $R+(x-1)c'$ (mod c), all of which are not equal to R. Is there a $1 \le j \le x-1$ and a $1 \le i_2 \le m$ such that the i_1-points form a rectangle with points in column $i_2 + jm$?

Since χ is a strong (c,c')-coloring, points in column i_2 and on the same row as the i_1-points are either colored *differently*, or both colors are in $[c']$. We consider both of these cases.

Case 1: In column i_2 the colors are B and G where $B \neq G$ (it is possible that $B = R$ or $G = R$ but not both). By the construction the points in column $i_2 + jm$ are colored $B + jc'$ (mod c) and $G + jc'$ (mod c). These points are colored differently, hence they cannot form a rectangle with the i_1-points.

$\cdots$	i_1	$\cdots$	i_2	$\cdots$	$\cdots$	i_1+jm	$\cdots$	i_2+jm	$\cdots$
$\cdots$	R	$\cdots$	B	$\cdots$	$\cdots$	$R+jc'$	$\cdots$	$B+jc'$	$\cdots$
$\cdots$	R	$\cdots$	G	$\cdots$	$\cdots$	$R+jc'$	$\cdots$	$G+jc'$	$\cdots$

Case 2: In column i_2 the colors are both B.

$\cdots$	i_1	$\cdots$	i_2	$\cdots$	$\cdots$	i_1+jm	$\cdots$	i_2+jm	$\cdots$
$\cdots$	R	$\cdots$	B	$\cdots$	$\cdots$	$R+jc'$	$\cdots$	$B+jc'$	$\cdots$
$\cdots$	R	$\cdots$	B	$\cdots$	$\cdots$	$R+jc'$	$\cdots$	$B+jc'$	$\cdots$

We have $R, B \in [c']$. By the construction the points in column $i_2 + jm$ are both colored $B + jc'$ (mod c). We show that $R \not\equiv B + jc'$ (mod c). Since $1 \le j \le x-1$ we have

$$c' \le jc' \le (x-1)c'.$$

Hence

$$B + c' \le B + jc' \le B + (x-1)c'.$$

Since $B \in [c']$ we have $B + (x-1)c' \le xc'$. Hence

$$B + c' \le B + jc' \le xc'.$$

By the definition of x we have $xc' \le c$. Since $B \in [c']$ we have $B + c' \ge c' + 1$. Hence

$$c' + 1 \le B + jc' \le c.$$

Since $R \in [c']$ we have that $R \not\equiv B + jc'$. ∎

4.2 Using Combinatorics and Strong (c,c')-Colorings

Theorem 4.6 *Let $c \ge 2$.*

1. *There is a strong c-coloring of $G_{c+1,\binom{c+1}{2}}$.*

2. *There is a c-coloring of $G_{c+1,m}$ where $m = c\binom{c+1}{2}$.*

Proof:
1) We first do an example of our construction. In the $c = 5$ case we obtain the following coloring.

5	5	5	5	5	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	5	5	5	5	2	2	2	2	2	2
1	5	2	2	2	5	2	2	2	5	5	5	3	3	3
2	2	5	3	3	2	5	3	3	5	3	3	5	5	4
3	3	3	5	4	3	3	5	4	3	5	4	5	4	5
4	4	4	4	5	4	4	4	5	4	4	5	4	5	5

Here is our general construction. Index the columns by the $\binom{[c+1]}{2}$. Color rows of column $\{x, y\}$, $x < y$, as follows.

1. Color rows x and y with color c.
2. On the other spots use the colors $\{1, 2, 3, \ldots, c-1\}$ in increasing order (the actual order does not matter).

2) This follows from Lemma 4.5 with $c = c$ and $c' = 1$, and Part (1) of this theorem. ▮

The next theorem generalizes Theorem 4.6.

Theorem 4.7 *Let $c, c' \in \mathsf{N}$ with $c \geq 2$ and $1 \leq c' \leq c$.*

1. *There is a strong (c, c')-coloring of $G_{c+c',m}$ where $m = \binom{c+c'}{2}$.*
2. *There is a c-coloring of $G_{c+c',m'}$ where $m' = \lfloor c/c' \rfloor \binom{c+c'}{2}$.*

To prove Theorem 4.7, we will use a partition of $\binom{[2n]}{2}$ into perfect matchings of $[2n]$ for certain values of n. Each perfect matching thus has size n.

We first give some examples and then a general lemma.

Example 4.8

1. If $n = 3$, $2n = 6$, $2n - 1 = 5$. We show a partition of $\binom{[6]}{2}$ into 5 parts of size 3. We first pair up the elements as follows, each number in the top row being paired with the number below it:

1	2	3
6	5	4

   This corresponds to $\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$. This is our first part of size 3.

   We keep 1 fixed and keep rotating the other numbers clockwise to obtain the following parts.

$$\left| \begin{array}{c|c|c} 1 & 6 & 2 \\ 5 & 4 & 3 \end{array} \right|$$

$$\left| \begin{array}{c|c|c} 1 & 5 & 6 \\ 4 & 3 & 2 \end{array} \right|$$

$$\left| \begin{array}{c|c|c} 1 & 4 & 5 \\ 3 & 2 & 6 \end{array} \right|$$

$$\left| \begin{array}{c|c|c} 1 & 3 & 4 \\ 2 & 6 & 5 \end{array} \right|$$

Note that the first pair went $\{1, 5\}$, $\{1, 4\}$, $\{1, 3\}$, $\{1, 2\}$. That is, 1 was fixed but the other element decreased by 1. Also note that the second and third pair had both elements decrease by 1 except 2 goes to 6. This partition is a special case of a general construction we will have later. The same applies to the next example.

2. If $n = 4$, $2n = 8$, $2n - 1 = 7$. Here is a partition of $\binom{[8]}{2}$ into 7 parts of size 4.

   We keep 1 fixed and keep rotating the other numbers clockwise to obtain the following parts.

$$\left| \begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{array} \right|$$

$$\left| \begin{array}{c|c|c|c} 1 & 8 & 2 & 3 \\ 7 & 6 & 5 & 4 \end{array} \right|$$

$$\left| \begin{array}{c|c|c|c} 1 & 7 & 8 & 2 \\ 6 & 5 & 4 & 3 \end{array} \right|$$

$$\left| \begin{array}{c|c|c|c} 1 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 \end{array} \right|$$

$$\left| \begin{array}{c|c|c|c} 1 & 5 & 6 & 7 \\ 4 & 3 & 2 & 8 \end{array} \right|$$

$$\left| \begin{array}{c|c|c|c} 1 & 4 & 5 & 6 \\ 3 & 2 & 8 & 7 \end{array} \right|$$

$$\left| \begin{array}{c|c|c|c} 1 & 3 & 4 & 5 \\ 2 & 8 & 7 & 6 \end{array} \right|$$

The next lemma shows that such partitions always exist. The lemma (and the examples above) is based on the Wikipedia entry on Round Robin tournaments. We present a proof for completeness.

Lemma 4.9 *Let $n \in \mathsf{N}$. $\binom{[2n]}{2}$ can be partitioned into $2n-1$ sets $P_1, \ldots, P_{2n-1}$, each of size n, such that each P_i is itself a partition of $[2n]$ into pairs (i.e., a perfect matching).*

Proof: Following the examples above, we define the cyclic permutation ρ on $\{2, 3, \ldots, 2n\}$ as follows:

$$\rho(x) = \begin{cases} x-1 & \text{if } 2 < x \leq 2n, \\ 2n & \text{if } x = 2, \end{cases}$$

for all $x \in \{2, 3, \ldots, 2n\}$. Then for each $1 \leq i \leq 2n-1$, we define

$$P_i = \{\ \{1, 2n-i+1\}\ \} \cup \{\ \{\rho^{(i-1)}(j), \rho^{(i-1)}(2n-j+1)\} \mid 2 \leq j \leq n\}\ ,$$

noting that $2n-i+1 = \rho^{(i-1)}(2n)$. It is not too hard to see that each P_i contains exactly n pairwise disjoint pairs, so it suffices to show that no pair appears in two different P_i. Clearly, no pair of the form $\{1, x\}$ can appear in more than one P_i. Suppose $\{x, y\}$ appears in both P_i and P_j for some $i < j$, where $2 \leq x, y \leq 2n$. Then without loss of generality, x appears in the top row of P_i with y just below it. If x is still in the top row of P_j, then x has shifted to the right and y to the left, and so x and y are not vertically aligned in P_j, which means that $\{x, y\} \notin P_j$. So it must be that x is on the bottom row of P_j with y just above it. But for this to happen, x and y would have to rotate different amounts from P_i to P_j (one an even distance and the other an odd distance), but they rotate the same amount, namely, $j-i$ spaces—contradiction. Thus the P_i are as required. ▮

Proof: [Proof of Theorem 4.7]
1) Here is our general construction. We split into two cases.

Case 1: $c + c'$ is even. Then $c + c' = 2n$ for some n. Since $c' \leq c$, we also have $c' \leq n$. Let $P_1, \ldots, P_{2n-1}$ be the partition of $[2n]$ of Lemma 4.9. Index the elements of each P_i as $p_{i,j}$ for $1 \leq j \leq n$, that is, $P_i = \{p_{i,1}, p_{i,2}, \ldots, p_{i,n}\}$. We break up the columns into $2n-1$ blocks of n columns each (note that $n(2n-1) = \binom{2n}{2}$). We color the j^{th} column in the i^{th} block as follows:

- Assign color 1 to the two elements of $p_{i,(j+1) \bmod n}$,
- Assign color 2 to the two elements of $p_{i,(j+2) \bmod n}$,
- $\vdots$
- Assign color c' to the two elements of $p_{i,(j+c') \bmod n}$,
- Assign the colors $c'+1, \ldots, c$ one each to the rest of the elements in the column in increasing order.

Suppose some pair $p_{i,k} = \{x, y\}$ is monochrome in two separate columns. Then both these columns must be in the i^{th} block, the j_1^{st} column (colored c_1) and j_2nd column (colored c_2), say. Then we must have

$$k = (j_1 + c_1) \bmod n = (j_2 + c_2) \bmod n.$$

Since $j_1 \neq j_2$, we must have $c_1 \neq c_2$.

Case 2: $c + c'$ is odd. Then we choose a simpler partition. Let $c + c' = 2n + 1$ for some n. Since $c' < c$, we also have $c' \leq n$. For $1 \leq i \leq 2n + 1$ and $1 \leq j \leq n$, define

$$p_{i,j} = \{(i + j) \bmod (2n + 1), (i - j) \bmod (2n + 1)\} \in \binom{[2n+1]}{2},$$

and let

$$P_i = \{p_{i,1}, \ldots, p_{i,n}\}.$$

It is not too hard to see that all the pairs within the same P_i are pairwise disjoint and that no pair is contained in more than one P_i.

We now proceed with exactly the same recipe as in Case 1, except that, noting that $\binom{2n+1}{2} = n(2n + 1)$, we group the columns into $2n + 1$ blocks of n columns each. We get a strong (c, c')-coloring just as in Case 1.

2) This follows from Lemma 4.5 and Part (1) of this theorem. ▮

Corollary 4.10 *For all $c \geq 2$ there is a c-coloring of $G_{2c,2c^2-c}$.*

4.3 Using Finite Fields and Strong c-Colorings

Def 4.11 Let X be a finite set and $q \in \mathsf{N}$, $q \geq 3$. Let $P \subseteq \binom{X}{q}$.

$$\text{pairs}(P) = \{\{a_1, a_2\} \in \binom{X}{2} : (\exists a_3, \ldots, a_q)[\{a_1, \ldots, a_q\} \in P]\}.$$

Example 4.12 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $q = 3$.

1. Let $P = \{\{1, 2, 6\}, \{1, 8, 9\}, \{2, 4, 6\}\}$. Then

$$\text{pairs}(P) = \{\{1, 2\}, \{1, 6\}, \{2, 6\}, \{1, 8\}, \{1, 9\}, \{8, 9\}, \{2, 4\}, \{4, 6\}\}$$

2. Let $P = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. Then

$$\text{pairs}(P) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{7, 8\}, \{7, 9\}, \{8, 9\}\}.$$

Lemma 4.13 *Let $c, m, r \in \mathsf{N}$. Assume that there exists $P_1, \ldots, P_m \subseteq \binom{[cr]}{r}$ such that the following hold.*

- *For all $1 \le j \le m$, P_j is a partition of $[cr]$ into c parts of size r.*
- *For all $1 \le j_1 < j_2 \le m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$.*

Then

1. *$G_{cr,m}$ is strongly c-colorable.*
2. *$G_{cr,cm}$ is c-colorable.*

Proof:
1)

We define a strong c-coloring COL of $G_{cr,m}$ using $P_1, \ldots, P_m$.

Let $1 \le j \le m$. Let

$$P_j = \{L_j^1, \ldots, L_j^c\}$$

where each L_j^i is a subset of r elements from $[cr]$.

Let $1 \le i \le cr$ and $1 \le j \le m$. Since P_j is a partition of $[cr]$ there exists a unique u such that $i \in L_j^u$. Define

$$COL(i, j) = u.$$

We show that this is a strong c-coloring. Assume, by way of contradiction, that there exists $1 \le i_1 < i_2 \le 2k$ and $1 \le j_1 < j_2 \le 2k - 1$ such that $COL(i_1, j_1) = COL(i_1, j_2) = u$ and $COL(i_2, j_1) = COL(i_2, j_2) = v$. By definition of the coloring we have

$$i_1 \in L_{j_1}^u, i_1 \in L_{j_2}^u, i_2 \in L_{j_1}^v, i_2 \in L_{j_2}^v$$

Then

$$\{i_1, i_2\} \in \text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}),$$

contradicting the second premise on the P's.

2) This follows from Part (1) and Lemma 4.5 with $c = c$ and $c' = 1$. ∎

The Round Robin partition of Lemma 4.9 is an example of a partition satisfying the premises of Lemma 4.13, where $c = n$, $r = 2$, and $m = 2n - 1 = 2c - 1$. The next theorem yields partitions with bigger values of r.

Theorem 4.14 *Let p be a prime and $s, d \in \mathsf{N}$. Let $c = p^{ds-s}$, $r = p^s$, and $m = \frac{p^{ds}-1}{p^s-1}$. Then $G_{cr,cm}$ is c-colorable.*

Proof: We show that there exists $P_1, \ldots, P_m$ satisfying the premise of Lemma 4.13. The result follows immediately.

Let F be the finite field on p^s elements. We identify $[cr]$ with the set F^d.

Def 4.15

1. Let $\vec{x} \in F^d$, $\vec{y} \in F^d - \{0^d\}$. Then
$$L_{\vec{x},\vec{y}} = \{\vec{x} + f\vec{y} \mid f \in F\}.$$
Sets of this form are called *lines*. Note that for all $\vec{x}, \vec{y}, a \in F$ with $a \neq 0$,
$$L_{\vec{x},\vec{y}} = L_{\vec{x},a\vec{y}}.$$

2. Two lines $L_{\vec{x},\vec{y}}$, $L_{\vec{z},\vec{w}}$ *have the same slope* if $\vec{y}$ is a multiple of $\vec{w}$.

The following are easy to prove and well-known.

- If L and L' are two distinct lines that have the same slope, then $L \cap L' = \emptyset$.
- If L and L' are two distinct lines with different slopes, then $|L \cap L'| \leq 1$.
- If L is a line then there are exactly $r = p^s$ points on L.
- If L is a line then there are exactly $c = p^{ds-s}$ lines that have the same slope as L (this includes L itself).
- There are exactly $\frac{p^{ds}-1}{p^s-1}$ different slopes.

We define $P_1, \ldots, P_m$ as follows.

1. Pick a line L. Let P_1 be the set of lines that have the same slope as L.
2. Assume that $P_1, \ldots, P_{j-1}$ have been defined and that $j \leq m$. Let L be a line that is not in $P_1 \cup \cdots \cup P_{j-1}$. Let P_j be the set of all lines that have the same slope as L.

We need to show that $P_1, \ldots, P_m$ satisfies the premises of Lemma 4.13

a) For all $1 \leq j \leq m$, P_j is a partition of $[cr]$ into c parts of size r. Let $L \in P_j$. Note that P_j is the set of all lines with the same slope as L. Clearly this partitions F^{sd} which is $[cr]$.

b) For all $1 \leq j_1 < j_2 \leq m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$. Let L_1 be any line in P_{j_1} and L_2 be any line in P_{j_2}. Since $|L_1 \cap L_2| \leq 1 < 2$ we have the result.

Note that each P_j has $c = p^{ds-s}$ sets (lines) in it, each set (line) has $r = p^s$ numbers (points), and there are $m = \frac{p^{ds}-1}{p^s-1}$ many P's. Hence the premises of Lemma 4.13 are satisfied. ∎

It is convenient to state the $s = 1$, $d = 2$ case of Theorem 4.14.

Corollary 4.16 *Let p be a prime.*

1. *There is a strong p-coloring of $G_{p^2,p+1}$.*
2. *There is a p-coloring of G_{p^2,p^2+p}.*

Note 4.17 It would be of interest to obtain a Lemma similar to Theorem 4.14 that does not need prime powers and possibly yields strong (c, c')-colorings.

5 Bounds on the Sizes of Obstruction Sets

5.1 An Upper Bound

Using the uncolorability bounds, we can obtain an upper-bound on the size of a c-colorable grid.

Theorem 5.1 *For all $c > 0$, G_{c^2+c,c^2+c} is not c-colorable.*

Proof: We apply Corollary 2.10 with $m = c^2 + c$ and $n = c^2 + c$. Note that

$$\begin{aligned} \left\lceil \frac{nm}{c} \right\rceil &= \left\lceil \frac{(c^2+c)(c^2+c)}{c} \right\rceil \\ &= (c+1)(c^2+c). \end{aligned}$$

Letting $q = c + 1$ and $r = 0$, we have

$$\begin{aligned} \frac{m(m-1) - 2qr}{q(q-1)} &= \frac{(c^2+c)(c^2+c-1)}{(c+1)c} \\ &= c^2 + c - 1 \\ &< c^2 + c \\ &= n. \end{aligned}$$

∎

Using this, we can obtain an upper-bound on the size of an obstruction set.

Theorem 5.2 *If $c > 0$, then $|\mathrm{OBS}_c| \leq 2c^2$.*

Proof: For each r, there can be at most one c-minimal grid of the form $G_{r,n}$. Likewise, there can be at most one c-minimal grid of the form $G_{n,r}$. If $r \leq c$ then for all n, $G_{r,n}$ and $G_{n,r}$ are trivially c-colorable and are, therefore, not c-minimal. Theorem 5.1 shows that for all $n, m > c^2 + c$, $G_{n,m}$ is not c-minimal. It follows that there can be at most two c-minimal grids for each integer r where $c < r \leq c^2 + c$. Therefore there are at most $2c^2$ c-minimal grids in OBS_c. ∎

5.2 A Lower Bound

To get a lower bound on $|\mathrm{OBS}_c|$, we will combine Corollary 2.9 and Theorem 4.7(2) with the following lemma:

Lemma 5.3 *Suppose that $G_{m_1,n}$ is c-colorable and $G_{m_2,n}$ is not c-colorable. Then there exists a grid $G_{x,y} \in \mathrm{OBS}_c$ such that $m_1 < x \leq m_2$ (and in addition, $y \leq n$).*

Proof: Given n, let x be least such that $G_{x,n}$ is not c-colorable. Clearly, $m_1 < x \le m_2$. Now given x as above, let y be least such that $G_{x,y}$ is not c-colorable. Clearly, $y \le n$ and $G_{x,y} \in \mathrm{OBS}_c$. ∎

Theorem 5.4 $|\mathrm{OBS}_c| \ge 2\sqrt{c}(1 - o(1))$.

Proof: For any $c \ge 2$ and any $1 \le c' \le c$ we can summarize Corollary 2.9 and Theorem 4.7(2) as follows:

$$G_{c+c',n} \text{ is } \begin{cases} c\text{-colorable} & \text{if } n \le \left\lfloor \frac{c}{c'} \right\rfloor \binom{c+c'}{2}, \\ \text{not } c\text{-colorable} & \text{if } n > \frac{c}{c'}\binom{c+c'}{2}. \end{cases}$$

(We won't use the fact here, but note that this is completely tight if c' divides c.)

Suppose $c' > 1$ and

$$\frac{c}{c'}\binom{c+c'}{2} < \left\lfloor \frac{c}{c'-1} \right\rfloor \binom{c+c'-1}{2}. \tag{1}$$

Then letting $n := \left\lfloor \frac{c}{c'-1} \right\rfloor \binom{c+c'-1}{2}$, we see that $G_{c+c'-1,n}$ is c-colorable, but $G_{c+c',n}$ is not. Then by Lemma 5.3, there is a grid $G_{c+c',y} \in \mathrm{OBS}_c$ for some y. So there are at least as many elements of OBS_c as there are values of c' satisfying Inequality (1)—actually twice as many, because $G_{n,m} \in \mathrm{OBS}_c$ iff $G_{m,n} \in OBS_c$.

Fix any real $\varepsilon > 0$. Clearly, Inequality (1) holds provided

$$\frac{c}{c'}\binom{c+c'}{2} \le \left(\frac{c}{c'-1} - 1\right)\binom{c+c'-1}{2}.$$

A rather tedious calculation reveals that if $2 \le c' \le (1-\varepsilon)\sqrt{c}$, then this latter inequality holds for all large enough c. Including the grid $G_{c+1,n} \in \mathrm{OBS}_c$ where $n = c\binom{c+1}{2} + 1$, we then get $|\mathrm{OBS}_c| \ge \lfloor (1-\varepsilon)\sqrt{c} \rfloor$ for all large enough c, and since ε was arbitrary, we therefore have $|\mathrm{OBS}_c| \ge \sqrt{c}(1 - o(1))$.

To double the count, we notice that $c + c' \le \left\lfloor \frac{c}{c'} \right\rfloor \binom{c+c'}{2}$, whence $G_{c+c',c+c'}$ is c-colorable by Theorem 4.7(2). This means that $G_{c+c',y} \in \mathrm{OBS}_c$ for some $y > c + c'$, and so we can count $G_{y,c+c'} \in \mathrm{OBS}_c$ as well without counting any grids twice. ∎

6 Which Grids Can be Properly 2-Colored?

Lemma 6.1

1. *$G_{7,3}$ and $G_{3,7}$ are not 2-colorable*
2. *$G_{5,5}$ is not 2-colorable.*
3. *$G_{7,2}$ and $G_{2,7}$ are 2-colorable (this is trivial).*

4. $G_{6,4}$ and $G_{4,6}$ are 2-colorable.

Proof:

We only consider grids of the form $G_{n,m}$ where $n \geq m$.

1,2)

The following table, along with Corollary 2.4, shows why $G_{7,3}$ and $G_{5,5}$ are not 2-colorable.

m	n	$z_{n,m}$	$\lceil \frac{nm}{c} \rceil$
3	7	11	11
5	5	13	13

Table 1: Uncolorability values for $c = 2$

3) $G_{7,2}$ is clearly 2-colorable.

4) $G_{6,4}$ is 2-colorable by Corollary 4.16 with $p = 2$.

∎

Theorem 6.2 $\mathrm{OBS}_2 = \{G_{7,3}, G_{5,5}, G_{3,7}\}$.

Proof:

$G_{7,3}$ is not 2-colorable by Lemma 6.1. $G_{6,3}$ is 2-colorable by Lemma 6.1. $G_{7,2}$ is 2-colorable by Lemma 6.1. Hence $G_{7,3}$ is 2-minimal. The proof for $G_{3,7}$ is similar.

$G_{5,5}$ is not 2-colorable by Lemma 6.1. $G_{5,4}$ and $G_{4,5}$ are 2-colorable by Lemma 6.1. Hence $G_{5,5}$ is 2-minimal.

We need to show that $G_{7,3}$, $G_{5,5}$, and $G_{3,7}$ are the only 2-minimal grids. We consider the different possible values of n with $m \leq n$ and then use symmetry.

n	$m \leq n$	comment
$1, 2, 3, 4$	any $m \leq n$	$G_{n,m}$ is 2-colorable by Lemma 6.1
n	$1, 2$	$G_{n,m}$ is 2-colorable by Lemma 6.1
5	$3, 4$	$G_{n,m}$ is 2-colorable by Lemma 6.1
5	5	$G_{5,5} \in \mathrm{OBS}_2$
6	$3, 4$	$G_{n,m}$ is 2-colorable by Lemma 6.1
6	$5, 6$	$G_{n,m}$ is not 2-minimal since $G_{5,5}$ not 2-colorable
7	3	$G_{7,3} \in \mathrm{OBS}_2$
$n \geq 7$	$4, \ldots, n$	$G_{n,m}$ is not 2-minimal since $G_{7,3}$ not 2-colorable

∎

The following chart indicates exactly which grids are 2-colorable. The entry for (n, m) is C if $G_{n,m}$ is 2-colorable, and N if $G_{n,m}$ is not 2-colorable.

	2	3	4	5	6	7	8
2	C	C	C	C	C	C	C
3	C	C	C	C	C	N	N
4	C	C	C	C	C	N	N
5	C	C	C	N	N	N	N
6	C	C	C	N	N	N	N
7	C	N	N	N	N	N	N
8	C	N	N	N	N	N	N

7 Which Grids Can be Properly 3-Colored?

Lemma 7.1

1. *$G_{19,4}$ and $G_{4,19}$ are not 3-colorable.*
2. *$G_{16,5}$ and $G_{5,16}$ are not 3-colorable.*
3. *$G_{13,7}$ and $G_{7,13}$ are not 3-colorable.*
4. *$G_{12,10}$ and $G_{10,12}$ are not 3-colorable.*
5. *$G_{11,11}$ is not 3-colorable.*
6. *$G_{19,3}$ and $G_{3,19}$ are 3-colorable (this is trivial).*
7. *$G_{18,4}$ and $G_{4,18}$ are 3-colorable.*
8. *$G_{15,6}$ and $G_{6,15}$ are 3-colorable.*
9. *$G_{12,9}$ and $G_{9,12}$ are 3-colorable.*

Proof: We just consider the grids $G_{n,m}$ were $n \geq m$.
$1, 2, 3, 4, 5)$

Table 2, along with Corollaries 2.4 and 2.8, shows why $G_{19,4}$, $G_{16,5}$, $G_{13,7}$, $G_{12,10}$, and $G_{11,11}$ are not 3-colorable.

m	n	$z_{n,m}$	$\lceil \frac{nm}{c} \rceil$	r	$\binom{m}{2}$	
4	19		26	7	6	apply Corollary 2.8
5	16		27	11	10	apply Corollary 2.8
7	13	31	31			apply Corollary 2.4
10	12	40	40			apply Corollary 2.4
11	11	41	41			apply Corollary 2.4

Table 2: Uncolorability values for $c = 3$

6) $G_{19,3}$ is clearly 3-colorable.

7) $G_{18,4}$ is 3-colorable by Theorem 4.6 with $c = 3$.

8) $G_{15,6}$ is 3-colorable by Corollary 4.10 with $c = 3$.

9) $G_{12,9}$ is 3-colorable by Corollary 4.16 with $p = 3$. ▮

Lemma 7.2 $G_{10,10}$ *is 3-colorable.*

Proof: This is the 3-coloring:

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

▮

Note 7.3 The coloring in Lemma 7.2 we found by first finding a size 34 Rectangle Free Subset of $G_{10,10}$ and then using that for one of the colors and doing trial and error (with a short computer program). It is an open problem to find a general theorem that has a corollary that $G_{10,10}$ is 3-colorable.

Lemma 7.4 *If* $A \subseteq G_{11,10}$ *and* A *is rectangle-free then* $|A| \le 36 = \lceil \frac{11 \cdot 10}{3} \rceil - 1$. *Hence* $G_{11,10}$ *is not 3-colorable.*

Proof:

We divide the proof into cases. Every case will either conclude that $|A| \le 36$ or A cannot exist.

For $1 \le j \le 10$ let x_j be the number of elements of A in column j. We assume

$$x_1 \ge \cdots \ge x_{10}.$$

1. $5 \leq x_1 \leq 11$.

   By Lemma 3.7 with $x = 5$, $n = 11$, $m = 10$ we have

   $$|A| \leq x+m-1+\text{maxrf}(n-x, m-1) \leq 5+10-1+\text{maxrf}(11-5, 10-1) \leq 14+\text{maxrf}(6, 9).$$

   By Lemma 12.1 we have $\text{maxrf}(6, 9) = 21$. Hence

   $$|A| \leq 14 + 21 = 35 \leq 36.$$

2. There exists k, $1 \leq k \leq 6$, such that $x_1 = \cdots = x_k = 4$ and $x_{k+1} \leq 3$. Then

   $$|A| = \sum_{j=1}^{10} x_j = (\sum_{j=1}^{k} x_j) + (\sum_{j=k+1}^{10} x_j) \leq 4k + 3(10-k) = 30 + k$$

   Since $k \leq 6$ this quantity is $\leq 30 + 6 = 36$. Hence $|A| \leq 36$.

3. $x_1 = \cdots = x_7 = 4$ and, for all, $1 \leq j_1 < j_2 < j_3 \leq 7$,

   $$|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 0.$$

   Let G' be the grid restricted to the first 7 columns. Let B be A restricted to G'. Since every column of G' has 4 elements of B, $|B| = 7 \times 4 = 28$. Since every row of G' has ≤ 2 elements of B, $|B| \leq 2 \times 11 = 22$. Therefore A does not exist.

4. $x_1 = \cdots = x_7 = 4$ and there exists $1 \leq j_1 < j_2 < j_3 \leq 7$ such that

   $$|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 1,$$

   but for all $1 \leq j_1 < j_2 < j_3 < j_4 \leq 7$

   $$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 0.$$

   By renumbering we can assume that

   $$|C_1 \cap C_2 \cap C_3| = 1$$

   and that the intersection is in row 11. Let G' be the grid restricted to the first 7 columns. Let B be A restricted to G'. The following picture portrays what is in the first 3 columns of G'.

	1	2	3	4	5	6	7
1	R						
2	R						
3	R						
4		R					
5		R					
6		R					
7			R				
8			R				
9			R				
10							
11	R	R	R				

Since there are no $1 \le j_1 < j_2 < j_3 < j_4 \le 7$ with

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 1,$$

there will be no other elements of A in row 11 of G'. Let N' be the number of $4 \le j \le 10$ with $x_j = 4$ in G'. Note that $N' = 4$.

Let G'' be G' with row 11 removed. (We do not include a picture— just remove the last row from the picture above.)

For $1 \le j \le 7$ let y_j be the number of elements in the j^{th} column of G''. Let N'' be the number of $4 \le j \le 7$ with $y_j = 4$. Since G' cannot use the 11^{th} row in columns $4, 5, 6, 7$, $N' = N''$.

The first three columns are disjoint. By Theorem 3.11.3 with $k = 3$, $y_1 = y_2 = y_3 = 3$ (remember that these are the columns G'') $p = 1$, $q = 1$, $n = 10$, $m = 7$, we have

$$N' = N'' \le y_3 = 3.$$

Since $N' = 4$ this is a contradiction. Hence A cannot exist.

5. $x_1 = \cdots = x_7 = 4$ and there exists $1 \le j_1 < j_2 < j_3 < j_4 \le 7$ such that

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 1.$$

By renumbering we can assume that

$$|C_1 \cap C_2 \cap C_3 \cap C_4| = 1.$$

By Lemma 3.5

$$16 = x_1 + x_2 + x_3 + x_4 \le 11 + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 11 + 6 - 4 + 1 = 14.$$

Hence A does not exist.

∎

Theorem 7.5

$$\mathrm{OBS}_3 = \{G_{19,4}, G_{16,5}, G_{13,7}, G_{11,10}, G_{10,11}, G_{7,13}, G_{5,16}, G_{4,19}\}.$$

Proof:

For each $G_{n,m}$ listed above (1) by Lemma 7.1 or 7.4 $G_{n,m}$ is not 3-colorable, (2) by Lemma 7.1 or 7.2 both $G_{n-1,m}$ and $G_{n,m-1}$ are 3-colorable. Hence all of the grids listed are in OBS_3. We need to show that no other grids are in OBS_3. This is a straightforward use of Lemmas 7.1, 7.2, and 7.4. The proof is similar to how Theorem 6.2 was proven. We leave the details to the reader. ∎

The following chart indicates exactly which grids are 3-colorable. The entry for (n, m) is C if $G_{n,m}$ is 3-colorable, and N if $G_{n,m}$ is not 3-colorable.

	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
3	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
4	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N
5	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
6	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
7	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N
8	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N
9	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N
10	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N
11	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N
12	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N
13	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N
16	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
17	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
18	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
19	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
20	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

8 Which Grids Can be Properly 4 Colored?

In the first section we give absolute results about which grids are 4 colorable. In the second section we give results that assume a conjecture.

8.1 Absolute Results

Theorem 8.1

1. *$G_{41,5}$ and $G_{5,41}$ are not 4-colorable.*
2. *$G_{31,6}$ and $G_{6,31}$ are not 4-colorable.*
3. *$G_{29,7}$ and $G_{7,29}$ are not 4-colorable.*
4. *$G_{25,9}$ and $G_{9,25}$ are not 4-colorable.*
5. *$G_{23,10}$ and $G_{10,23}$ are not 4-colorable.*
6. *$G_{22,11}$ and $G_{11,22}$ are not 4-colorable.*
7. *$G_{21,13}$ and $G_{13,21}$ are not 4-colorable.*
8. *$G_{20,17}$ and $G_{17,20}$ are not 4-colorable.*
9. *$G_{19,18}$ and $G_{18,19}$ are not 4-colorable.*
10. *$G_{41,4}$ and $G_{4,41}$ are 4-colorable (this is trivial).*
11. *$G_{40,5}$ and $G_{5,40}$ are 4-colorable.*
12. *$G_{30,6}$ and $G_{6,30}$ are 4-colorable.*
13. *$G_{28,8}$ and $G_{8,28}$ are 4-colorable.*
14. *$G_{20,16}$ and $G_{16,20}$ are 4-colorable.*

Proof:

We only consider grids $G_{n.m}$ where $n \geq m$.

1,2,3,4,5,6,7,8,9)

Table 3 along with Corollaries 2.4, 2.8 and 2.10 show why $G_{41,5}$, $G_{31,6}$, $G_{29,7}$, $G_{25,9}$, $G_{23,10}$, $G_{22,11}$, $G_{21,13}$, $G_{20,17}$, and $G_{19,18}$ are not 4-colorable.

10) $G_{41,4}$ is clearly 4-colorable.

11) $G_{40,5}$ is 4-colorable by Theorem 4.6 with $c = 4$.

12) $G_{30,6}$ is 4-colorable by Theorem 4.7 with $c = 4$ and $c' = 2$.

13) $G_{28,8}$ is 4-colorable by Theorem 4.14 with $p = 2$, $d = 3$, and $s = 1$.

14) $G_{20,16}$ is 4-colorable by Theorem 4.14 with $p = 2$, $d = 2$, and $s = 2$

∎

Lemma 8.2 *If $A \subseteq G_{19,17}$ and A is rectangle-free then $|A| \leq 80 = \lceil \frac{19 \cdot 17}{4} \rceil - 1$. Hence $G_{19,17}$ is not 4-colorable.*

m	n	$z_{n,m}$	$\lceil \frac{nm}{c} \rceil$	r	$\binom{m}{2}$	$\frac{m(m-1)-2qr}{q(q-1)}$	
5	41		52	11	10		apply Corollary 2.8
6	31		47	16	15		apply Corollary 2.8
7	29		51	22	21		apply Corollary 2.8
9	25	57	57				apply Corollary 2.4
10	23		58			21	apply Corollary 2.10
11	22		61			21	apply Corollary 2.10
13	21	69	69				apply Corollary 2.4
17	20	85	85				apply Corollary 2.4
18	19		86			18	apply Corollary 2.10

Table 3: Uncolorability values for $c = 4$

Proof: We divide the proof into cases. Every case will either conclude that $|A| \le 80$ or A cannot exist.

For $1 \le j \le 17$ let x_j be the number of elements of A in column j. We assume

$$x_1 \ge \cdots \ge x_{17}.$$

1. $6 \le x_1 \le 19$.

   By Lemma 3.7 with $x = 6$, $n = 19$, $m = 17$,

   $$|A| \le x+m-1+\mathrm{maxrf}(n-x, m-1) \le 6+17-1+\mathrm{maxrf}(19-6, 17-1) = 22+\mathrm{maxrf}(13, 16).$$

   Assume, by way of contradiction, that $|A| \ge 81$. Then $\mathrm{maxrf}(13, 16) \ge 59$ By Lemma 2.7 with $n = 16$, $m = 13$, $a = 59$, $q = 3$, $r = 11$

   $$16 \le \left\lfloor \frac{13 \times 12 - 2 \times 3 \times 11}{3 \times 2} \right\rfloor = 15.$$

   This is a contradiction.

2. There exists k, $1 \le k \le 12$, such that $x_1 = \cdots = x_k = 5$ and $x_{k+1} \le 4$. Then

   $$|A| = \sum_{j=1}^{17} x_j = (\sum_{j=1}^{k} x_j) + (\sum_{j=k+1}^{17} x_j) \le 5k + 4(17-k) = 68 + k.$$

   Since $k \le 12$ this quantity is $\le 68 + 12 = 80$. Hence $|A| \le 80$.

3. $x_1 = x_2 = \cdots = x_{13} = 5$. Look at the grid restricted to the first 13 columns. Let B be A restricted to that grid. Note that B is a rectangle-free subset of $G_{19,13}$ of size 65. By Lemma 2.7 with $n = 19$, $m = 13$, $a = 65$, $q = 3$, and $r = 8$ we have

$$19 \le \left\lfloor \frac{13 \times 12 - 2 \times 8 \times 3}{3 \times 2} \right\rfloor = 18.$$

This is a contradiction, hence A cannot exist.

▮

Lemma 8.3 *$G_{24,9}$ is 4-colorable.*

Proof: We show that $G_{9,6}$ is strongly $(4,1)$-colorable and then apply Lemma 4.5 with $c = 4$ and $c' = 1$.

The following is a strong 4-coloring of $G_{9,6}$.

	1	2	3	4	5	6
1	Y	R	R	Y	R	R
2	Y	B	B	R	Y	B
3	Y	G	G	B	B	Y
4	R	Y	G	Y	G	R
5	B	Y	R	B	Y	G
6	G	Y	B	G	R	Y
7	G	B	Y	Y	B	G
8	R	G	Y	G	Y	R
9	B	R	Y	R	G	Y

▮

Theorem 8.4

1. *The following grids are in* OBS$_4$*:*

   $G_{41,5}$, $G_{31,6}$, $G_{29,7}$, $G_{25,9}$, $G_{9,25}$, $G_{7,29}$, $G_{6,31}$, $G_{5,41}$.

2. *For each of the following grids it is not known if it is 4-colorable. These are the only such.* $G_{17,17}$, $G_{17,18}$, $G18, 17$, $G_{18,18}$. $G_{21,10}$, $G_{22,10}$, $G_{23,10}$, $G_{22,10}$.

3. *Exactly one of the following grids is in* OBS$_4$*:* $G_{23,10}$, $G_{22,10}$, $G_{21,10}$.

4. *Exactly one of the following grids is in* OBS$_4$*:* $G_{22,11}$, $G_{21,11}$, $G_{21,10}$.

5. *Exactly one of the following grids is in* OBS$_4$*:* $G_{22,11}$, $G_{22,10}$, $G_{21,10}$.

6. *Exactly one of the following grids is in* OBS_4*:* $G_{21,13}$, $G_{21,12}$, $G_{21,11}, G_{21,10}$.

7. *Exactly one of the following grids is in* OBS_4*:* $G_{19,17}$, $G_{18,17}$, $G_{17,17}$.

8. *If* $G_{19,17} \in \mathrm{OBS}_4$ *then it is possible that* $G_{18,18} \in \mathrm{OBS}_4$.

Proof: This is easily proven from Lemmas 8.1,8.2, and 8.3. For a visual aid see the following chart where we put a C in the (n, m) spot if $G_{n,m}$ is Colorable, an N if it is not colorable, and a U if it is not known. ∎

	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21
8	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
9	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
10	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
11	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
12	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	U
13	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
14	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
15	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
16	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
17	C	C	C	C	C	C	C	C	C	C	C	C	C	U	U	N	N	N
18	C	C	C	C	C	C	C	C	C	C	C	C	C	U	U	N	N	N
19	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
20	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
21	C	C	C	C	C	C	C	C	U	N	N	N	N	N	N	N	N	N
22	C	C	C	C	C	C	U	N	N	N	N	N	N	N	N	N	N	N
23	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N
24	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N
25	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
26	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
27	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
28	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
29	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
30	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
31	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
32	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
33	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
34	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
35	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
36	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
37	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
38	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
39	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
40	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
41	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

8.2 Assuming the Rectangle-Free Conjecture

We have the following conjecture which, if true, yields more 4-colorings and allows us to state exactly what OBS_4 is.

Rectangle-Free Conjecture (RFC): Let $n, m, c \geq 2$. If there exists a rectangle-free subset of $G_{n,m}$ of size $\lceil nm/c \rceil$ then $G_{n,m}$ is c-colorable.

Lemma 8.5 *There exists a rectangle-free subset of $G_{22,10}$ of size $55 = \lceil \frac{22 \cdot 10}{4} \rceil$. Hence, if RFC is true, there is a 4-coloring of $G_{22,10}$ and $G_{10,22}$.*

Proof:

Here is the rectangle-free set.

	01	02	03	04	05	06	07	08	09	10
1	R						R			
2		R					R			
3			R				R			
4				R			R			
5					R		R			
6						R	R			
7	R	R						R		
8			R	R				R		
9					R	R		R		
10		R	R						R	
11				R	R				R	
12	R					R			R	
13	R			R						R
14		R				R				R
15			R		R					R
16		R			R					
17	R		R							
18				R		R				
19			R			R				
20		R		R						
21	R				R					
22							R	R	R	R

$\blacksquare$

Lemma 8.6 *There exists a rectangle-free subset of $G_{21,12}$ of size $63 = \lceil \frac{21 \cdot 12}{4} \rceil$. Hence, if RFC is true, there is a 4-coloring of $G_{21,12}$ and $G_{12,21}$.*

Proof:

Two grids are equivalent if you permute the rows and columns of one to get the other one. We show two grids that are not equivalent.

	01	02	03	04	05	06	07	08	09	10	11	12
1	R	R										
2	R		R									
3		R	R									
4			R	R	R							
5		R		R		R						
6	R				R	R						
7						R	R	R				
8					R		R		R			
9				R				R	R			
10						R				R	R	
11					R					R		R
12				R							R	R
13			R			R			R			R
14		R			R			R			R	
15	R			R			R			R		
16			R				R				R	
17		R							R	R		
18	R							R				R
19			R					R		R		
20		R					R					R
21	R								R		R	

	01	02	03	04	05	06	07	08	09	10	11	12
1	R	R										
2	R		R									
3		R	R									
4			R			R	R				R	
5		R			R				R	R		
6	R			R				R				R
7			R	R					R			
8			R				R					R
9			R			R				R		
10		R			R							R
11		R				R		R				
12		R							R		R	
13	R				R		R					
14	R							R		R		
15	R			R							R	
16				R	R	R						
17							R	R	R			
18										R	R	R
19						R			R			R
20					R			R			R	
21				R			R			R		

$\blacksquare$

Lemma 8.7 *There exists a rectangle-free subset of* $G_{18,18}$ *of size* $81 = \left\lceil \frac{18 \cdot 18}{4} \right\rceil$. *Hence, if RFC is true, there is a 4-coloring of* $G_{18,18}$.

Proof:

Here is the rectangle-free set.

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18
1		R		R										R		R	R	
2	R	R								R	R		R					
3	R								R						R	R		R
4						R			R			R	R	R				
5		R	R			R												R
6	R			R		R	R											
7							R	R		R				R				R
8			R				R		R		R						R	
9		R			R		R					R			R			
10				R							R	R						R
11	R		R		R									R				
12			R	R				R					R		R			
13					R	R		R			R					R		
14	R							R				R					R	
15				R	R				R	R								
16						R				R					R		R	
17			R							R		R				R		
18					R								R				R	R

∎

Note 8.8 If the 5^{th} row and the 2^{nd} column were removed then this would be a rectangle free set of $G_{17,17}$ of size 74. Note that $\lceil \frac{17\times 17}{4} \rceil = 73$. Hence if we had a weaker version of RFC then we would have that $G_{17,17}$ is 4-colorable.

Theorem 8.9 *Assume RFC is true. Then*

$$\mathrm{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \bigcup$$

$$\{G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}$$

Proof: For each $G_{n,m}$ listed above (1) by Lemma 8.1 or 8.2 $G_{n,m}$ is not 4-colorable, (2) by Lemmas 8.1, 8.3, 8.5, 8.6, or 8.7, $G_{n-1,m}$ and $G_{n,m-1}$ are 4-colorable. Hence all of the grids listed are in OBS_4. We need to show that no other grids are in OBS_4. This is a straightforward use of the lemmas listed above. The proof is similar to how Theorem 6.2 was proven. We leave the details to the reader. ∎

9 Application to Bipartite Ramsey Numbers

We state the Bipartite Ramsey Theorem. See [5] for history, details, and proof.

Def 9.1 $K_{a,b}$ is the bipartite graph that has a vertices on the left, b vertices on the right, an edge between every left and right vertex, and no other edges.

Theorem 9.2 *For all a, c there exists $n = BR(a, c)$ such that for all c-colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$.*

The following theorem is easily seen to be equivalent to this.

Theorem 9.3 *For all a, c there exists $n = BR(a, c)$ so that for all c-colorings of $G_{n,n}$ there will be a monochromatic $a \times a$ submatrix.*

In this paper we are c-coloring $G_{n,m}$ and looking for a 2×2 monochromatic submatrix. We have the following theorems which, except where noted, seem to be new.

Theorem 9.4

1. $BR(2, 2) = 5$. *(This was also shown in [10].)*
2. $BR(2, 3) = 11$.
3. $17 \leq BR(2, 4) \leq 19$.
4. $BR(2, c) \leq c^2 + c$.
5. *If p is a prime and $s \in \mathsf{N}$ then* $BR(2, p^s) \geq p^{2s}$.
6. *For almost all c,* $BR(2, c) \geq c^2 - 2c^{1.525} + c^{1.05}$.

Proof:
1) By Lemma 6.1 $G_{5,5}$ is not 2-colorable and $G_{4,4}$ is 2-colorable.
2) By Lemma 7.4 $G_{11,11}$ is not 3-colorable. By Lemma 7.2 $G_{10,10}$ is 2-colorable.
3) By Lemma 8.2 $G_{19,19}$ is not 4-colorable. By Lemma 8.1 $G_{16,16}$ is 4-colorable.
4) By Theorem 5.1 G_{c^2+c,c^2+c} is not c-colorable.
5) By Theorem 4.14 $G_{cr,cm}$ is c-colorable where $c = p^s$, $r = p^s$, and $m = \frac{p^{2s}-1}{p^s-1}$. Note that $m \leq p^s$. Hence $G_{p^{2s},p^{2s}}$ is p^s-colorable.
6)

Baker, Harman, and Pintz [1] (see [8] for a survey) showed that, for almost all c, there is a prime between c and $c - c^{0.525}$. Let p be that prime. By part 5 with $s = 1$, $BR(2, p) \geq p^2$. Hence

$$BR(2, c) \geq BR(2, p) \geq p^2 \geq (c - c^{0.525})^2 \geq c^2 - 2c^{1.525} + c^{1.05}.$$

∎

10 Open Questions

1. Find OBS_4. We feel this is possible since we are so close. A clever computer program may be needed.
2. Refine our tools so that our ugly proofs can be corollaries of our tools.
3. Find an algorithm that will, given c, find OBS_c or $|\mathrm{OBS}_c|$ quickly.
4. We know that $2\sqrt{c}(1-o(1)) \le |\mathrm{OBS}_c| \le 2c^2$. Bring these bounds closer together.
5. Is the Rectangle-Free Conjecture True? If so then this may help us find c-colorings. If not then this may open up new techniques for proving that a grid is not c-colorable.

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12 Appendix: Exact values of $\mathrm{maxrf}(n,m)$ for $0 \le m \le 6$, $m \le n$

Lemma 12.1

0) For $m \ge 0$, $\mathrm{maxrf}(0,m) = 0$.

1) For $m \ge 1$, $\mathrm{maxrf}(1,m) = m$.

2) For $m \ge 2$, $\mathrm{maxrf}(2,m) = m+1$.

3) For $m \ge 3$, $\mathrm{maxrf}(3,m) = m+3$.

4)

$$\mathrm{maxrf}(4,m) = \begin{cases} m+5 & \text{if } 4 \le m \le 5 \\ m+6 & \text{if } m \ge 6 \end{cases}$$

5)

$$\text{maxrf}(5, m) = \begin{cases} 12 \ \textit{if} \ m = 5 \\ m + 8 \ \textit{if} \ 6 \leq m \leq 7 \\ m + 9 \ \textit{if} \ 8 \leq m \leq 9 \\ m + 10 \ \textit{if} \ m \geq 10 \end{cases}$$

6)

$$\text{maxrf}(6, m) = \begin{cases} 2m + 4 \ \textit{if} \ 6 \leq m \leq 7 \\ 19 \ \textit{if} \ m = 8 \\ m + 12 \ \textit{if} \ 9 \leq m \leq 10 \\ m + 13 \ \textit{if} \ 11 \leq m \leq 12 \\ m + 14 \ \textit{if} \ 13 \leq m \leq 14 \\ m + 15 \ \textit{if} \ m \geq 15 \end{cases}$$

Proof:

Lemma 2.7 will provide all of the upper bounds. The lower bounds are obtained by actually exhibiting rectangle-free sets of the appropriate size. We do this for the case of $\text{maxrf}(6, m)$. Our technique applies to all of the other cases.

Case 1: $\text{maxrf}(6, m)$ **where** $6 \leq m \leq 7$ **and** $m = 8$**:** Fill the first four columns with 3 elements (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 12 are blocked. Hence we can fill the next $15 - 12 = 3$ columns with two elements each, and the remaining column (if $m = 8$) with 1 element. The picture below shows the result for $\text{maxrf}(6, 8) = 19$; however, if you just look at the first 6 (7) columns you get the result for $\text{maxrf}(6, 6)$ ($\text{maxrf}(6, 7)$).

R		R		R			R
R			R		R		
R	R					R	
	R	R			R		
	R		R	R			
		R	R			R	

Case 2: $\text{maxrf}(6, m)$ **where** $9 \leq m \leq 10$**:** Fill the first three columns with 3 elements each (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 9 are blocked. Hence we can fill the next $15 - 9 = 6$ columns with two elements each and the remaining column (if $m = 10$) with 1 element. The picture below shows the result for $\text{maxrf}(6, 10) = 22$; however, if you just look at the first 9 columns you get the result $\text{maxrf}(6, 9) = 21$.

R		R		R					
R					R		R	R	
R	R					R			
	R	R			R				
	R		R	R			R		
		R	R			R		R	R

Case 3: maxrf$(6, m)$ **where** $11 \leq m \leq 12$**:** Fill the first two columns with 3 elements each (they overlap). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 6 are blocked. Hence we can fill the next $15 - 6 = 9$ columns with two elements each and the remaining column (if $m = 12$) with 1 element. The picture below shows the result for maxrf$(6, 12) = 25$; however, if you just look at the first 11 columns you get the result maxrf$(6, 11) = 24$.

R		R		R						R	
R					R		R	R			
R	R					R					
	R	R			R				R		
	R		R	R			R				
			R			R		R	R	R	R

Case 4: maxrf$(6, m)$ **where** $13 \leq m \leq 14$**:** Fill the first column with 3 elements. This column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs. Hence we can fill the next $15 - 3 = 12$ columns with two elements each and the remaining column (if $m = 14$) with 1 element. We omit the picture.

Case 5: maxrf$(6, m)$ **where** $m \geq 15$**:** Fill the first $\binom{6}{2} = 15$ columns with two elements each in a way so that each column has a distinct pair. Fill the remaining $m - 15$ columns with one element each. The result is a rectangle-free set of size $30 + m - 15 = m + 15$. ∎

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